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THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
DEPARTMENT OF MATHEMATICS

Technical Report

The Sound Speeds of a Charged Fluid

RALPH ABRAHAM

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THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE • ANN ARBOR

T H E U N I V E R S I T Y O F M I C H I G A N
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THE SOUND SPEEDS OF A CHARGED FLUID

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1. INTRODUCTION

We consider a charged perfect fluid with finite electrical conductivity, no heat conductivity, and variable dielectric capacity and magnetic permeability in the space-time of general relativity amid arbitrary Maxwellian fields. From the description of these fields by a pair of bivectors (skew symmetric tensors of order two) and a decomposition theorem for bivectors, a new description is obtained which generalizes the classical scheme. From this new scheme the characteristic equation is found, which shows that the fluid supports two modified sound waves and a modified contact surface.

2. THE BIVECTOR PAIR DESCRIPTION

In the scheme of Pham Mau Quan,[1] the fluid is described by the parameters:

- ρ: energy density
- σ: specific entropy density
- $p(\rho,\sigma)$: pressure
- $\lambda(\rho,\sigma)$: dilectric capacity
- $\mu(\rho,\sigma)$: magnetic permeability
- π: electrical conductivity
- δ: net electric charge density
- u_i: unit time-like 4-velocity vector
- J_i: net electric charge 4-current vector
- G_{ij}: magnetic field-electric displacement bivector
- H_{ij}: electric field-magnetic induction bivector.

The physical equations are:
the constitutive (or linking) equations:

(2.1) $$G_{rs} = 1/\mu \quad H_{rs} + \frac{1-\lambda\mu}{\mu}\left(H_{mr}\, u^m\, u_s - u_r\, H_{ms}\, u^m\right)$$

the homogeneous Maxwell equations:

(2.2) $$\nabla_m H_{rs} + \nabla_r H_{sm} + \nabla_s H_{mr} = 0$$

the inhomogeneous Maxwell equations:

(2.3) $$\nabla_r G^{rs} = J^s$$

Ohm's law for the finitely conductive case, without Hall current:

(2.4) $$J_r = \delta u_r + \pi\, H_{mr}\, u^m$$

the adiabatic assumption, expressing the constancy of entropy along streamlines:

(2.5) $u^m \nabla_m \sigma = 0$

and the relativistic conservation laws,

(2.6) $\nabla_r T^{rs} = 0$

where T_{rs} is the energy-momentum tensor

(2.7) $T_{rs} = \rho u_r u_s + p\,(u_r u_s - g_{rs}) + \tau_{rs} - (1-\lambda\mu)\tau_{rm}\, u^m\, u_s$

and τ_{rs} is the Maxwell stress tensor

(2.8) $\tau_{rs} = \frac{1}{4} G^{mn}\, H_{mn}\, g_{rs} - G_{mr}\, H^{m}_{\cdot s}$

Computing the divergence of the energy-momentum tensor (2.7) the conservation equations (2.6) yield

$$\begin{aligned}&(\rho+p)u^s\, \nabla_r u^r + (\rho+p)u^r\, \nabla_r u^s + u^r u^s\, (\nabla_r\rho + \nabla_r p) - \nabla^s\, p \\ &+ \nabla_r\, \tau^{rs} - (1-\lambda\mu)\, (u_m u^s\, \nabla_r\, \tau^{rm} + \tau^{rm}\, u^s\, \nabla_r\, u_m + \tau^{rm}\, u_m\, \nabla_r\, u^s) \\ &+ \tau^{rm}\, u_m\, u^s\, \nabla_r\, (\lambda\mu) = 0\end{aligned} \tag{2.9}$$

Forming the scalar product of this equation with u_s, and taking account of the fact that $u_s u^s = 1$ by hypothesis, one obtains the energy conservation equation

$$\begin{aligned}&(\rho+p)\, \nabla_r\, u^r + u^r\, \nabla_r\rho + \lambda\mu\, u_s\, \nabla_r\, \tau^{rs} \\ &- (1-\lambda\mu)\tau^{rs}\, \nabla_r\, u_s + \tau^{rm}\, u_m\, \nabla_r\, (\lambda\mu) = 0\end{aligned} \tag{2.10}$$

