HAMILTONIAN CATASTROPHES

by

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In the five years since the publication of Foundations of Mechanics [FM] there have been a number of results extending our understanding of the qualitative structure of conservative mechanical systems, and I would like to take this opportunity to bring up to date the picture presented in that book, and to correct several mistakes. I am indebted to Clark Robinson, Ken Meyer, and Floris Takens for valuable discussions over these years, as well as their published papers, which contain most of the results I am going to describe, and to Joel Robbin and Floris Takens for criticizing this manuscript.

Over these years, I have been deeply concerned with the social problems of the world, the misuse of technology, and the basic question: is mathematics worth doing. My experiences and reflections during this period leave me with the conviction that much of mathematics, including the ideas of this lecture in particular, are part of the intellectual wealth of mankind, essential to our evolution and survival. I may add, however, the qualification that the relevance and value of this kind of wealth is dependent upon applications, which up to now, have been shamefully neglected. All wealth can be used, misused, or neglected.

CONTENTS

PART I. GENERIC PROPERTIES

1. Introduction. The book [FM] has three principal parts: formal mechanics (Chapters 3 and 4), qualitative theory (Chapter 6), and applications to the restricted three-body problem (Chapter 7). In the first of these, it now seems to me that the weakest parts are the treatment of generating functions and groups of symmetries. These subjects are now well understood, thanks mainly to the work of Weinstein [20] and Smale [12, 13] respectively, which Joel Robbin has just explained in his talks. The second part, on the generic qualitative behaviour, was described in the book as "a science fiction story about the future of the subject". Like space travel, most of this story, suitably corrected of course, has now become fact, through the work of Robinson [9, 10, 11] and Meyer [3, 4] especially. It is mainly these two developments which I want to describe in this lecture. The third part of the book, on the restricted problem of three-bodies, is much simplified by the observation that the Poisson bracket $\{\ell, g\}$ appearing in most of the equations is zero, as many readers have informed me. The rather complicated proof can be found in Brouwer and Clemence [22]. Also, this area has been greatly enriched by the work of Smale [12, 13].

2. Five years ago. Consider a manifold M of dimension $2n$ with symplectic form ω, and a function $H : M \longrightarrow \mathbb{R}$. The symplectic gradient of H, $X_H = \mathrm{grad}_\omega(H)$, is a globally Hamiltonian vectorfield on M, and

$$\mathrm{grad}_\omega : \mathcal{C}^{r+1}(M, \mathbb{R}) \longrightarrow \mathfrak{X}^r(M)$$

is linear, with one-dimensional kernel, if M is connected. Here $\mathcal{C}^{r+1}(M, \mathbb{R})$ denotes the C^{r+1} real functions, and $\mathfrak{X}^r(M)$ the C^r vectorfields, on M. The image of grad_ω is $\mathfrak{X}^r_H(M)$, the space of globally Hamiltonian vectorfields on M. We will denote this space, with the C^r Whitney topology, by $\mathcal{H}^r$. This is a Baire space, so a residual subset (a countable intersection of open dense sets) is dense. A property $P(X)$ for $X \in \mathcal{H}^r$ is C^r generic if $\{X \in \mathcal{H}^r \mid P(X) \text{ true}\}$ is residual in $\mathcal{H}^r$. In 1966 I conjectured that six properties were generic in this sense:

H1 : on characteristic exponents of critical points [FM, p. 180]

H2 : on characteristic multipliers of closed orbits

H5 : on the lack of first integrals other than the energy

H6 : on stability of the critical set [FM, p. 182]

T : on the twist of closed orbits [FM, p. 185], and

S : on structural stability within energy surfaces [FM, p. 187].

The first of these was mistakenly called a Theorem. I could have added to this list three others:

H3 : transversal intersection of stable and unstable manifolds.

H4 : The closed orbits are dense.

H7 : homoclinic points are dense in compact energy surfaces.

3. Recent results. Since 1966, there have appeared results relating to all of these conjectures. The property H1 was proved generic by Buchner [1], and independently by Robinson [9] in 1968. Robinson [8] and Meyer at the same

time explained why H2 could not be generic, and Robinson showed that a modified property, which I will call R2, is generic. In addition, he proved that H5 and H6 are generic, but S is not (see also [27]). Meyer and Palmore [6] provided examples of the typical behaviour of R2 systems: bifurcation of closed orbits. Then, in 1970, Meyer [3] gave a complete classification of the bifurcations arising generically in the case of two degrees of freedom (n=2, dimension of m = 4). That is, he defined a property I will call M1 for $X \in \mathcal{H}^r$, for r sufficiently large, indicated it was C^r-generic, and classified the bifurcations of M1 systems. This property is difficult to describe, but it is similar to condition T . In 1971, Meyer [4] showed that a stronger condition, I will call it M2 , is generic, and used it to decide the orbital stability of the bifurcating orbits of his previous classification. All conditions of a general type, including R2 , M1, M2 , and T were proven to be generic by Takens [14] in 1970. Simultaneously, Robinson proved that T is generic, and H3 is not. He proposed a corrected property, R3 , and proved it is generic. Property R3 differs from H3 more or less as R2 differs from H2 , and I will omit the definition [10]. Also, Pugh proved the C^1 closing lemma for Hamiltonian systems [30], and established H4 as a generic property , and Takens proved H7 and related properties are generic [15] .

There may be other results unknown to me, but here I would like especially to discuss the work of Robinson and Meyer relating to H2 and bifurcations of orbit cylinders, to complete a general picture of the generic pathology of Hamiltonian systems which I tried unsuccessfully to describe in 1966.

4. Orbit Cylinders. Now I will consider a fixed Hamiltonian system $X_H \in \mathcal{H}^r$ on a symplectic manifold M of dimension $2n$, with $r \geq 1$, and a closed orbit $\gamma \subset M$. Then for all points $m \in \gamma$, $H(m) = c$, a constant, and $H^{-1}(c)$ contains a hypersurface Σ_c, with $\gamma \subset \Sigma_c$. Also, a neighborhood of $\gamma \subset M$ is foliated by such energy hypersurfaces $\Sigma_e \subset H^{-1}(e)$, for $e \in (c - \delta, c + \delta)$. Let S be a local transversal section of X_H at $m \in \gamma$, having dimension $2n-1$, and Θ the associated Poincaré map. The eigenvalues of $T_m\Theta$, which are independent of $m \in \gamma$ and S, are the characteristic multipliers, or CM's, of X_H at γ. Because of the symplectic eigenvalue theorem [FM, 13.16] on the quadruplet symmetry of the spectrum of a symplectic linear transformation, these CM's occur in quadruplets $(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$ of equal multiplicity, and $\lambda = 1$ is always a CM, with odd multiplicity. Following Robinson, the principal characteristic multipliers, or PCM's of X_H at γ are defined as follows: the CM 1 of multiplicity $2k + 1$ is a PCM of multiplicity k, the CM -1 of multiplicity $2k$ is a PCM of multiplicity k. To a unimodular CM pair $(\lambda; \bar{\lambda}; |\lambda| = 1, \operatorname{Re}(\lambda) > 0)$ of multiplicity k corresponds the single PCM λ with multiplicity k. To a real CM pair $(\lambda, \lambda^{-1}; |\operatorname{Re}(\lambda)| > 1)$ of multiplicity k corresponds the single PCM λ with multiplicity k. And finally, to a CM quadruplet $(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}; |\lambda| > 1, \operatorname{Re}(\lambda) > 0)$ of multiplicity k corresponds a pair of PCM's $(\lambda, \bar{\lambda})$ of multiplicity k. Counting multiplicities, the $2n-1 = 2(n-1) + 1$ CM's have been replaced by $(n-1)$ PCM's (see Figure 1). The theorem on cylinders of closed orbits [FM] may now be expressed this way.

Definition. An energy cylinder of X_H is an embedding $\Gamma : S^1 \times (a, b) \longrightarrow M$ such that for all $e \in (a, b)$, $\gamma_e = \Gamma[S^1 \times \{e\}]$ is a closed orbit of X_H of energy $H[\gamma_e] = e$, and Γ is transversal to the energy surface Σ_e. That is, $H \circ \Gamma$ has no critical point.

Energy Cylinder Theorem. If γ is a closed orbit of X_H, then γ is contained in an energy cylinder if and only if 1 is not a PCM of γ.

One could ask how these cylinders terminate, and now this question may be answered more fully than in 1966. Next, I will recall the definition of property H2. Let $\lambda_1,\ldots,\lambda_k$ be the unimodular PCM's of γ, and write $\lambda_j = \exp(2\pi i\alpha_j)$, $\alpha_j \in [0, \frac{1}{2}]$, $j = 1,\ldots,k$. For a positive integer N, γ is N-elementary if its unimodular PCM's have multiplicity one, and for all integers $p_1,\ldots,p_k \in [-N, N]$,

$$\sum_{j=1}^{k} p_j\alpha_j \text{ an integer} \Rightarrow p_1 = \ldots = p_k = 0$$

Further, γ is $\aleph$-elementary if it is N-elementary for all N. Finally, X_H has property H2 if all its closed orbits are $\aleph$-elementary. All of this is review. But now it is easy to see why H2 cannot be generic, as Robinson and Meyer pointed out in 1968, and undoubtedly this has been known to specialists of celestial mechanics for quite a long time. For if γ_c is an $\aleph$-elementary closed orbit of X_H, then 1 is not among its PCM's, so the Energy Cylinder Theorem applies, and γ belongs to cylinder of closed orbits $\{\gamma_e\}$ parameterized by the energy, $\gamma_e \subset \Sigma_e$. The PCM's of γ_e vary continuously with e, and thus also the transverse frequencies $\{\alpha_j(e)\}$, which may therefore be $\mathbb{Z}$-dependant for nearly all (that is, a dense set of) values $e \in (c-\epsilon, c+\epsilon)$. Perturbations of H do not improve the situation.

