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Piecewise Differentiable Manifolds and the Space-Time of General Relativity

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CONTENTS

1. Introduction. From the cosmological point of view the space-time universe is generally taken to be an indefinitely differentiable or smooth manifold (sub-stratum). In the local point of view of general relativity, discontinuities must be superimposed on the sub-stratum to account for the singularities of the physical parameters which arise, for example, in the study of shock waves in fluids, or of the gravitational field of the sun. These parameters are related to the geometry of the universe by the EINSTEIN field equations, and therefore induce geometric discontinuities. These discontinuities in space-time prove very useful in the applications of general relativity, especially to fluid dynamics, and have been studied by a number of authors, including O'BRIEN and SYNGE [1], TAUB [2], LICHNEROWICZ [3], and COBURN [4]. In this paper we shall make a very general study of manifolds which admit the simple types of discontinuities which are important in general relativity, in the language of modern differential

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geometry. The global point of view is emphasized throughout the paper and index computations are used only rarely when they prove more convenient than the global notation.

The next five sections are devoted to the definitions of manifolds, tensors, metric, and connection with discontinuities, and the establishment of their basic properties. Only a few elementary notions of differential geometry and multi-linear algebra are assumed. The next three sections, 7, 8 and 9, constitute an excursion into the theory of hypersurfaces which results in a global derivation of the GAUSS and CODAZZI equations in the language of the earlier sections. In section 10, we return to the manifolds with discontinuities, and apply the GAUSS and CODAZZI equations to the problem of recognizing the discontinuities of the curvature tensors of the manifold. All continuity conditions for these tensors result, including a generalized form of the O'BRIEN-SYNGE condition for the EINSTEIN tensor.

In the final section a model for the relativistic universe is defined. Inspired by the model of LICHNEROWICZ, it is shown to possess the following properties:

(1) The superiority and invariance of the light speed.

(2) The principle of geodesics.

(3) A continuity condition of SCHWARZSCHILD, which has been interpreted as relating the gravitational field to its sources.

(4) SCHWARZSCHILD's solar model arises as a special case.

The fact the LICHNEROWICZ' model lacks the property (4) has been pointed out by TAUB, and our model avoids this difficulty by incorporating an idea of ISREAL [5].

It is probable that all of this work could be expressed in the index notation. We have used the modern notation not only because many of the general arguments are simplified, but also because a much more complete understanding of relativity results from the global expression, which applies to the universe as a whole rather than to only a single coordinate neighborhood.

2. Piecewise differentiable manifolds.

We shall define the manifold by means of overlapping systems of local coordinates.

Definition 2.1. Suppose U is an open subset of Euclidean m-space, R^m, V a subset of R^n, and $f : U \to V$ by

$$f(u^1, \cdots, u^m) = (f^1, \cdots, f^n).$$

Then f is *of class* C^0 if it is continuous, *of class* C^r $(0 < r \leqq \infty)$ if the f^i and their partials of orders $1, \cdots, r$ exist and are continuous, a C^r *diffeomorphism* if f and f^{-1} are both defined and of class C^r. If S is a hypersurface in R^m defined by

$$g(u^1, \cdots, u^m) = 0$$

where g is real valued and of class C^1, let $S(c)$ denote the neighboring hypersurface defined by $g = c$. A function or diffeomorphism f is *piecewise of class* C^r $(0 \leqq r \leqq \infty)$ in U if and only if: (a) f is defined and of class C^r in $U - S$, where S is a finite union of closed hypersurfaces in U, S_i ; and (b) the functions $f \mid S_i(c)$ converge uniformly to bounded limits on S_i as c tends to zero through positive or negative values. The S_i are the *discontinuity hypersurfaces* of f. If f is piecewise of class C^0 in U we say it is *of class* C^{-1} in U. If f is of class C^r, and simultaneously piecewise of class C^s in U, we say it is *of class* (C^r, C^s) in U.

Definition 2.2. If M_n is a HAUSDORFF n-manifold, a (C^r, C^s) *coordinate system* on M_n is a collection of pairs (U_a, x_a) where $\{U_a\}$ constitutes an open covering of M_n, x_a is a homeomorphism of U_a into R^n, and if $U_a \cap U_b \neq 0$, then $x_a \circ x_b^{-1}$ is a (C^r, C^s) diffeomorphism on $x_b(U_a \cap U_b)$. Two (C^r, C^s) coordinate systems are equivalent if and only if their union is again a (C^r, C^s) coordinate system. A (C^r, C^s) *differentiable structure* on M_n is an equivalence class of (C^r, C^s) coordinate systems. *A piecewise differentiable n-manifold of class* (C^r, C^s) is a HAUSDORFF n-manifold M_n together with a (C^r, C^s) differentiable structure, $\mathfrak{A}$. A coordinate system on M_n is *admissible* on $(M_n, \mathfrak{A})$ if and only if it belongs to $\mathfrak{A}$. An element (U, x) of an admissible coordinate system is an *admissible chart.*