If now (2.9) is simplified by subtracting (2.10) times u_s, the equations of motion are obtained,

$$\begin{aligned}&(\rho+p)u^r\, \nabla_r\, u^s + (u^r u^s - g^{rs})\nabla_r p + \nabla_r\, \tau^{rs} \\ &- u_m u^s\, \nabla_r\, \tau^{rm} - (1-\lambda\mu)\tau^{rm}\, u_m\, \nabla_r\, u^s = 0\end{aligned} \tag{2.11}$$

3. A DIGRESSION ON BIVECTORS

Let $\underline{e}_{rsmn}$ be the covariant indicator tensor density of weight -1, having value 1, 0, or -1 according as (rsmn) is an even, repeated, or odd permutation of (1234), and e_{rsmn} the associated absolute tensor

$$(3.1) \qquad e_{rsmn} = \sqrt{-g}\, \underline{e}_{rsmn}$$

where g is the determinant of the metric, g_{rs}. If A_{rs} is an arbitrary bivector, define its <u>dual</u> (or adjoint) by

$$(3.2) \qquad {}^*A_{rs} = \frac{1}{2} e_{rsmn} A^{mn}$$

If b_r is a non-null vector, and $A_{rs}b^r = 0$, say that the bivector A_{rs} is <u>orthogonal</u> to the vector b_r. If vectors a_r and b_r exist such that $A_{rs} = a_r b_s - b_r a_s$, say that A_{rs} is a <u>simple</u> (or decomposable) bivector.

THEOREM. If a bivector A_{rs} is orthogonal to a unit vector a_r, then both A_{rs} and ${}^*A_{rs}$ are simple, and further

$$(3.3) \qquad {}^*A_{rs} = s\,\{a_r\,({}^*A_{ms}\,a^m) - ({}^*A_{mr}\,a^m)a_s\}$$

where $s = a^r a_r$.

Proof. It is known[2] that every non-zero bivector in a V_4 is of rank 4 (or 2), and can be written as a sum of two (or one) simple bivectors (its leaves or blades). Suppose A_{rs} is of rank 4, so we may write

$$(3.4) \qquad A_{rs} = (b_r c_s - c_r b_s) + (v_r w_s - w_r v_s)$$

where b_r, c_r, v_r, and w_r are linearly independent, and suppose further that A_{rs} is orthogonal to a_r. Then from (3.4) we have

$$(3.5) \qquad (a^r b_r)c_s - (a^r c_s)b_s - (a^r v_r)w_s - (a^r w_r)v_s = 0$$

As b_r, c_r, v_r, and w_r are linearly independent by hypothesis, the coefficients of (3.5) all vanish, and thus $a_r = 0$. Therefore a bivector of rank 4 is orthogonal to no vector, and so if A_{rs} is orthogonal to a_r, non-null, then A_{rs} is simple, and we may write

$$(3.6) \qquad A_{rs} = b_r c_s - c_r b_s$$

for some vectors b_r and c_r. Taking the dual of (3.6) according to (3.2), we have

$$(3.7) \qquad {}^*A_{rs} = e_{rsmn} b^m c^n$$

which is clearly orthogonal to both b_r and c_r. At a generic point p, b_r and c_r may be taken orthogonal to each other. Contracting (3.6) with a^r, we have by assumption

$$(3.8) \qquad (a^r b_r)c_s - (a^r c_r)b_s = 0$$

so at p, a_r, b_r, and c_r form an orthogonal triple. We see from (3.7) that

$$(3.9) \qquad {*A_{rs}}a^s = e_{rsmn}a^s b^m c^n$$

which is clearly non-zero, so that $*A_{rs}$ is not orthogonal to a_r, and thus there exists a vector d_r orthogonal to a_r such that

$$(3.10) \qquad *A_{rs} = a_r d_s - d_r a_s$$

Forming a scalar product of (3.10) with a_r, we find (as $s = a^r a_r$)

$$(3.11) \qquad d_s = s{*A_{rs}}a^r$$

This completes the proof.