5. Generic conditions of Robinson. The first improvement in the treatment of orbit cylinders is the 0-elementary condition of Robinson. This concerns the question: what happens when 1 occurs as a PCM. Let m, γ, Σ_c, S

and θ be as before. Then the vector subspace

$$V_m = T_m S \cap \operatorname{Ker} dH(m) < T_m S$$

is the tangent space at m of the energy subsurface $S \cap \Sigma_c$ within S. Following Robinson, we say γ is <u>0-elementary</u> if the image of

$$T_m\theta - I : T_m S \to T_m S$$

contains V_m, where I is the identity of $T_m S$. If 1 is not a PCM, then γ is 0-elementary. But if 1 is a PCM, γ may still be 0-elementary (or not). Thus 0-elementary is a weaker condition than $1 \notin$ PCM, and the following is a stronger result than the Energy Cylinder Theorem of [FM].

<u>Orbit Cylinder Theorem of Robinson</u> [9].

<u>If γ is a 0-elementary closed orbit, then there is a cylinder $\Gamma : S^1 \times (-\delta, \delta) \to M$ such that $\gamma_\lambda = \Gamma[S^1 \times \{\lambda\}]$ is a closed orbit, and $\gamma_0 = \gamma$.</u>

Here the parameter λ may not be assumed to be the energy unless γ satisfies the stronger condition $1 \notin$ PCM. This theorem is also proved by Meyer [3] in the special case of $n=2$ degrees of freedom. As Robinson has shown that the condition <u>R2-0</u> : <u>all closed orbits are 0-elementary</u> is a C^2-generic property, I may suppose from now on that in the system X_H, all closed orbits lie in orbit cylinders. Further, for such an orbit cylinder $\{\gamma_\lambda\}$, I will assume that a curve

$$\sigma : [-\delta, \delta] \to \operatorname{Symp}(\mathbb{R}^{2n-2}) : \lambda \to A_\lambda$$

has been constructed so that the spectrum of the symplectic linear transformation

A_λ is the set of CMs of γ_λ, with one 1 removed. This construction is tedious but possible. Then in Symp($\mathbb{R}^{2n-2}$), let B_N denote the bad set of transformations A such that the unimodular eigenvalues of A, say

$$\exp(\pm 2\pi i\alpha_1),\ldots, \exp(\pm 2\pi i\alpha_k),\ \alpha_j \in [0, \tfrac{1}{2}]$$

(each repeated according to multiplicity) have linearly dependant transverse frequencies over $[-N, N]$. That is, there are integers $p_1,\ldots,p_k \in [-N, N]$, not all zero, such that

$$p_1 \alpha_1+\ldots+ p_1 \alpha_k \text{ is an integer.}$$

At this point it is necessary to know that B_N is a semi-algebraic set without interior, and therefore a union of submanifolds of positive codimension. A careful proof is found in Buchner [1] . The orbit cylinder Γ is said to be <u>N-elementary</u> if the symplectic curve σ is transversal to the submanifolds comprising the bad set B_N. This implies that σ will intersect B_N only in isolated values of the parameter $\lambda \in (-\delta, \delta)$, and only in strate of B_N of codimension one. Now Robinson's definition: X_H satisfies <u>R2</u> if every closed orbit is O-elementary, and every orbit cylinder is N-elementary, for all positive integers N.

<u>Theorem of Robinson</u> [9]. <u>Property R2 is C^r-generic</u> , for $r \geq 2$.

Actually, Robinson's conditions are a little stronger, as he proved independence for all of the PCM's , not only the unimodular ones. However, I think that this version contains the essense of his result. In fact, the bad behaviour of Hamiltonian systems stems from resonances of the unimodular transverse frequencies only. And according to Robinson's theorem, resonance of order N $(\sigma(\lambda) \in B_N)$ occurs only at isolated closed orbits within a given orbit cylinder. As N increases, these may become more numerous, and a very high order resonance could occur for nearly all (that is, a dense set of) orbits in the cylinder. Perhaps it is not too early for me to admit that conjecture H2 was a bit optimistic.

PART II. BIFURCATIONS

6. Introduction. The principal pathology of the qualitative theory of Hamiltonian systems involves the origination and termination of orbit cylinders, and their incidence upon each other. These phenomena, called bifurcations, occur also in one-parameter families of general dynamical systems. The theory of bifurcations is now only in its infancy, and its importance in applications is established by a number of recent developments, especially the work of René Thom, about which I will speak in my next lecture. The work of Meyer [3] is a first step toward a taxonomy of generic bifurcations of orbit cylinders of Hamiltonian systems. He gives a complete zoo in the case of $n=2$ degrees of freedom, where an orbit cylinder has only one PCM, which is an Nth root of unity if it is in the bad set B_N . I will dodge the meaning of generic in this context, except to say that it involves properties M1 and M2 for the Hamiltonian system X_H , defined in terms of higher derivatives of the flow, and these conditions are proved to be C^r-generic, for r sufficiently large, by Takens [14]. Thus with $\dim(M) = 4$, and $X_H \in \mathcal{H}^r$ satisfying H1 , R2 , M1 , and M2 , I am going to describe geometrically the configurations of orbit cylinders which can occur.

7. The burst and reincarnation.

First, an orbit cylinder can originate or terminate at a critical point. This bifurcation, a metaphor for asexual creation, was known to Liapounov, and is described in the Liapounov Stability Theorem [FM; 29.3]. It is similar to the Hopf bifurcation in the context of one-parameter families of vectorfields, which I will describe in my next lecture.

Let $m \in M$ be an μ-elementary critical point of X_H. Two cases arise (as n=2) : either m is a saddle-center, with CMs $(\exp(\pm 2\pi i\alpha), \mu, \mu^{-1})$ with $\alpha \in (0, \frac{1}{2})$ and irrational, $\mu > 1$ real, or m is a pure center, with CMs $(\exp(\pm 2\pi i\alpha), \exp(\pm 2\pi i\beta))$ with $\alpha, \beta \in (0, \frac{1}{2})$, and irrationally related. Take the saddle-center case first, and let $A < T_m M$ be the eigenspace of $\exp(\pm 2\pi i\alpha)$. By the Liapounov Theorem, there is a 2-dimensional submanifold $C \subset M$ tangent to A at m, consisting entirely of closed orbits of transverse frequency approximately α, the center manifold of m. The center manifold must be an orbit cylinder therefore, closed by the point m, as shown in Figure 2. As X_H is assumed R2, this cylinder is N-elementary for all N. Suppose this cylinder is parameterized as $\{\gamma_\lambda\} = \Gamma$, with γ_λ tending to m as $\lambda > 0$ tends to zero. The flow normal to γ_λ, for λ sufficiently small, is governed by the CMs (μ, μ^{-1}) of m, as $C = \Gamma \cup \{m\}$ is tangent to A at m. Thus the CM μ_λ of γ_λ approaches μ as λ tends to zero, and therefore is eventually real. Thus a disk in C around m consists entirely of closed orbits of hyperbolic type (real PCM), closed by m. As in the qualitative view only elliptic orbit cylinders (unimodular PCM) are significant, because of their generic orbital stability [FM; 30.4, here generic means property T] this case is of no great significance/

Now consider the second possibility, in which m is a pure center. Let $A < T_m M$ be as before, and $B < T_m M$ be similarly the eigenspace of the CMs $\exp(\pm 2\pi i\beta)$. The Liapounov construction now applies to both A and B, so we have two orbit cylinders (not intersecting) closed by m, comprising the two subcenter manifolds of m, say

$$C_i = \Gamma_i \cup \{m\} \ , \ \Gamma_i = \{\gamma_\lambda^i\} \ , \ i = \alpha, \beta \ .$$

Here the PCM of γ_λ^α approaches $\exp(2\pi i\alpha)$ as λ tends to zero. That is, the transverse frequency of γ_λ^α approaches α. Similarly, the transverse frequency of γ_λ^β approaches β. If the period of a closed orbit γ is τ, then its orbital frequency is $2\pi/\tau$. The orbital and transverse frequencies must not be confused. In this case, the transverse frequency of γ_λ^α approaches the orbital frequency of γ_λ^β, and vice versa. Eventually, both γ_λ^α and γ_λ^β are of elliptic type, and therefore of qualitative significance. As PCM $\neq 1$ in either case, the parameter λ can be taken to be the energy, which may have at m either a local extremum or a saddle point. In the first case, in each energy surface Σ_e near m, we have two elliptic closed orbits γ_e^α and γ_e^β, collapsing to m as e aproaches its critical value $c = H(m)$, as shown in Figure 3. The energy surfaces are ellipsoids around m. From the catastrophe point of view, this represents the simultaneous creation (or annihilation in vacuo of twin stable oscillations of (possibly) large orbital and transverse frequencies, with amplitude increasing from (or decreasing to) zero as e passes its critical value c at m, the relative extremum, the burst catastrophe.

In the saddle case, the energy surfaces are hyperboloidal, and each contains a closed orbit, γ_e^α for $e > c$, γ_e^β for $e < c$. As e passes c, γ_e^β shrinks, dies, and is reborn as γ_e^α, the reincarnation catastrophe.

In the case of $n=3$ degrees of freedom, we would have three possibilities for the CMs of m : one, two, or three subcenter manifolds or orbit cylinders at m. But only in the case of a pure center (three orbit cylinders) would any (and thus all) of the cylinders be of elliptic (stable) type. And in this case, the critical point m could be either an extremum or a saddle. If extremal, three qualitatively significant closed orbits are simultaneously created (or annihilated) as e increases past c the burst .

If a saddle, one closed orbit dies, and two are born (or vice versa) as e passes c, reincarnation. For n degrees of freedom, we may have in the pure center case k cylinders dying, and reincarnated as $n-k$ cylinders.