Henceforth *piecewise differentiable manifold* shall mean an infinituple $(M_n, \mathfrak{A}^1, \mathfrak{A}^2, \cdots, \mathfrak{A}^\infty)$ where M_n is a HAUSDORFF n-manifold, $\mathfrak{A}^\infty$ is a (C^∞, C^∞) differentiable structure on M_n, and $\mathfrak{A}^r$ is the unique (C^r, C^∞) differentiable structure on M_n containing $\mathfrak{A}^\infty$, $1 \leqq r < \infty$.

If $(M_n, \mathfrak{A}^1, \cdots, \mathfrak{A}^\infty)$ is a piecewise differentiable manifold, clearly $\mathfrak{A}^1 \supset \mathfrak{A}^2 \supset \cdots \supset \mathfrak{A}^\infty$. This sequence of structures may be used to extend the notion of (C^r, C^s) differentiability to real valued functions on a piecewise differentiable manifold.

Definition 2.3. If $f : M_n \to R$, and (U, x) is an admissible chart on $(M_n, \mathfrak{A}^1)$, then $f_U^* = f \circ x^{-1}$ is the *induced function of f on* (U, x). We say that f is *of class* (C^r, C^s) *with respect to* A, an admissible coordinate system, if and only if all of the induced functions of f on the charts of A are of class (C^r, C^s) in the sense of 2.1.

Suppose an n-manifold is covered by two coordinate systems, A and B, with A of class C^∞. Then every point p of the domain of f is covered by at least one chart (U, x) of A and at least one chart (V, y) of B, and so in a neighborhood of p we have

$$f_V^* = f_U^* \circ (x \circ y^{-1}).$$

Differentiating both sides k times with respect to the coordinates y^i by the composite function rule, we see that the partials of order k of f_V^* will exist or be continuous if the partials of order k of f_U^* with respect to the x^i's and simultaneously those of $(x \circ y^{-1})$ with respect to the y^i's exist or are continuous. Thus the differentiability of f with respect to B is determined by the differentiability of f with respect to A and the equivalence class of $A \cup B$, and we have proved the following.

Theorem 2.4. *If A and B are coordinate systems on M_n, A of class C^∞, $A \cup B$ of class (C^r, C^s), and f is of class (C^m, C^n) with respect to A, then f is of class (C^a, C^b) with respect to B, with $a = \min(r, m)$, $b = \min(s, n)$.*

Thus on a piecewise differentiable manifold, functions which are of class (C^r, C^∞), ($r = -1, 0$, or 1), with respect to any admissible coordinate system have the same class with respect to all admissible coordinate systems. Further, functions of class (C^r, C^∞), $(2 \leqq r \leqq \infty)$, with respect to any element of $\mathfrak{A}^r$ have the same class for all elements of $\mathfrak{A}^r$. These remarks justify the next definition.

Definition 2.5. On a piecewise differentiable manifold $(M_n, \mathfrak{A}^1, \mathfrak{A}^2, \cdots)$ let $\mathfrak{F}^r$ ($r = -1$, 0 or 1) be the set of real valued functions defined on M_n which are of class (C^r, C^∞) with respect to elements of $\mathfrak{A}^1$, let $\mathfrak{F}^r$ $(2 \leqq r \leqq \infty)$ be the set of those of class (C^r, C^∞) with respect to elements of $\mathfrak{A}^r$. The elements of $\mathfrak{F}^r$ $(-1 \leqq r \leqq \infty)$ are called *C^r functions*, piecewise C^∞ being tacitly understood. We shall generally write $\mathfrak{F}$ instead of $\mathfrak{F}^{-1}$.

It is evident from this definition that the sets $\mathfrak{F}^r$ are commutative rings with identity under pointwise addition and multiplication, and are linearly ordered by inclusion. A C^r function $(r \geqq 2)$ will not have C^r induced functions in all admissible coordinate systems, but any added discontinuities may be considered artificial in the sense that they can be transformed away by a change of coordinate systems.

3. Piecewise differentiable tensors. We shall define vectorfields as derivations (R-linear functions obeying LEIBNITZ' rule) on the functions of the previous section, and then obtain the tensors in the usual algebraic way.