COROLLARY. If A_{rs} is an arbitrary bivector and a_r is any time-like unit vector, then

$$(3.12) \qquad A_{rs} = a_r(A_{ks}a^k) - (A_{kr}a^k)a_s - e_{rsmn}a^m({*A^{kn}}a_k)$$

Proof. Let

$$(3.13) \qquad B_{rs} = A_{rs} - (a_r A_{ks}a^k - A_{kr}a^k a_s)$$

Then B_{rs} is clearly orthogonal to a_r when $s = 1$, so by (3.3) we have

$$(3.14) \qquad *B_{rs} = a_r({*B_{ks}}a^k) - ({*B_{kr}}a^k)a_s$$

It is evident from (3.13) and (3.2) that ${*B_{kr}}a^k = {*A_{kr}}a^k$, so taking the dual of (3.14), we have

$$(3.15) \qquad **B_{rs} = e_{rsmn}a^m({*A^{kn}}a_k)$$

Now as $**B_{rs} = -B_{rs}$,[3] substitution for B_{rs} from (3.13) and a simplification yield the equation (3.12).

4. THE VECTOR TETRAD DESCRIPTION

The bivector pair description of the Maxwellian fields is to be interpreted thus. If an observer should move in space-time with the fluid, that is with velocity u_r, the classical Maxwellian fields observed by him are supposed to be:

(4.1a) $H_s = {}^*G_{rs}u^r$: magnetic field intensity,

(4.1b) $D_s = G_{rs}u^r$: electric displacement,

(4.1c) $E_s = H_{rs}u^r$: electric field intensity,

(4.1d) $B_s = {}^*H_{rs}u^r$: magnetic induction.

These 4-vectors are all orthogonal to u_r, and hence lie in the spatial section of the observer, and may be thought of as the classical 3-vectors of the same names.

These equations translate the bivector pair into a description of the Maxwellian fields seen by an observer in terms of the classical tetrad of vectors. The corollary of the last section provides an inversion of this translation, for if u_r takes the role of a_r, and G_{rs}, H_{rs}, ${}^*G_{rs}$, and ${}^*H_{rs}$ successively the role of A_{rs} in (3.12), (4.1) provides

(4.2a) $G_{rs} = u_rD_s - D_ru_s - e_{rsmn}\, u^mH^n$

(4.2b) $H_{rs} = u_rE_s - E_ru_s - e_{rsmn}\, u^mB^n$

(4.2c) ${}^*G_{rs} = u_rH_s - H_ru_s - e_{rsmn}\, u^mD^n$

(4.2d) ${}^*H_{rs} = u_rB_s - B_ru_s - e_{rsmn}\, u^mE^n$

The translation equations of (4.1) and (4.2) may now be used to transcribe the physical equations (2.1) to (2.11) into the vector tetrad description of an observer moving with the fluid.

Forming the scalar product of (2.1) and its dual with u_r, the <u>constitutive equations</u> are obtained in the classical form:

(4.3a) $D_s = \lambda E_s$

(4.3b) $B_s = \mu H_s$

From (4.2d) the <u>homogeneous Maxwell equations</u> (2.2) (or $\nabla_r\, {}^*H^{rs} = 0$) may be written

$$(4.4)\quad \nabla_r (u^r B^s - B^r u^s - e^{rsmn} u_m E_n) = 0$$

and from (4.2a) the inhomogeneous Maxwell equations (2.3) (or $\nabla_r G^{rs} = J^s$) are

$$(4.5)\quad \nabla_r (u^r D^s - D^r u^s - e^{rsmn} u_m H_n) = J^s$$

Using (4.1c) the Ohm's law (2.4) becomes

$$(4.6)\quad J_r = \delta u_r + \pi E_r$$

The adiabatic assumption (2.5) is not changed by the translation. To continue the translation, we must compute the term $G_{mr}H^{ms}$ which appears in (2.8). By raising the indices of (4.2b) and contracting with (4.2a) we obtain

$$(4.7)\quad G_{mr}H^{ms} = (u_m D_r - D_m u_r - e_{mrcd} u^c H^d)(u^m E^s - E^m u^s - e^{msab} u_a B_b)$$