8. Creation. In the case of n=2 degrees of freedom, a closed orbit γ has only one PCM μ, either real ($|\mu| > 1$, the hyperbolic case), unimodular ($\mu = \exp(2\pi i\alpha)$, $\alpha \in (0, \frac{1}{2})$, the elliptic case), or both ($\mu = \pm 1$, the degenerate cases.) Thus for n=2, the bad set B_N corresponds to $\mu = \exp(2\pi i\alpha)$ with $\alpha \in [0, \frac{1}{2}]$, and there is a non-zero integer $p \in [-N, N]$ such that $p\alpha$ is an integer, or $\alpha = q/p$. Without loss of generality, as α is non-negative, we may assume p is positive, and q non-negative, so μ is a p-th root of unity. In other words, for an orbit cylinder $\{\gamma_\lambda\}$ in the case n=2, we have non-H2 behaviour whenever the PCM μ_λ is a p-th root of unity, p = 1,2,..., etc. In the rest of this lecture I will consider these cases one at a time, and describe the results of Meyer, who classified all the generic phenomena which arise with n=2. He calls these the generic p-bifurcations, and in this section I consider the first case, p = 1, which Meyer calls an extremal closed orbit. Thus we have a 0-elementary orbit cylinder $\{\gamma_\lambda\}$, with PCM μ_λ, and γ_0 is extremal, or $\mu_0 = 1$. By the Energy Cylinder Theorem, the orbit cylinder is tangent to the energy surface Σ_c, $c = H(\gamma_0)$, so λ cannot be the energy of γ_λ. Generically, Meyer shows that this occurs only when μ_λ changes transversally from real to unimodular values, passing through 1 at $\lambda = 0$, so γ_λ changes suddenly from hyperbolic to elliptic type (or vice versa) as λ increases through zero, as shown in Figure 4. Also, he shows that the parameter can be chosen such that the period of γ_λ is $\tau_\lambda = \tau_0 + \lambda$, and that γ_0 is orbitally unstable.

From the catastrophe point of view, with the energy e as parameter, the vector-field $X_H | \Sigma_e = X_e$ suddenly develops an unstable periodic orbit γ_0 for e = c, of large amplitude, presumably by a Pugh catastrophe: The closing of a recurrent orbit. For e > c, this extremal orbit γ_0 splits into two closed orbits γ_e^- and γ_e^+ , where $\gamma_e^- = \gamma_\lambda$ for some $\lambda < 0$, and is hyperbolic, and $\gamma_e^+ = \gamma_\lambda$ for some $\lambda > 0$, and is elliptic. As only γ_e^+ is qualitatively "visible" , a single elliptic closed orbit has suddenly made its appearance in the phase portrait of X_e , as e increased past $e = c$, and nearby is its phantom dual, γ_e^- , which is qualitatively invisible. Alternatively, the process could be read in reverse, as the instantaneous annihilation of a large closed orbit, through cancellation by a phantom dual. I therefore call this phenomenon the creation (or annihilation) catastrophe .

9. Subtle division. Next let p=2 , the 2-bifurcation or transitional orbit of Meyer. As 1 is not a PCM in this case (or in fact in any of the remaining cases) the orbit cylinder may be parameterized by the energy according to the energy cylinder theorem. Thus $\Gamma = \{\gamma_e\}$, $\mu_e = \text{PCM}(\gamma_e)$, and $\mu_c = -1$. Generically, according to Meyer, the transitional orbit γ_c occurs only for a transversal change of μ_e from unimodular to real values, through the common point -1 , and γ_e undergoes "transition" from elliptic to hyperbolic type as e increases through c , or vice versa. This aspect is similar to the extremal orbit of the creation catastrophe, but μ_e moves through -1 instead of +1 . But in this case energy is the parameter, and there is a further pathology, in the incidence at $\gamma_c \subset \Gamma$ of another orbit cylinder $\Delta = \{\delta_e \mid e > c\}$. Two cases arise. In the first, δ_e is of elliptic type. As e approaches c from above, δ_e tends to a double covering of γ_c , and the orbital frequency of δ_e approaches half the orbital frequency of γ_c . Thus we may consider δ_e a sub-harmonic of γ_c , approaching resonance, as shown in Figure 5. Meyer has shown that the transitional orbit γ_c is orbitally

stable. Thus from the catastrophe point of view, as e increases through c, we have the significant orbit γ_e replaced by the qualitatively visible sub-harmonic δ_e, while γ_e itself becomes invisible. Qualitatively, the behaviour is not changed very much, as δ_e is approximately a double covering of γ_c, so the orbital frequency and amplitude of the oscillation are not catastrophically changed. Only later, as e increases considerably, will it become apparent to an observer that δ_e has doubled its period because δ_e is no longer running twice around in a neighbourhood of γ_e. Hence I call this phenomenon subtle division. Read the other way, with energy decreasing with time, a visible oscillation doubles over itself, and resonates with a phantom oscillation having twice its orbital frequency, a subtle doubling. These are two versions of the first of the two cases arising generically when the PCM is -1.

10. Murder. In this case, the arriving sub-harmonic orbit cylinder $\Delta = \{\delta_e\}$ is of hyperbolic type, and approaches from the other side of Σ_c, that is, along the elliptic part of Γ. Therefore, the configuration is identical to subtle division, with "elliptic" and "hyperbolic" interchanged everywhere in Figure 5, and the energy parameter reversed. Thus γ_e changes from elliptic type $(e < c)$ to hyperbolic type $(e > c)$ at γ_c, the transitional orbit, which in this case is orbitally unstable. The hyperbolic sub-harmonic δ_e $(e < c)$ approaches a double covering of γ_e as e increases to c, and the orbit cylinder Δ terminates at γ_c, as shown in Figure 6. From the catastrophe point of view, only γ_e, for $e < c$, is visible. The phantom killer δ_e approaches the stable orbit γ_e at half its orbital frequency. At $e = c$, resonance occurs, δ_e dissappears, and γ_e dies. For $e > c$, γ_e persists only as a ghost, so I call it the murder

catastrophe. In this case also, the transition can be interpreted with time (energy) reversed. Thus a ghost (γ_e) suddenly materializes (becomes elliptic), and emits a phantom sub-harmonic (δ_e , hyperbolic), the materialization catastrophe.

11. The phantom kisses. The cases already described cover transitions between hyperbolic and elliptic states (PCM = ± 1) and the remaining p-bifurcations (PCM = p-th root of unity, p = 3,4,...) all must occur along orbit cylinders, parameterized by the energy, which remain elliptic during the bifurcation. The case p=3 and one of the two cases with p=4 are very similar. The orbit cylinder $\Gamma = \{\gamma_e\}$ is elliptic, and γ_e has PCM $\exp(2\pi i \alpha_e)$. For $e = c$, the transverse frequency is $\alpha_c = \frac{1}{3}$ [resp. $\frac{1}{4}$], and generically, α_e passes transversally through this value. Nearby are two other orbit cylinders

$$\Delta^+ = \{\delta_e : e \in (c-\epsilon, c)\}$$

$$\Delta^- = \{\delta_e : e \in (c, c+\epsilon)\}$$

of hyperbolic type. As e approaches c from above or below, δ_e approaches a triple [resp. quadruple] covering of γ_e , as shown in Figure 7. Both cylinders terminate at γ_c , which is orbitally unstable. The set of closed orbits

$$\Delta = \Delta^+ \cup \gamma_c \cup \Delta^- = \{\delta_e : e \in (c-\epsilon, c+\epsilon)\}$$

with $\delta_c = \gamma_c$ may be considered an orbit "cylinder" , degenerate at $e = c$. In the catastrophe view, the phantom δ_e approaches γ_e , resonates to its third [resp. fourth] harmonic, kisses Γ at $\gamma_c = \delta_c$, an excited state, undergoes subtle frequency division (falls to its ground state) and departs again. The actor γ_e loses his stability momentarily, but recovers immediately. An observer sees nothing but this momentary instability, if anything.

12. Emission. The second case of 4-bifurcation, and all cases of p-bifurcation for $p > 4$ (there is generically only one phenomenon for each $p > 4$, according to Meyer) are similar. Again $\Gamma = \{\gamma_e\}$ is an elliptic orbit cylinder, and the transverse frequency α_e passes transversally through $\alpha_c = \frac{1}{4}$ [resp. $\frac{q}{p}$, $p > 4$, $1 \le q \le [p/2]$]. There are two nearby orbit cylinders, an elliptic one $\Delta = \{\delta_e\}$ and a hyperbolic one $E = \{\epsilon_e\}$, both defined only for $e > c$. As e approaches c from above, both δ_e and ϵ_e approach a p-fold covering of γ_e, Δ and E terminating at γ_c, as shown in Figure 8. The critical orbit γ_c is stable. Here, from the catastrophe view, something significant happens. As e increases through c, the stable orbit γ_e splits into two stable orbits, γ_e and δ_e, the latter at a subtly divided (sub-harmonic) orbital frequency. Although no change is observed in the principle actor γ_e, a new actor is emitted from γ_c, along with a phantom twin, ϵ_e. Hence, I call this the emission catastrophe, following Thom. Read in reverse, it is the absorbtion catastrophe.

CONCLUSION: The general pathology.

Finally, I would like to combine these results and the invariant tori of Kolmogorov, Arnol'd, and Moser, in a single picture. Suppose M has dimension four, $H : M \to \mathbb{R}$ is C^r, $r \geq 1$, and the Hamiltonian system X_H satisfies all of the generic conditions envisioned so far. The X_H has a set C of isolated critical points which are of hyperbolic (real or complex), saddle-center, or generic pure-center type, or $C = C_h \cup C_s \cup C_c$. Each point $m \in C$ has a different energy $H(m)$. The complement $M \setminus C$ is foliated by energy surfaces Σ_e of dimension three. A point $m \in C_h$ has two dimensional insets and outsets, and no center set. A point $m \in C_s$ has one dimensional insets and outsets, and a two dimensional center, which is a hyperbolic orbit cylinder. A point $m \in C_c$ has two sub-centers of dimension two, each an elliptic orbit cylinder - the burst catastrophe. The remaining closed orbits comprise orbit cylinders which originate and terminate on these center cylinders, or on each other. Each orbit cylinder Γ may be tangent to an energy surface only at isolated values of its cylinder parameter - the creation catastrophes. At each of these catastrophic tangencies, the cylinder changes from hyperbolic to elliptic type, or vice versa. The creation catastrophes of a given cylinder divide it into bands of closed orbits, parameterized by the energy. At isolated closed orbits in these bands, there may occur either subtle division (arrival of an elliptic cylinder of half the orbital frequency) or murder (arrival of a hyperbolic cylinder of half the orbital frequency). In either case, the original band changes from elliptic to hyperbolic type. Omitting the transitional orbits of all three types, the components of the rest of the original cylinder Γ is a union of connected orbit bands, each completely elliptic or hyperbolic. The hyperbolic bands have no further catastrophes. Let E be an elliptic band of Γ.