Definition 3.1. Let $\mathfrak{X}^r$ be the set of all derivations of $\mathfrak{F}^{r+1}$ into $\mathfrak{F}^r$, $-1 \leqq r \leqq \infty$, together with the compositions

(a) $\mathfrak{X}^{r+1} \times \mathfrak{X}^{r+1} \to \mathfrak{X}^r$ by $[X, Y] = XY - YX$

(b) $\mathfrak{F}^r \times \mathfrak{X}^r \to \mathfrak{X}^r$ by $(fX)g = f(Xg)$.

The elements of $\mathfrak{X}^r$ are the (C^r, C^∞) *contravariant vectorfields*, or briefly *C^r vectors*. We shall generally write $\mathfrak{X}$ for $\mathfrak{X}^{-1}$, and refer to its elements as *vectors*.

Lemma 3.2. *Let U be an open set of M_n, and f a (C^r, C^∞) function of U considered as a manifold. Then for every point of U there is an open neighborhood V in U and a function $\bar{f} \in \mathfrak{F}^r$ which extends f from V to M_n, that is, which agrees with f in V.*

Proof. For any point p there exists an admissible chart of a C^∞ coordinate system (V', x) such that $X(V') = C(3)$, the open n-ball of radius 3 and center at the origin, in R^n. If $p \in U$, we may take $V' \subset U$. It is well known that such a coordinate chart admits a characteristic function [6]. That is, there exists a function $\varphi \in \mathfrak{F}^\infty$ such that $\varphi \circ x^{-1}(\overline{C(1)}) = 1$, $\varphi \circ x^{-1}(C(2) - \overline{C(1)})$ is between

0 and 1, and $\varphi \circ x^{-1}(R^n - C(2)) = 0$. Let $V = x^{-1}(C(1))$, and $\bar{f} = \varphi \cdot f$ in V', zero elsewhere. Then clearly $\bar{f} \in \mathfrak{F}^r$, and $\bar{f}$ agrees with f in V, so $\bar{f}$ extends f from V to M_n.

Theorem 3.3. *$\mathfrak{X}^r$ is a locally free unitary $\mathfrak{F}^r$-module with commuting local $\mathfrak{F}^r$-bases.*

Proof. It is immediate from 3.1 that $\mathfrak{X}^r$ is a unitary $\mathfrak{F}^r$-module. We need only demonstrate a basis. Consider any $p \in M_n$, and any $f \in \mathfrak{F}^0$. Suppose (U, x) is an admissible chart from $\mathfrak{A}^\infty$ and $x(p) = (x^1(p), \cdots, x^n(p))$. Let $f^* = f \circ x^{-1}$. Then f^* is of class C^∞ in $xU - S$, where S is a finite union of closed hypersurfaces of xU. If $x(p)$ is in $xU - S$, a standard argument using TAYLOR's formula shows that, for any derivation X,

$$(Xf)(p) = \sum_{i=1}^{n} X(x^i) \frac{\partial f}{\partial x^i}(p). \tag{1}$$

Note that X produces a discontinuity in f if and only if it produces one in an x^i. If $x(p)$ is in a hypersurface of S across which f^* is of class C^1 and Xf is continuous, then (1) still holds, as it holds in a neighborhood of p and both sides are continuous. If $x(p)$ is in a hypersurface of S across which f^* is not of class C^1, then (1) holds in a neighborhood of p, both sides converge uniformly, the right hand side has a discontinuity at p, and so Xf is not defined at p. We conclude that the equality

$$X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial x^i} \tag{2}$$

holds whenever the action of X on a function is defined. We have assumed $(U, x) \in A \in \mathfrak{A}^\infty$, so that the x^i are of class C^∞ with respect to A in U, and thus extend to C^∞ functions on M_n by 3.2. Hence $\bar{x}^i \in \mathfrak{F}^{r-1}$ for every r, $X \in \mathfrak{X}^r$ implies $X(\bar{x}^i) \in \mathfrak{F}^r$, and p has a neighborhood V in which $\bar{x}^i$ agrees with x^i. We may extend $\partial/\partial x^i$ to a global derivation by setting $\bar{\partial}_i = \bar{1}\ \partial/\partial x^i$, where $\bar{1}$ is the extension of 1. Then in V, we have

$$X = \sum_{i=1}^{n} X(\bar{x}^i)\, \bar{\partial}_i\,, \tag{3}$$

so $\{\bar{\partial}_i\}$ is a local $\mathfrak{F}^r$-basis of global vectors for $\mathfrak{X}^r$, and thus $\mathfrak{X}^r$ is a locally free $\mathfrak{F}^r$-module. Further, it is evident that $[\bar{\partial}_i, \bar{\partial}_j] = 0$ in V, so the basis $\{\bar{\partial}_i\}$ commutes in V. As the point p is generic, this completes the proof.