If we now define Poynting's vector by

$$(4.8)\quad P_r = e_{rsmn} E^s H^m u^n$$

recall[4]

$$(4.9)\quad e_{mrcd} e^{msab} = -\delta^s_r \delta^a_c \delta^b_d - \delta^s_c \delta^a_d \delta^b_r - \delta^s_d \delta^a_r \delta^b_c + \delta^s_c \delta^a_r \delta^b_d + \delta^s_d \delta^a_c \delta^b_r + \delta^s_r \delta^a_d \delta^b_c$$

and perform the indicated contractions in (4.7), we obtain, after lowering the index s, and taking account of (4.3b) and (4.7),

$$(4.10)\quad G_{mr}H^{m}_{\cdot s} = (D_m E^m + H_m B^m) u_r u_s - H_m B^m g_{rs} + B_r H_s + D_r E_s - \lambda\mu u_r P_s - P_r u_s$$

Contracting this equation with g^{rs}, we get

$$(4.11)\quad G_{mn}H^{mn} = 2(D_m E^m - H_m B^m)$$

We may now substitute (4.10) and (4.11) into the Maxwell stress tensor (2.8), obtaining

$$(4.12)\quad \tau_{rs} = \Lambda(2u_r u_s - g_{rs}) - \mu H_r H_s - \lambda E_r E_s + \lambda\mu u_r P_s + P_r u_s$$

where we have written Λ for the electromagnetic energy density, that is,

$$(4.13)\quad \Lambda = -1/2\,(D_m E^m + H_m B^m)$$

For the energy-momentum tensor, we substitute (4.12) into (2.7), obtaining directly

$$(4.14)\quad T_{rs} = (\rho+p)u_r u_s - pg_{rs} + \Lambda\{(1+\lambda\mu)u_r u_s - g_{rs}\} - \mu H_r H_s - \lambda E_r E_s + \lambda\mu(u_r P_s + P_r u_s)$$

For the non-inductive case, $\lambda\mu = 1$, this reduces to the energy-momentum tensor used by Fourès-Bruhat.[5] To complete this scheme by translating the equations of conservation of energy (2.10) and of momentum (2.11) we must compute the divergence of the Maxwell stress tensor (4.12). For this computation we now make a digression to establish an identity for $H^{mn}\,\nabla_s G_{mn}$.

Differentiating the linking equation (2.1) term by term and contracting the result with H^{mn}, we obtain

$$(4.15)\quad H^{mn}\nabla_s G_{mn} = (H^{mn}H_{mn} - 2H^{km}u_k H_{nm}u^n)\,\nabla_s(1/\mu) + 2H^{km}u_k H_{nm}u^n\nabla_s\lambda + 1/\mu\, H^{mn}\nabla_s H_{mn} + 2\,\frac{1-\lambda\mu}{\mu}\,H^{mn}(u^k u_n\,\nabla_s H_{km} + H_{km}u_n\,\nabla_s u^k + H_{km}u^k\,\nabla_s u_n)$$

Now collecting the terms containing derivatives of H_{rs} and rearranging indices, the coefficient is seen to be the right side of the linking equation (2.1) and thus (4.15) is seen to be, if $H^{km}u_k$ is replaced by E^m,

$$(4.16)\quad H^{mn}\nabla_s G_{mn} = (H^{mn}H_{mn} - 2E^m E_m)\,\nabla_s(1/\mu) + 2E^m E_m\,\nabla_s\lambda + G^{mn}\nabla_s H_{mn} + 4\,\frac{1-\lambda\mu}{\mu}\,E_m H^{mk}\,\nabla_s u_k$$