Then at isolated orbits in E, hyperbolic cylinders may touch E in one of the kisses, resonating to either their third or fourth octaves at contact. On the rest of E, the transverse frequency α_e of $\gamma_e = E \cap \Sigma_e$ passes continuously through rational and irrational values. At each rational value, the emission of a pair of sub-harmonic cylinders occurs, one elliptic and one hyperbolic. A few of these are illustrated in Figure 9, within a three dimensional submanifold $S \subset M$, which is transversal to the cylinder E ($S \cap E$ is a curve) and such that S is a transversal section for each orbit $\gamma_e \subset E$. The energy subsurfaces $S_e = S \cap \Sigma_e$ are drawn as horizontal subspaces, and the curve $S \cap E$ is a vertical line. Each S_e is a transversal section for γ_e within Σ_e, so closed orbits δ_e of an emitted cylinder Δ appear as periodic orbits of the Poincaré section map of S_e. As e varies, the periodic points corresponding to δ_e trace curves in S, which appear as ribs of an upside-down umbrella. At each e with α_e rational, such an umbrella originates, with $2p$ ribs, if α_e is a p-th root of unity. The ribs alternate elliptic/hyperbolic and are tangent to S_e at $X_e = \gamma_e \cap S$, according to Meyer [3]. Only a few umbrellas are drawn in Figure 9, but in fact, S is practically full of them. See also Figure 10.

Now fix an elliptic closed orbit $\gamma_e \subset E$. As X_H satisfies property T, the Poincaré section S_e has many concentric invariant circles around $x_e = \gamma_e \cap S$, corresponding to invariant tori around γ_e, on which X_H is an irrational rotation, according to the Moser Stability Theorem. The ribs of emitted umbrellas for $x_c = \gamma_c \cap S$ below X_e pierce the annuli bounded by the Moser circles. As x_c gets closer to x_c, the ribs emitted from x_c pierce progressively smaller annuli. For a given p, there will be a finite number of umbrellas of 2p ribs, so the ribs of the umbrella of x_c must get increasingly numerous as x_c approaches x_e .

Finally sweep the whole picture around the orbit cylinder E , to return to M - difficult to visualize? This is the garden variety elliptic orbit band.

Now consider a hyperbolic orbit band $H \subset \Gamma$. Then $\gamma_e \subset H \cap \Sigma_e$ is a normal hyperbolic closed orbit of a flow on a three dimensional manifold, and therefore has two dimensional inset and outset (that is, stable and unstable manifold) intersecting transversally through γ_e . By Robinson's generic condition R3 , these insets and outsets, for all hyperbolic orbits in Σ_e , intersect transversally within Σ_e . The hyperbolic bands, like H , originate and terminate only at saddle-center critical points by bursting on elliptic bands via kissing, murder, or emission, or by transition to an elliptic band through creation, subtle division, or murder.

A few possibilities for a typical orbit cylinder are shown in schematic form in Figure 11. Here, orbit cylinders are shown as curves - dotted for hyperbolic, solid for elliptic. Energy surfaces appear as horizontal curves.

This then, is the typical behaviour of a Hamiltonian system with two degrees of freedom. For higher dimensions, the burst phenomenon has already been described. The transitional catastrophes are similar to the cases described. Another kind of transition occurs already for three degrees of freedom. We may have a closed orbit with its four CMs arranged in a symplectic quadruplet, which collapses onto the unit circle, as shown in Figure 12. An example has been studied by Robinson [9]. The p-bifurcations in higher dimensions should admit many new catastrophes. Worse, catastrophes must be expected whenever the transverse frequencies are rationally related, not only when one of them is a root of unity.

At present, the study of Hamiltonian bifurcations is in its infancy, and to my knowledge, practically nothing is known for $n > 2$. However, the development of technical tools has begun. The techniques used by Meyer for $n=2$ are based on the Poincaré generating function. This is the subject of two papers by Weinstein [19, 20], as you know from Joel Robbin's talk. Also, recent work of Takens [15] attacks this problem, and the normal forms of Robinson [11] and of Burgonne-Cushman [21] should be helpful. At least, it seems there is much more information and activity now than in 1966, and I hope I may be forgiven the naivity of my ideas of that time, and that this review will compensate somewhat for the weaknesses of <u>Foundations of Mechanics</u>.

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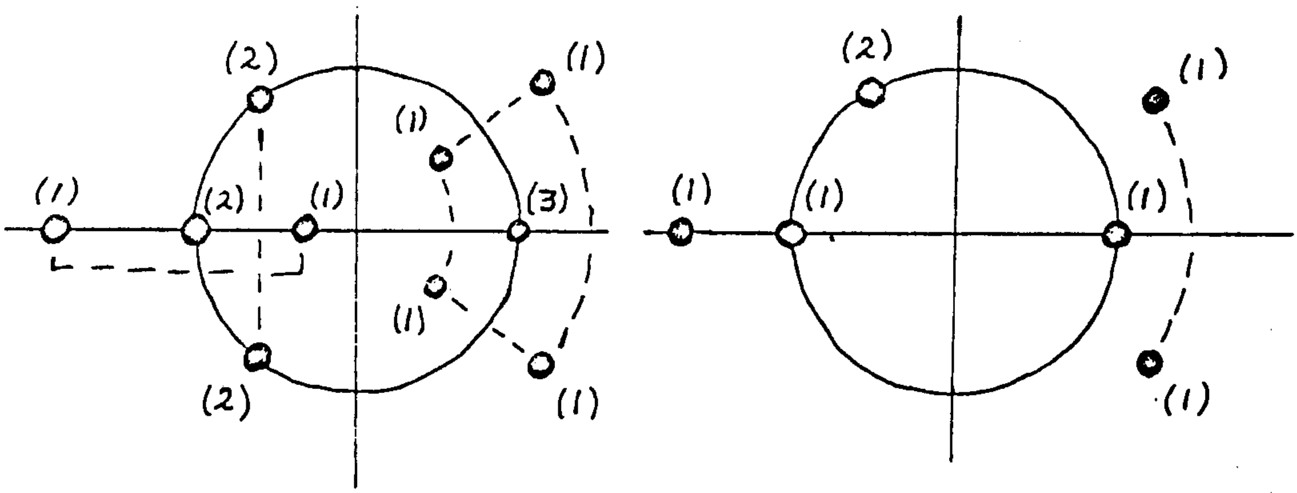

2n-1 = 15 CMs

n-1 = 7 PCMs

FIGURE 1

A typical symplectic spectrum, for a closed orbit of a Hamiltonian system, multiplicities in parenthesis, for eight degrees of freedom. See Section 4.

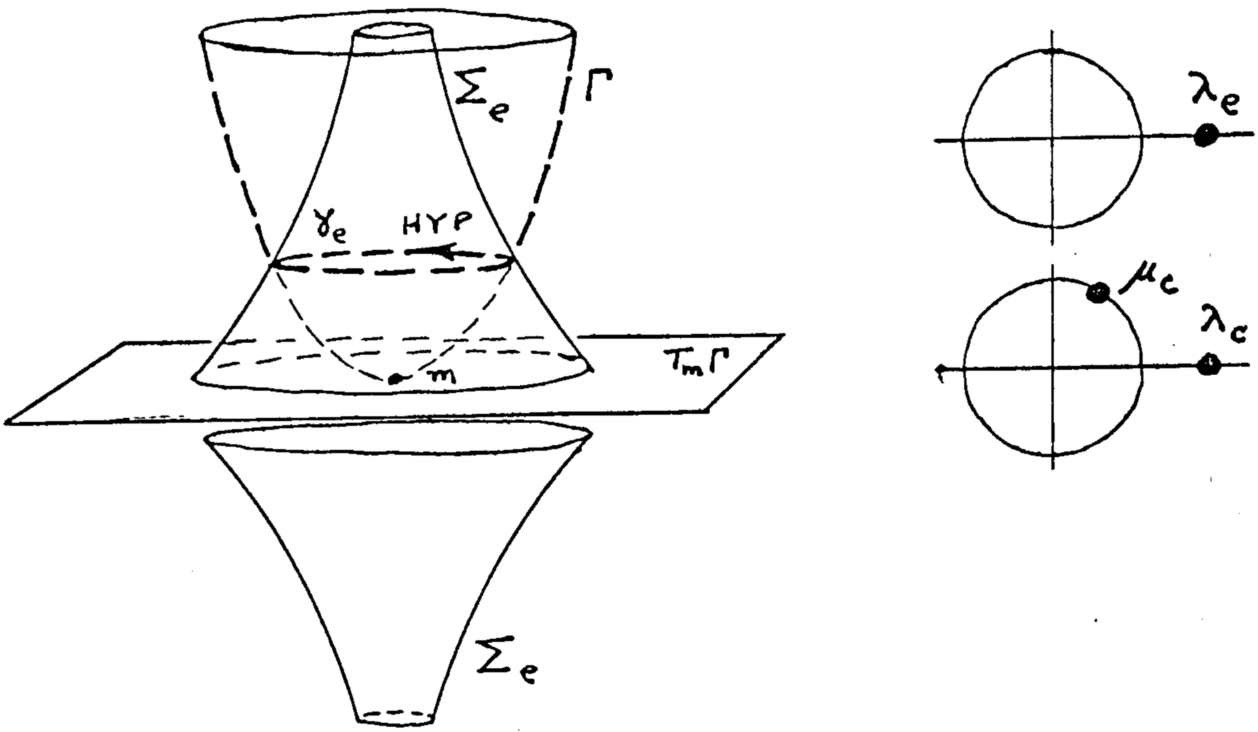

FIGURE 2

Schematic diagram of the Liapounov bifurcation, in n=2 degrees of freedom, for a single hyperbolic orbit cylinder, incident at a critical point of saddle-center type, along the center. A three dimensional energy surface is shown as a two dimensional surface of rotation. The PCMs of the critical point are shown below the PCM of the approaching closed orbits. See section 7. Here Γ is shown dashed because it is hyperbolic, and therefore qualitatively invisible.