The local basis exhibited above is shared by all of the modules $\mathfrak{X}^r \mid U$, and

$$\mathfrak{F}^{-1} \supset \mathfrak{F}^0 \supset \mathfrak{F}^1 \supset \cdots.$$

Hence

$$\mathfrak{X}^{-1} \mid U \supset \mathfrak{X}^0 \mid U \supset \mathfrak{X}^1 \mid U \supset \cdots,$$

and it is natural to consider

$$\mathfrak{X}^{-1} \supset \mathfrak{X}^0 \supset \mathfrak{X}^1 \supset \cdots.$$

Obviously if X is a C^r vector there is a C^{r+1} coordinate system (namely A above) in which all of the local components $X(x^i)$ are in $\mathfrak{F}^r$. It is not difficult to see that the converse is also true, so we have:

Theorem 3.4. *A vector is C^r if and only if the manifold admits a C^{r+1} coordinate system in which all of the local components are C^r functions.*

Definition 3.5. Let $*\mathfrak{X}^r$ be the set of $\mathfrak{F}^r$-linear functions of $\mathfrak{X}^r$ into $\mathfrak{F}^r$ $(-1 \leqq r \leqq \infty)$, together with the composition $\mathfrak{F}^r \times *\mathfrak{X}^r \to *\mathfrak{X}^r$ defined by $(f\theta)(X) = f(\theta(X))$. The elements of $*\mathfrak{X}^r$ are the (C^r, C^∞) *covariant vectorfields*, or briefly C^r *forms*. We shall generally write $*\mathfrak{X}$ for $*\mathfrak{X}^{-1}$ and refer to its elements as *forms*.

If $\{X_i\}$ is a basis for $\mathfrak{X}^r \mid U$ there is a dual basis $\{\theta^i\}$ for $*\mathfrak{X}^r \mid U$, so $*\mathfrak{X}^r$ is obviously a locally free unitary $\mathfrak{F}^r$-module. If $\phi \in *\mathfrak{X}^r$, then in U we have

$$\phi = \sum_{i=1}^{n} f_i \theta^i. \tag{4}$$

Applying ϕ to the local basis vectors X_i , we find by the duality condition that $f_i = \phi(X_i)$. If A is an admissible coordinate system of $\mathfrak{A}^r$, then in each chart (U, x) of A the vectors $\tilde{\partial}_i$ form a local $\mathfrak{F}^r$-basis for $\mathfrak{X}^r$. The functions $f_i = \phi(\tilde{\partial}_i)$ in U are the *canonical components* of ϕ in U. ϕ is C^r, $\phi(\tilde{\partial}_i)$ are C^r functions. We claim that the converse holds, that is if $A \in \mathfrak{A}^r$ and the canonical components of ϕ are C^r functions for all the charts (U, x) of A, then ϕ is a C^r form. In fact if Y is a C^r vector, we have in U:

$$Y = \sum_{i=1}^{n} y^i \tilde{\partial}_i .$$

And so from (4),

$$\phi(Y) = \sum_{i=1}^{n} f_i y^i. \tag{5}$$

The y^i are C^r functions by 3.4, so if the f_i are C^r functions then $\phi(Y)$ is a C^r function in U, so is in $*\mathfrak{X}^r \mid U$. As this is so for all $(U, x) \in A \in \mathfrak{A}^{r+1}$, see that $\phi(Y) \in \mathfrak{F}^r$ by 2.3, so ϕ is a C^r form. Thus, considering $*\mathfrak{X}^{r+1} \subset *\mathfrak{X}^r$, we have proved the following.

Theorem 3.6. *The $*\mathfrak{X}^r$ are locally free $\mathfrak{F}^r$-modules, and a form is C^r if and only if the manifold admits a C^{r+1} coordinate system in which all of the local canonical components are C^r functions.*

Definition 3.7. Let

$$\mathfrak{J}^r(p, q) = \overbrace{\mathfrak{X}^r \otimes \cdots \otimes \mathfrak{X}^r}^{(p\text{ factors})} \otimes \overbrace{*\mathfrak{X}^r \otimes \cdots \otimes *\mathfrak{X}^r}^{(q\text{ factors})}$$

together with the composition $\mathfrak{F}^r \times \mathfrak{J}^r(p, q) \to \mathfrak{J}^r(p, q)$ defined by

$$(fT)(\theta, \cdots, X) = f(T(\theta, \cdots, X)).$$

We identify $\mathfrak{F}^r \otimes \mathfrak{I}^r(p, q)$ with $T^r(p, q)$ by the natural isomorphism $f \otimes T \leftrightarrow fT$. The elements of $\mathfrak{I}^r(p, q)$ are the (C^r, C^∞) *tensorfields of type* (p, q) or contravariant order p and covariant order q, or briefly C^r *tensors*. We shall generally write $\mathfrak{I}(p, q)$ for $\mathfrak{I}^{-1}(p, q)$.