We wish now to translate the right side of this identity to the vector tetrad description, so we must replace H_{mn} by means of (4.2b). From that expression we find

$$(4.17)\quad H_{mn}H^{mn} = 2E_m E^m + e_{mnab}e^{mncd}u^a B^b u_c B_d$$

It is known that[6]

$$(4.18)\quad e_{mnab}e^{mncd} = -2\,(\delta^c_a\delta^d_b - \delta^c_b\delta^d_a)$$

and therefore (4.17) may be written

$$(4.19)\quad H^{mn}H_{mn} - 2E^m E_m = -2B^m B_m$$

Also, forming the scalar product of (4.2b) with E^r and raising the index of the result, we find by (4.8),

$$(4.20)\quad E_m H^{mk} = -E^m E_m u^k - \mu P^k$$

Recalling that $B^m B_m = \mu^2\ H^m H_m$ and that $u^k\nabla_s u_k = 0$, we may substitute (4.19) and (4.20) into the first and last coefficients of (4.16), respectively, obtaining finally the identity

$$(4.21) \quad H^{mn}\nabla_s G_{mn} = 2E^m E_m \nabla_s \lambda - 2H^m H_m \nabla_s \mu - 4(1-\lambda\mu)P^k \nabla_s u_k + G^{mn}\nabla_s H_{mn}$$

We now return to the program and the divergence of the Maxwell stress tensor. Taking the divergence term by term of (2.8), we get

$$(4.22) \quad \nabla_r \tau^r{}_s = 1/4\ H_{mn} \nabla_s G^{mn} + 1/4\ G^{mn} \nabla_s H_{mn} - (\nabla_r G^{mr}) H_{ms} - G^{mr} \nabla_r H_{ms}$$

and replacing the first term by (4.21), we have

$$(4.23) \quad \begin{aligned} \nabla_r \tau^r{}_s = {} & 1/2\ E^m E_m \nabla_s \lambda - 1/2\ H^m H_m \nabla_s \mu - (1-\lambda\mu) P^k \nabla_s u_k + \\ & + 1/2\ G^{mn} \nabla_s H_{mn} + (\nabla_r G^{rm}) H_{ms} - G^{mr} \nabla_r H_{ms} \end{aligned}$$

It is not hard to show that the sum of the fourth and sixth terms of the right side of (4.23) vanish by virtue of the homogeneous Maxwell equation (2.2).[7] The fifth term, which is interpreted by Pham Mau Quan as the Lorentz force,[8] may be written $J^m H_{ms}$ according to the inhomogeneous Maxwell equation (2.3). Contracting equations (2.4) and (4.2b) and simplifying with (4.8) we find that our assumption for the current yields for the Lorentz force

$$(4.24) \quad J^m H_{ms} = \delta E_s - \pi E^m E_m u_s - \pi\mu P_s$$

and thus (4.23) becomes

$$(4.25) \quad \begin{aligned} \nabla_r \tau^r{}_s = {} & \delta E_s - \pi E^m E_m u_s - \pi\mu P_s + 1/2\ E^m E_m \nabla_s \lambda - 1/2\ H^m H_m \nabla_s \mu - \\ & - (1-\lambda\mu) P^k \nabla_s u_k \end{aligned}$$