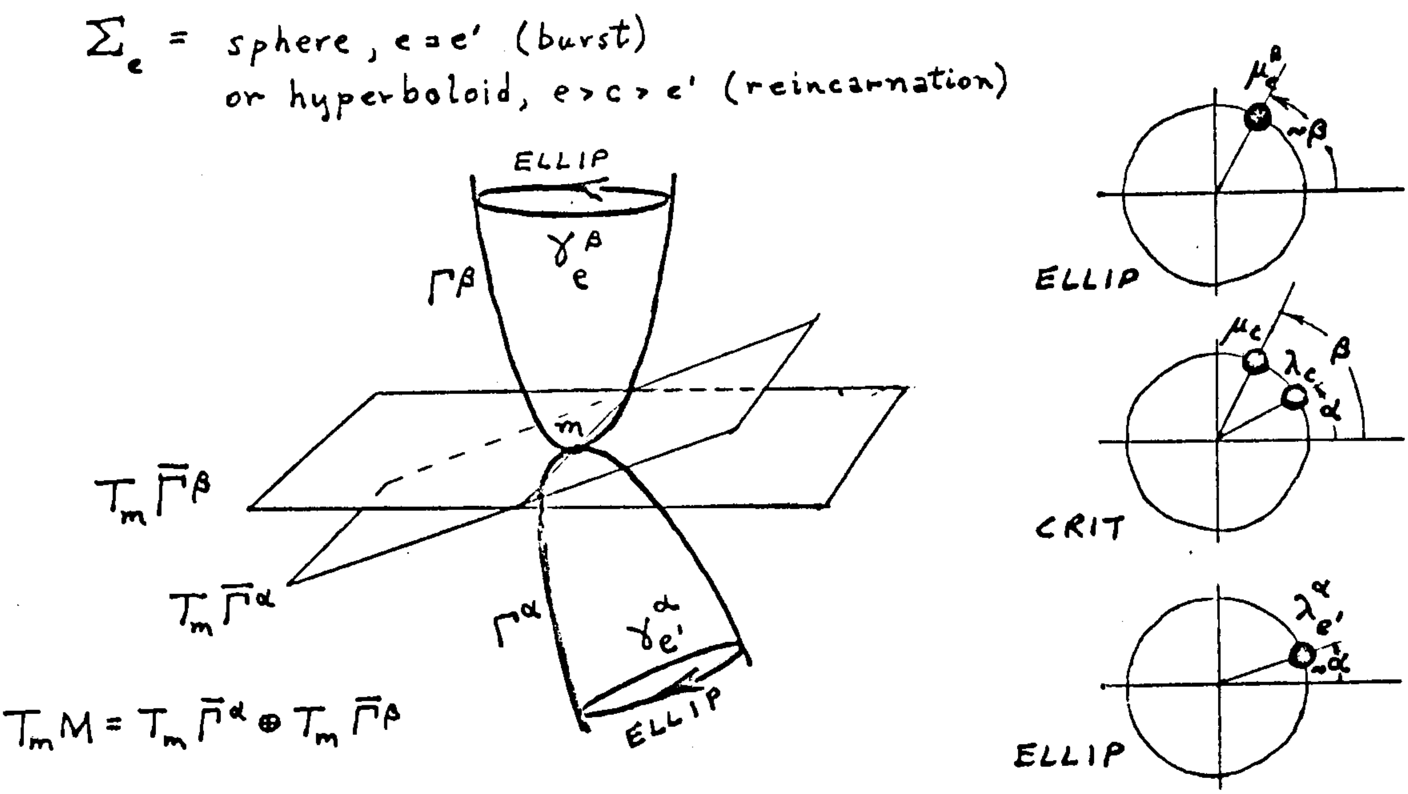

FIGURE 3: THE BURST, AND REINCARNATION

Schematic diagram of the Liapounov bifurcation in n=2 degrees of freedom, for a pair of elliptic orbit cylinders, incident at a critical point of pure-center type, along the sub-centers. In four dimensions, they don't intersect. Notations as in Figure 2.

The PCM of γ_e^β is shown above the PCMs of m , with the PM of γ_e^α at the bottom. See Section 7.

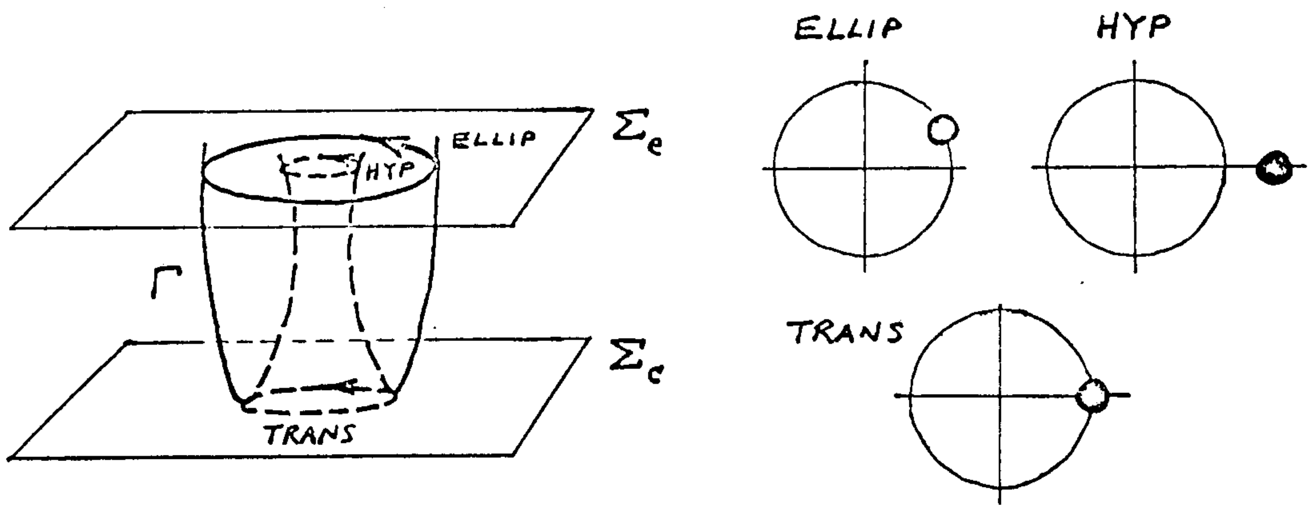

FIGURE 4: CREATION

Schematic diagram of a hyperbolic to elliptic transition via tangency of an orbit cylinder (here shown as a surface of revolution, hyperbolic dashed, elliptic solid) to an energy surface (here, a horizontal plane). The PCM for the elliptic, transitional, and hyperbolic cases are shown alongside. The transitional orbit is unstable, and therefore belongs to the dashed partion of Γ . See Section 8.

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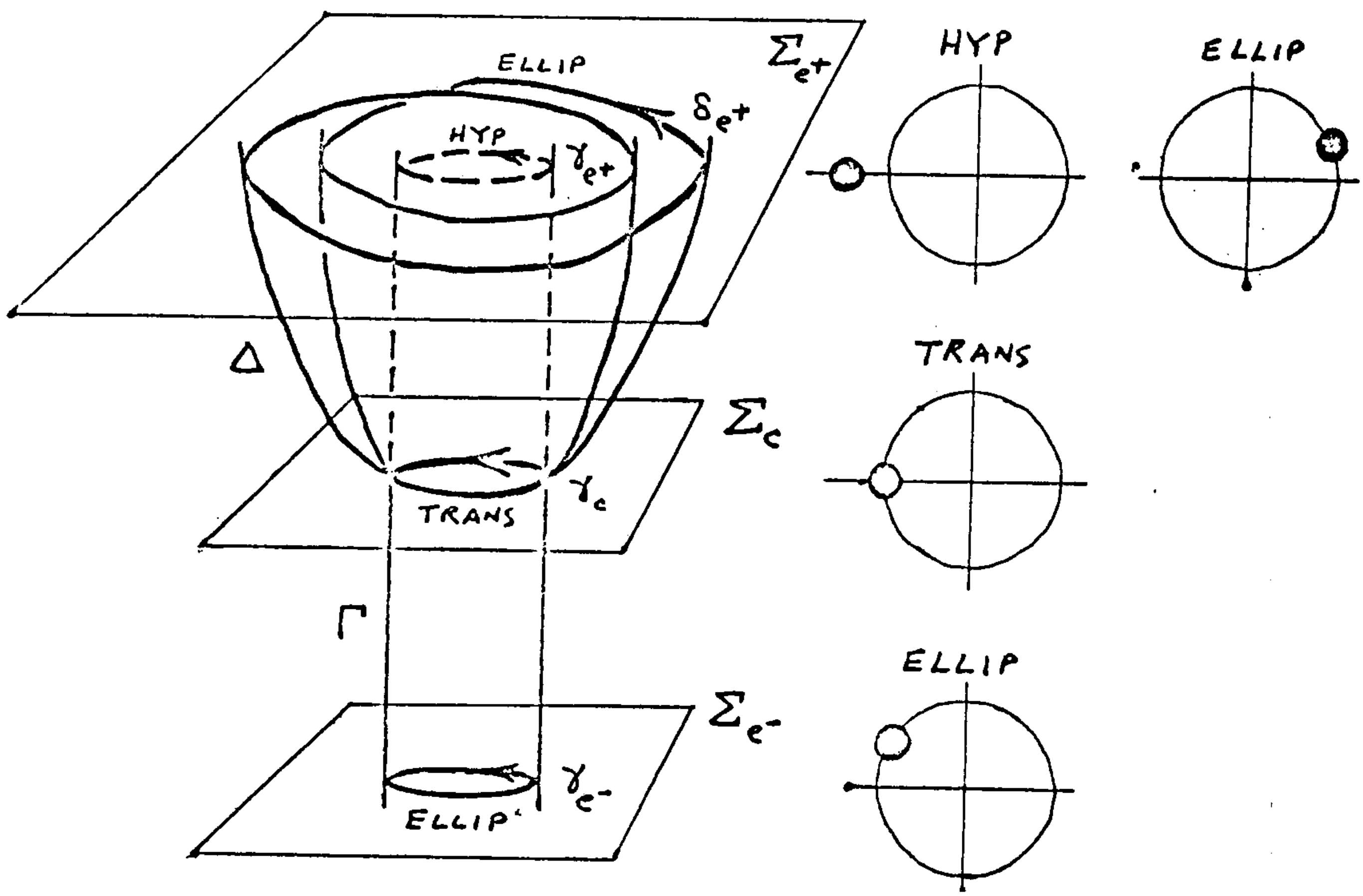

FIGURE 5: SUBTLE DIVISION

Schematic of an elliptic to hyperbolic transition via crossing of PCM through -1, with emission of a subtly halved elliptic cylinder. See Section 9. The transitional orbit is stable.