This is the divergence of the Maxwell stress tensor expressed in the vector tetrad description, and substitution of it into the relativistic conservation equations (2.6), along with (4.12), completes the transcription. Specifically, forming the scalar product of (4.12) with u^s, we get

$$(4.26) \quad \tau_{rs} u^s = \Lambda u_r + P_r$$

Substitution of (4.12), (4.25), and (4.26) in (2.10) and (2.11) yields the <u>equation of conservation</u> of energy:

$$(4.27) \quad \begin{aligned} & u^r \nabla_r \rho - 1/2\ \mu^2 H^m H_m u^r \nabla_r \lambda - (1/2\ \lambda^2 E^m E_m + \lambda\mu H^m H_m) u^r \nabla_r \mu + P^r \nabla_r (\lambda\mu) - \\ & - \lambda\mu\pi E^m E_m + [\rho + p + (1-\lambda\mu)\Lambda] \nabla_r u^r - 2(1-\lambda\mu)\Lambda^{rs} \nabla_r u_s = 0 \end{aligned}$$

where we have written

$$(4.28) \quad \Lambda_{rs} = -\frac{1}{2}(\mu H_r H_s + \lambda E_r E_s)$$

and the <u>equations of motion</u>:

$$(4.29)\quad \begin{aligned}&([\rho + p - (1-\lambda\mu)\Lambda]u^r - (1-\lambda\mu)P^r)\nabla_r u^s + \delta E^s - \pi\mu P^s + (u^r u^s - g^{rs})\\ &(\nabla_r p - 1/2\, E^m E_m\, \nabla_r\lambda + 1/2\, H^m H_m\, \nabla_r\mu + (1-\lambda\mu)P^m\, \nabla_r u_m) = 0\end{aligned}$$

This completes the description of the charged fluid in terms of the classical fields, for the bivectors have been eliminated from all of the physical equations.

5. THE CHARACTERISTIC SPEEDS

In this section we shall obtain the characteristic equation of the hyperbolic system comprising the equations of motion (4.29) and conservation (4.27) of the vector tetrad description together with the adiabatic assumption (2.5).

Suppose S is a time-like hypersurface in space-time, and x^i, $i = 0,1,2,3$, are Gaussian coordinates[9] for S, that is the equation of S is $x^0 = 0$ and the metric takes the form:

$$g_{rs} = \left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & g_{\alpha\beta} & \\ 0 & & & \end{array}\right) \tag{5.1}$$

and that the physical equations are expressed in these coordinates. Further, suppose the physical parameters are continuous across S, but may have smooth jump discontinuities in their partial derivatives. Finally, suppose the space-time defined in such a way that Hadamard's theorem obtains, that is, if F, a physical parameter, is continuous across S, then so also are its covariant derivatives tangent to S, that is

$$[\nabla_\alpha F] = 0 \tag{5.2}$$

where [F] indicates the jump of F across S, and α takes values 1, 2, and 3 only. If F is a C^1 function of ρ and σ, we have

$$\nabla_s F = \frac{\partial F}{\partial \rho} \nabla_s \rho + \frac{\partial F}{\partial \sigma} \nabla_s \sigma \tag{5.3}$$

and forming the scalar product with u^s, we find by virtue of the adiabatic assumption (2.5) that

$$u^s \nabla_s F = \frac{\partial F}{\partial \rho} u^s \nabla_s \rho \tag{5.4}$$

Taking the jump of this equation, we get by means of (5.2)

$$u^0 [\nabla_0 F] = \frac{\partial F}{\partial \rho} u^0 [\nabla_0 \rho] \tag{5.5}$$

and <u>assuming throughout this section that $u^0 \neq 0$</u>, we have

$$[\nabla_0 F] = \frac{\partial F}{\partial \rho} [\nabla_0 \rho] \tag{5.6}$$

For weak shocks an analogous result may be obtained by taking the entropy to be continuous across S, and as the adiabatic assumption is the only auxiliary equation to be added to the conservation equations to obtain the square linear system in this scheme, it may be shown that this system is one for which weak shocks propagate along characteristics.