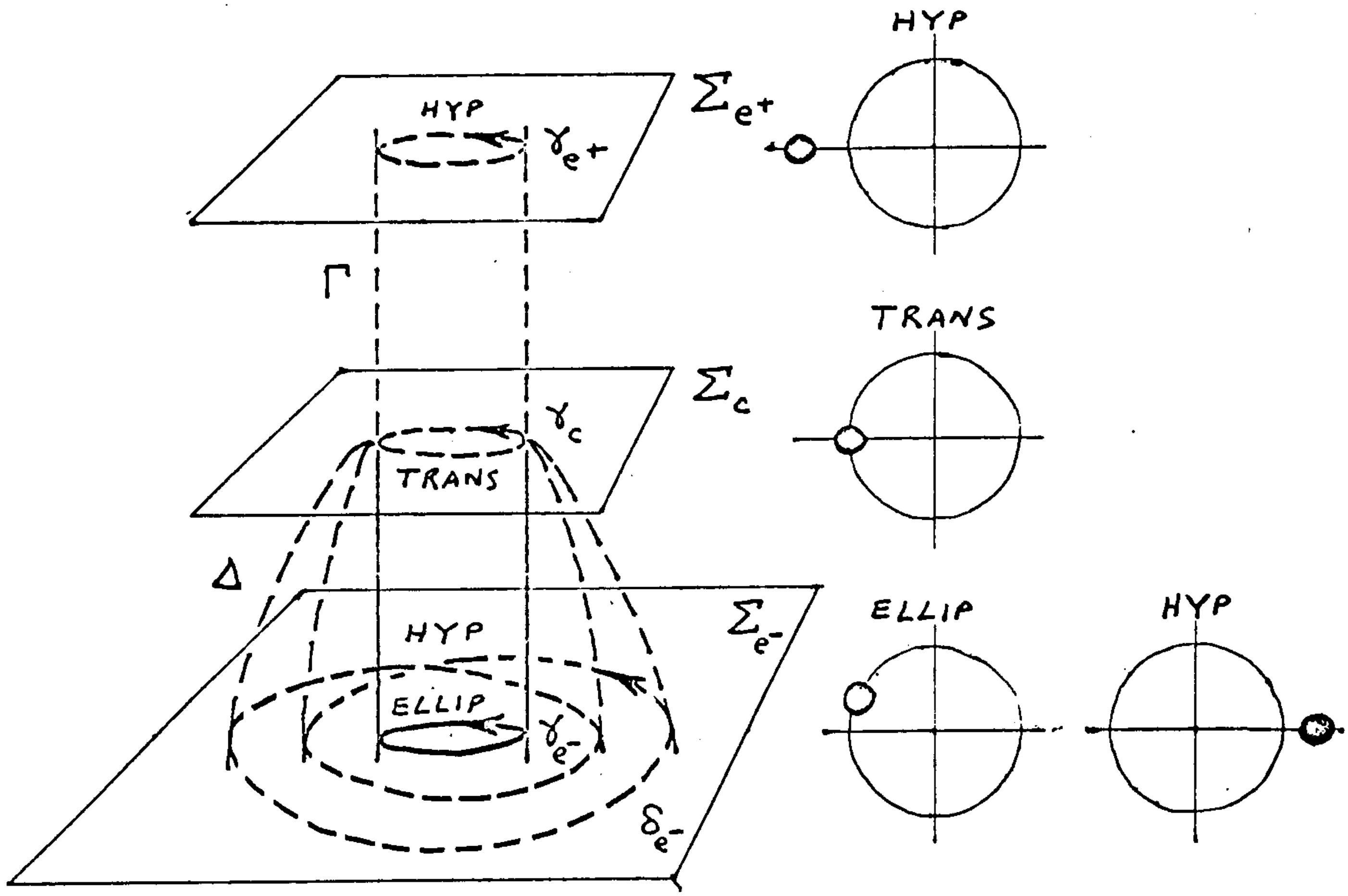

FIGURE 6: MURDER

An elliptic to hyperbolic transition via crossing of PCM through -1, with absorbtion of a sub-harmonic hyperbolic cylinder. The transitional orbit is unstable. See Section 10.

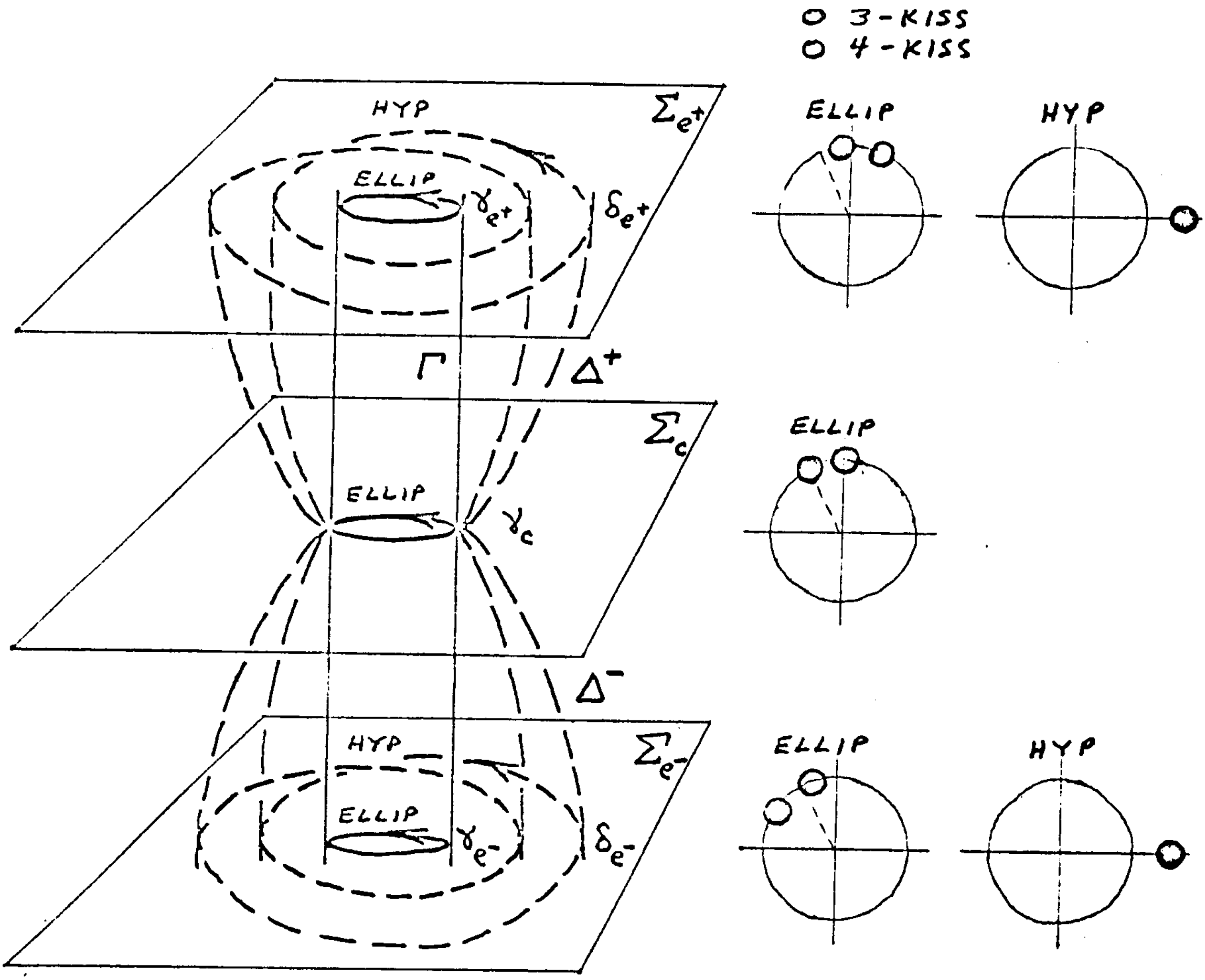

FIGURE 7: PHANTOM KISS

Schematic of a crossing of a PCM past $\exp(2\pi i \alpha)$, $\alpha = 1/3$, a 3-bifurcation, with a kiss by a hyperbolic sub-harmonic. The original elliptic cylinder is unperturbed. The 4-kiss, in which α passes 1/4, is identical, except that the osculating sub-harmonic has one-fourth the orbital frequency, instead of one-third as shown, of the elliptic cylinder. See Section 11.