Using (5.4) and (5.6), the jump of the energy conservation equation (4.27) is

$$(5.7) \quad \left(Au^0 + \frac{\partial(\lambda\mu)}{\partial\rho} P^0\right)[\nabla_0\rho] + B[\nabla_0 u^0] - 2(1-\lambda\mu)\Lambda^{0s}[\nabla_0 u_s] = 0$$

where we have written

$$(5.8) \quad A = 1 + 1/2\, \mu^2 H^2 \frac{\partial\lambda}{\partial\rho} + (1/2\, \lambda^2 E^2 + \lambda\mu H^2) \frac{\partial\mu}{\partial\rho}$$

$$(5.9) \quad B = \rho + p + (1-\lambda\mu)\Lambda$$

Similarly, the jumps of the equations of motion (4.29) are

$$(5.10) \quad \begin{aligned} &(Cu^0 + (1-\lambda\mu)P^0)\, [\nabla_0 u^s] + (u^0 u^s - g^{0s}) \\ &\qquad (D[\nabla_0\rho] + (1-\lambda\mu)P^m[\nabla_0 u_m]) = 0 \end{aligned}$$

where

$$(5.11) \quad C = \rho + p - (1-\lambda\mu)\Lambda$$

$$(5.12) \quad D = \frac{\partial p}{\partial\rho} + 1/2\, E^2 \frac{\partial\lambda}{\partial\rho} - 1/2\, H^2 \frac{\partial\mu}{\partial\rho}$$

For s = 0, (5.10) is

$$(5.13) \quad \begin{aligned} &(Cu^0 + (1-\lambda\mu)P^0)\, [\nabla_0 u^0] + (u^{0^2} + 1) \\ &\qquad (D[\nabla_0\rho] + (1-\lambda\mu)P\, [\nabla_0 u_m]) = 0 \end{aligned}$$

and contracting (5.10) with P_s and Λ^0_s, we obtain

$$(5.14) \quad \begin{aligned} &(Cu^0 + (1-\lambda\mu)P^0)P^m[\nabla_0 u_m] - P^0 \\ &\qquad (D[\nabla_0\rho] + (1-\lambda\mu)P^m[\nabla_0 u_m]) = 0 \end{aligned}$$

$$(5.15) \quad \begin{aligned} &(Cu^0 + (1-\lambda\mu)P^0)\Lambda^{0m}\, [\nabla_0 u_m] - \Lambda^{00} \\ &\qquad (D[\nabla_0\rho] + (1-\lambda\mu)P^m[\nabla_0 u_m]) = 0 \end{aligned}$$

The equations (5.7), (5.13), (5.14), and (5.15) are the components of the jump of (2.6) in the directions of u^s, the normal to the discontinuity hypersurface, the Poynting vector, and Λ^{0s}, respectively, and hence are in general linearly independent. They comprise a square linear homogeneous system which we may express in matrix form:

$$(5.16)\quad \begin{bmatrix} Au^0 + \frac{\partial(\lambda\mu)}{\partial\rho}P^0 & B & 0 & -2(1-\lambda\mu) \\ D(u^{0^2}+1) & Cu^0+(1-\lambda\mu)P^0 & (1-\lambda\mu)(u^{0^2}+1) & 0 \\ -P^0D & 0 & Cu^0 & 0 \\ -\Lambda^{00}D & 0 & -(1-\lambda\mu)\Lambda^{00} & Cu^0+(1-\lambda\mu)P^0 \end{bmatrix} \begin{bmatrix} [\nabla_0\rho] \\ [\nabla_0 u_m] \\ P^m[\nabla_0 u_m] \\ \Lambda^{0m}[\nabla_0 u_m] \end{bmatrix} = 0$$

The characteristic equation expresses the vanishing of the determinant of the matrix of coefficients of the system (5.16), so evaluating the determinant and equating to zero, we obtain after some simplification the characteristic equation in the form

$$(5.17)\quad \{Cu^0 + (1-\lambda\mu)P^0\}^2 \{(AC-BD)u^{0^2} + C\frac{\partial(\lambda\mu)}{\partial\rho}P^0u^0 - -D[B + 2(1-\lambda\mu)\Lambda^{00}]\} = 0$$

The first factor yields a root

$$(5.18)\quad u^0 = \frac{(1-\lambda\mu)P^0}{\rho + p - (1-\lambda\mu)\Lambda}$$

Now it is known that if an observer with velocity v_r passes through a time-like hypersurface with normal vector N_r, the instantaneous image of the hypersurface appears to him to be moving with a velocity U given by

$$(5.19)\quad U = \frac{|v^rN_r|}{\sqrt{(v^rN_r)^2 - N^rN_r}}$$

If N_r is taken to be a unit vector, and v_r is u_r, then U may be interpreted as the speed of propagation of the spatial sections of the hypersurface with respect to the charged fluid, and (5.19) becomes in Gaussian coordinates

$$(5.20)\quad U = \frac{u_0}{\sqrt{u_0^2 + 1}}$$

Thus the characteristic root (5.18) corresponds to a characteristic wave front moving through the fluid with the characteristic speed

$$(5.21)\quad U = \frac{(1-\lambda\mu)P^0}{\rho + p - (1-\lambda\mu)\Lambda}\left\{1 + \left[\frac{(1-\lambda\mu)P^0}{\rho + p - (1-\lambda\mu)\Lambda}\right]^2\right\}^{-1/2}$$

Now it is evident that if the index of refraction of the fluid approaches unity, that is, $\lambda\mu$ tends to zero, the speed of propagation of this characteristic tends to zero, so this discontinuity may be interpreted as the electro-magnetic analogue of the contact surface or stationary discontinuity. Its speed of propagation with respect to the fluid is generally small, and varies approximately linearly with P^0, the flux of electro-magnetic energy through the discontinuity hypersurface.

The second factor of (5.17) yields two roots and hence two propagation speeds. Inspection discloses that these speeds will coincide if either the index of refraction is independent of ρ, $\partial(\lambda\mu)/\partial\rho = 0$, or $P^0 = 0$. If both λ and μ are constant, the vanishing of the second factor of (5.17) may be expressed:

$$(5.22) \quad {u^0}^2 = \frac{\partial p}{\partial \rho} \frac{\rho + p + (1 - \lambda\mu)\ (\Lambda + 2\Lambda^{00})}{\rho + p - (1-\lambda\mu)\Lambda - \{\rho + p + (1-\lambda\mu)\Lambda\}\ [\partial p/\partial\rho]}$$

which corresponds to a characteristic propagating with speed, by (5.20)

$$(5.23) \quad U = \sqrt{\frac{\partial p}{\partial \rho}}\ \{1 - 2(1-\lambda\mu)Z\}^{-1/2}$$

where we have written

$$(5.24) \quad Z = \frac{\Lambda + \Lambda^{00}\left(1 - \frac{\partial p}{\partial \rho}\right)}{\rho + p + (1-\lambda\mu)(\Lambda + 2\Lambda^{00})}$$

Finally, if the index of refraction is sufficiently close to unity, that is, if $(1 - \lambda\mu)$ is sufficiently small, (5.23) yields the approximation

$$(5.25) \quad U = \sqrt{\frac{\partial p}{\partial \rho}}\ \{1 + (1-\lambda\mu)Z\}$$

This is the ordinary sound speed of the fluid, except for a small correction due to the electro-magnetic interaction, and we therefore may interpret the characteristic equation (5.17) by saying the first factor corresponds to a corrected contact surface, and the second to a corrected sound wave. Both of the corrections go to zero with the optical density $(1 - \lambda\mu)$.

It is interesting that no characteristics of purely electro-magnetic origin, such as Alven's wave, occur as characteristics of the scheme. This is due to the assumptions of the scheme for bounded conductivity, and the corresponding Ohm's law, (2.4). In the infinitely conductive scheme these two hypotheses are dropped, and equation (4.24) is replaced in the analysis by the inhomogeneous Maxwell equation, contracted with H_{ms}.[10-13] This means in effect that the current J_r may be discontinuous even across a discontinuity hypersurface of first order, as it is no longer related by Ohm's law to continuous parameters.[13,14] The characteristic system of this modified scheme has been found by Fourès-Bruhat to yield the hydromagnetic waves of the classical theory.[5]

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