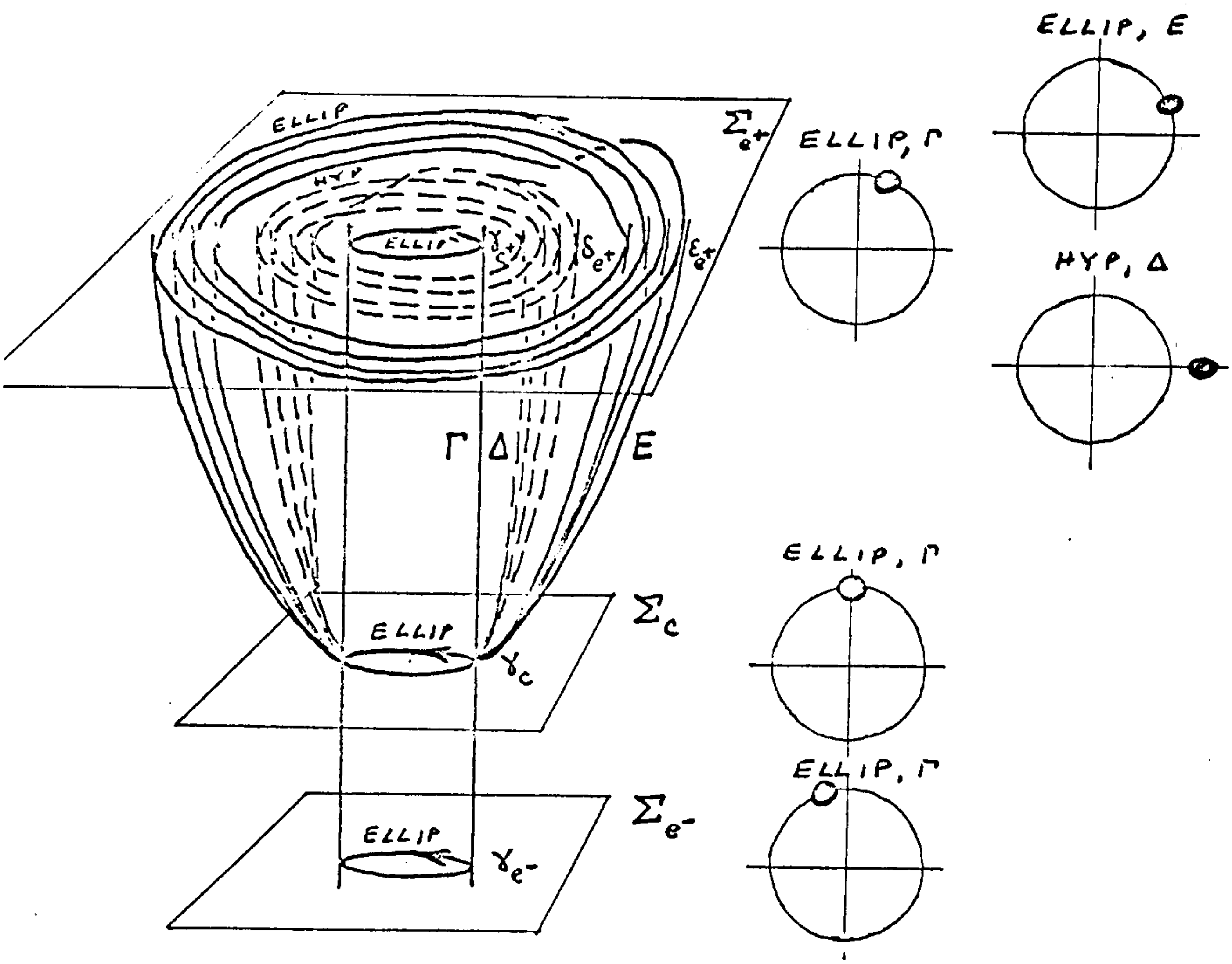

FIGURE 8: EMISSION

Schematic of a crossing of a PCM past $\exp(2\pi i\alpha)$, a p-th root of unity, $p \geq 4$, with emission of elliptic and hyperbolic sub-harmonics of one-p-th the orbital frequency of the original elliptic cylinder, shown here for $p = 4$. See Section 12.

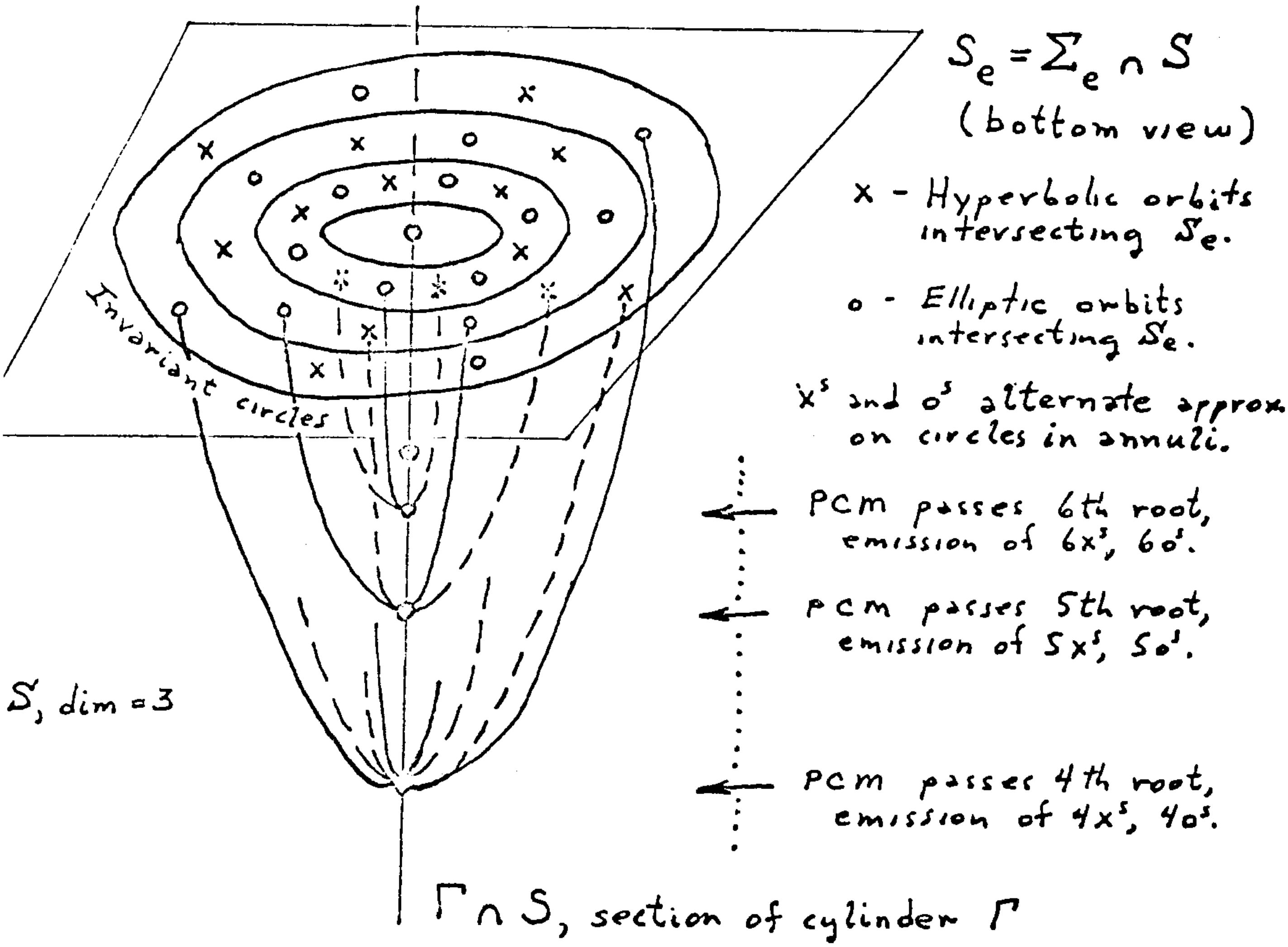

FIGURE 9: NESTED UMBRELLAS

Pictorial sketch of a cross-section S of an elliptic orbit cylinder, Γ, showing the loci of successive emitted sub-harmonic cylinders. The energy surfaces $S_e = \Sigma_e \cap S$, shown as horizontal planes, are actually two-dimensional. The intersection $S \cap \Gamma$, shown as a vertical line, is actually one-dimensional. Pictorials of all the catastrophes, shown shematically (four-dimensions represented as three) in the previous illustrations, could be made in this manner. See the Conclusion. Only a few ribs of three umbrellas are shown. There is a countable set of umbrellas.

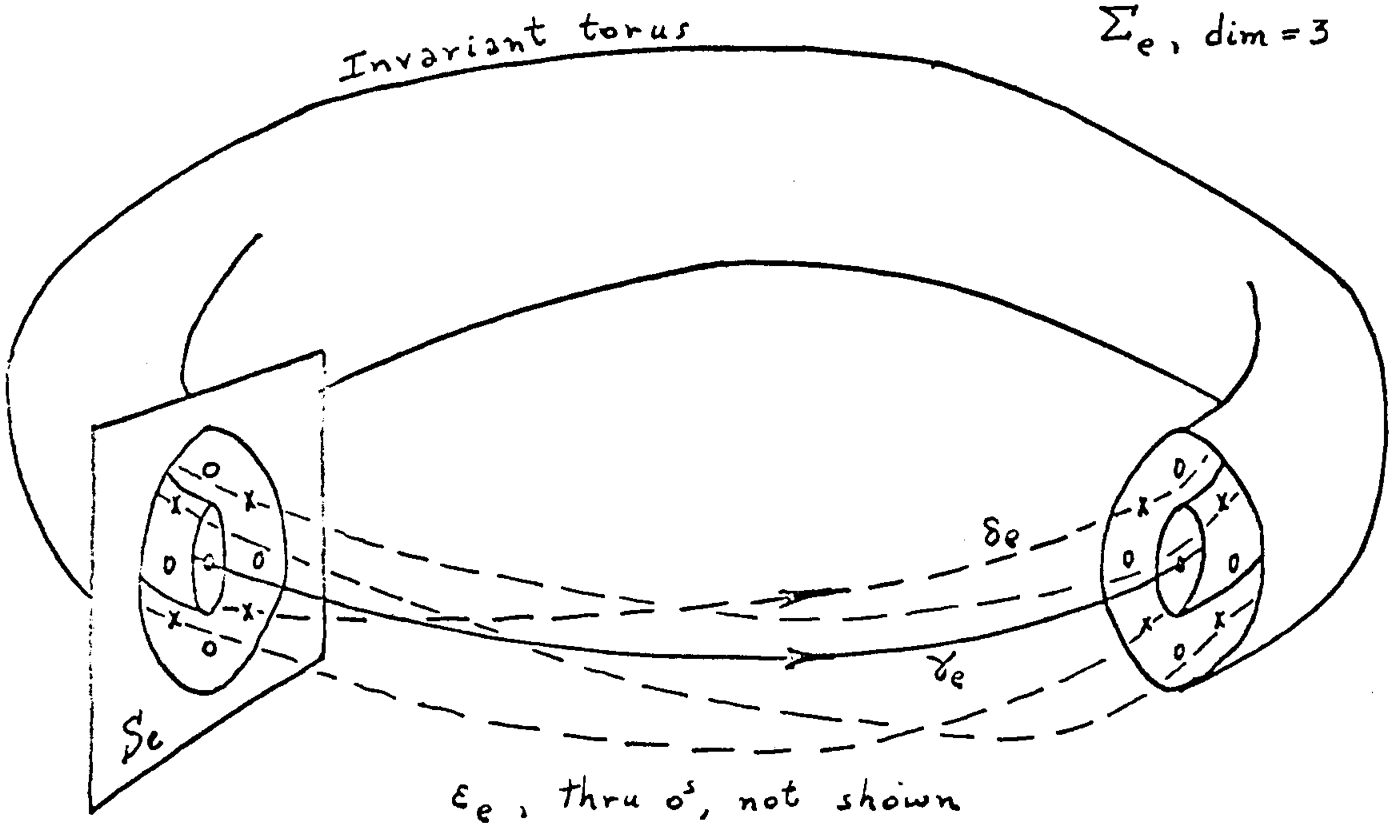

FIGURE 10: NESTED TORI

Another pictorial sketch of an elliptic orbit, γ_e , this time, a single three-dimensional energy surface Σ_e is shown. As e moves, this figure translates in four-space to generate the orbit cylinders Γ and those of the ribs of the umbrellas. For example, as e decreases, suppose γ_e is stationary within Σ_e , as ϵ_e approaches, faster and faster, a p-fold covering of γ_e . This point of view is the basis of all of the catastrophe interpretations of the preceeding bifurcations.

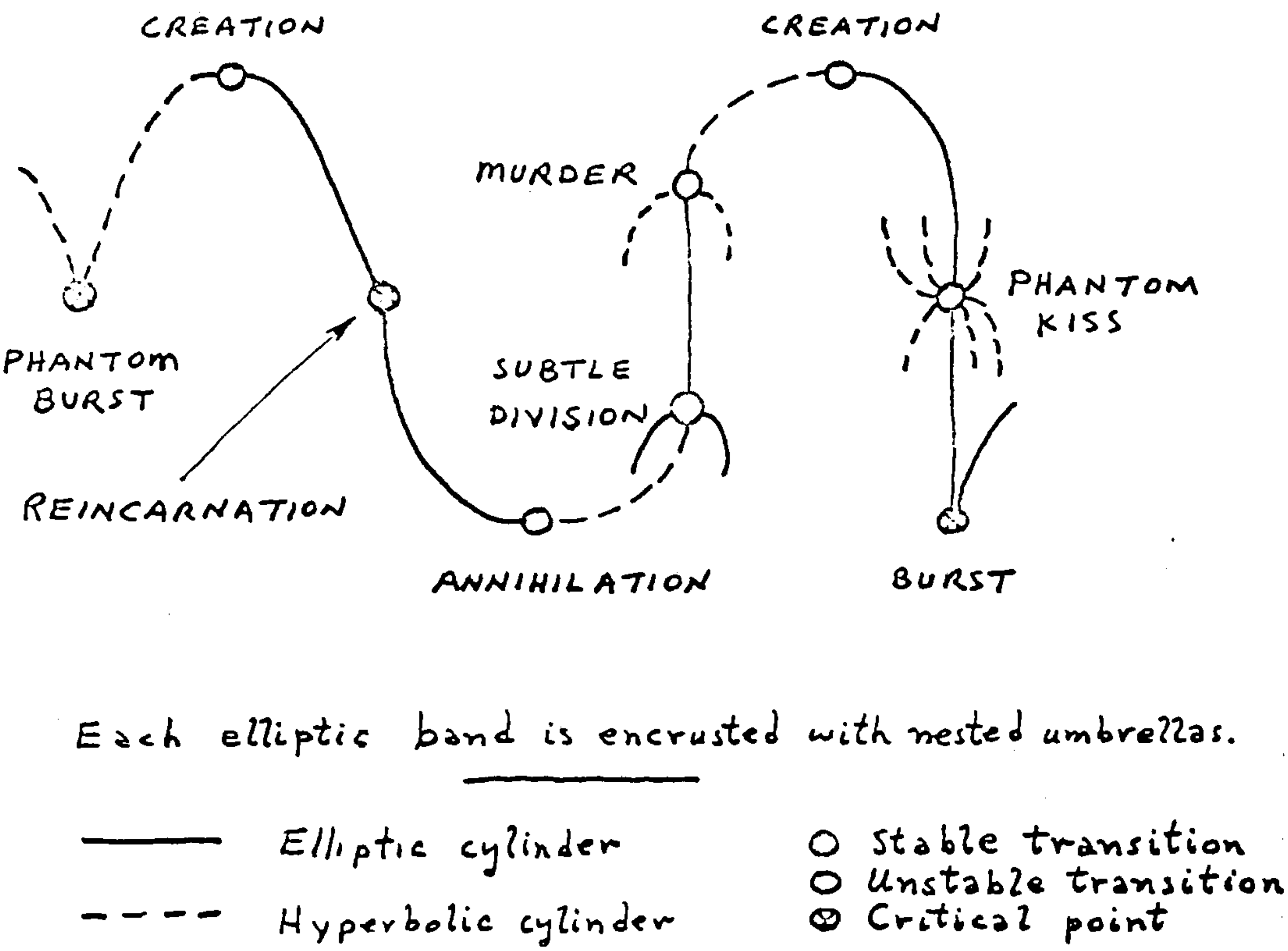

FIGURE 11: SECTION OF A TYPICAL CYLINDER

Schematic of a typical cylinder, from birth (here from a saddle-center critical point, initially hyperbolic) through successive catastrophes to death at a burst. Only a few of the many possibilities are shown. All orbit cylinders are represented as curves. Energy surfaces are imagined to be horizontal planes, except near the critical point. See the Conclusion. This indicates the generic pathology in the case of n=2 degrees of freedom.

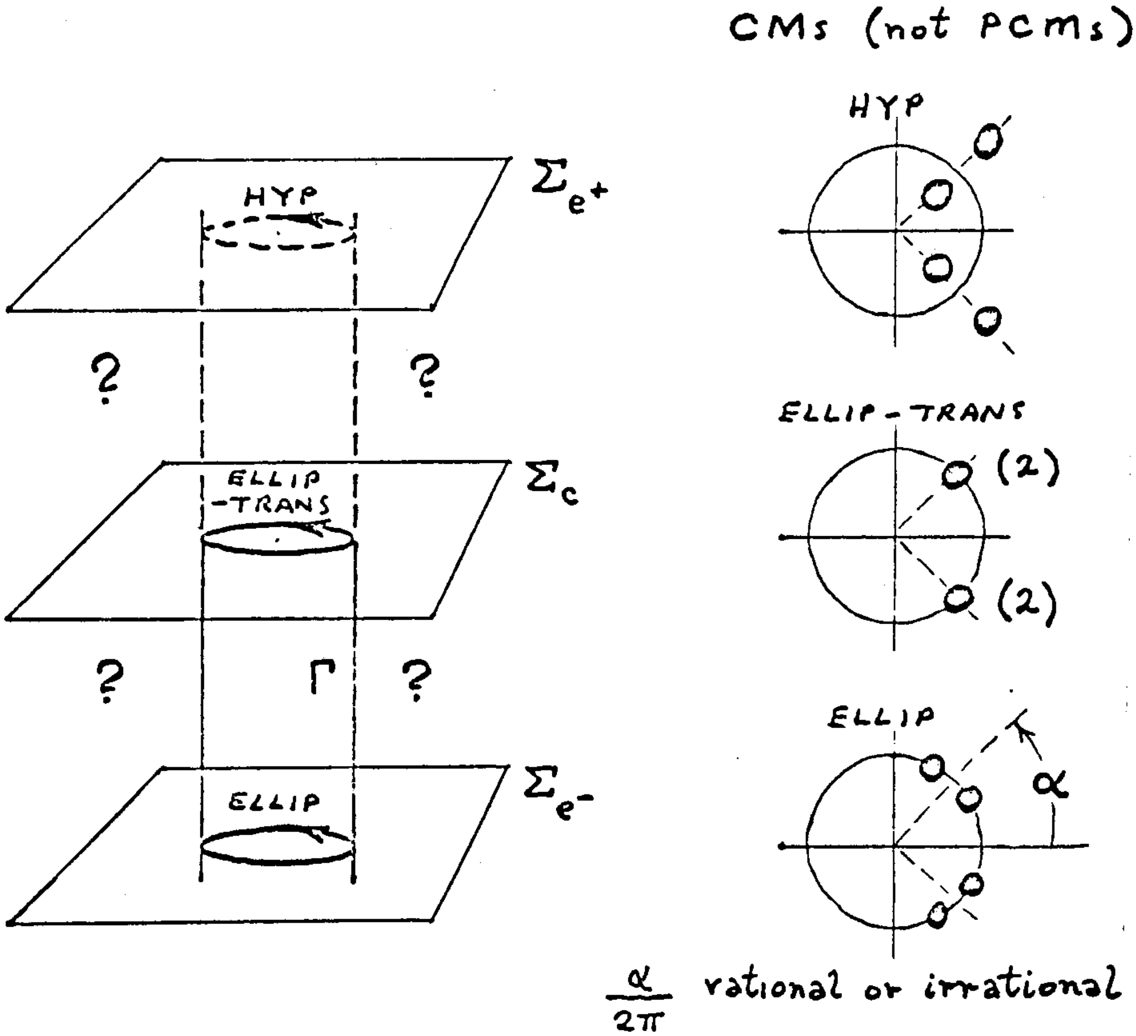

FIGURE 12 : KREIN ROBINSON BIFURCATION

All the preceeding catastrophes have been established for n=2 degrees of freedom. For larger n , they can also be expected. And in addition, there is one new phenomenon to be expected, which is typified by this bifurcation of a quadruplet of CMs, which can occur for n=3 . The incidence of other orbit cylinders in this case is not yet known. Further bifurcations can also be expected, which are caused by rational dependence of CMs belonging to distinct quadruplets or pairs. See the Conclusion. This represents the current frontier of knowledge of the qualitative theory of Hamiltonian systems.