Consider $(U, x) \in A \in \mathfrak{A}^{r+1}$. We have shown (3.4) that $\{X_i = \tilde{\partial}/\partial x_i\}$ is an $\mathfrak{F}^r$-basis for $\mathfrak{X}^r \mid U$, and (3.6) that the dual $\{\theta^i\}$ is an $\mathfrak{F}^r$-basis for $*\mathfrak{X}^r \mid U$. It follows that

$$\{X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_q}\}$$

is an $\mathfrak{F}^r$-basis for $\mathfrak{I}^r(p, q)U$, the *canonical basis* in (U, x). Thus if $(U, x) \in A \in A^\infty$, the canonical basis in (U, x) is shared by all of the free modules $\mathfrak{I}^r(p, q) \mid U$, ($p, q$ fixed), so again it is natural to consider $\mathfrak{I}^{r+1}(p, q) \subset \mathfrak{I}^r(p, q)$. This inclusion might have been included in the definition.

Analogous to equations (4) and (5), we have for $T \in \mathfrak{I}(p, q) \mid U$,

$$T = \sum_{i_1, \cdots j_q = 1}^{n} t^{i_1 \cdots i_p}_{j_1 \cdots j_q} X_{i_1} \otimes \cdots \otimes \theta^{j_q}, \tag{6}$$

in which the functions $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$, the *canonical components* of T in (U, x) are given by

$$t^{i_1 \cdots i_p}_{j_1 \cdots j_q} = T(\theta^{i_1}, \cdots, \theta^{i_p}, X_{j_1}, \cdots, X_{j_q}). \tag{7}$$

Applying T to $(\phi^{r_1}, \cdots, Y_{s_q})$ where

$$\phi^{r_j} = \sum_{i=1}^{n} f^{r_j}_i \phi^i, \qquad Y_{s_k} = \sum_{i=1}^{n} y^i_{s_k} X_i \ ,$$

we find

$$T(\phi^{r_1}, \cdots, Y_{s_q}) = \sum_{i_1, \cdots, j_q}^{n} t^{i_1 \cdots i_p}_{j_1 \cdots j_q} f^{r_1}_{i_1} \cdots y^{j_q}_{s_q} . \tag{8}$$

An argument based on these equations, similar to that preceding 3.6, proves the next theorem.

Theorem 3.8. *The $\mathfrak{I}^r(p, q)$ are locally free unitary $\mathfrak{F}^r$-modules, and a tensor is C^r if and only if the manifold admits a C^{r+1} coordinate system in which all of its canonical components are C^r functions.*

Definition 3.9. The contraction operator is a function,

$$C^r_s : \mathfrak{I}(p, q) \to \mathfrak{I}(p - 1, q - 1), \qquad 0 < r \leqq q, \qquad 0 < s \leqq p,$$

defined by

$$C^r_s(Y_1 \otimes \cdots \otimes Y_p \otimes \phi^1 \otimes \cdots \otimes \phi^q)$$
$$= \phi^r(Y_s)(\cdots \otimes Y_{s-1} \otimes Y_{s+1} \otimes \cdots \otimes \phi^{r-1} \otimes \phi^{r+1} \otimes \cdots)$$

and extended R-linearly.

It is evident that contraction preserves the differentiability of a tensor.

Definition 3.10. Let I be the identity of End $(\mathfrak{X})$. With it we identify in the usual way the (1, 1)-tensor δ defined by $\delta(\phi, Y) = \phi(I(Y)) = \phi(Y)$. This tensor, *Kronecker's delta*, is obviously C^∞.

Note that the components of a C^r tensor are C^r in any C^{r+1} coordinate system, but need not be in an arbitrary admissible coordinate system. The additional discontinuities may be considered artificial in the sense that they may be transformed away by a change of coordinate system.

4. Connection. Let ∇ denote a homomorphism of $\mathfrak{I}(p, q)$ into $\mathfrak{I}(p, q+1)$, for all (p, q). Then ∇ induces a function from $\mathfrak{X} \times \mathfrak{I}(p, q)$ into $\mathfrak{I}(p, q)$, for all (p, q) by $(X, t) \to \nabla_X t$, where

$$\nabla_X t(\varphi^1, \cdots, \varphi^p; Y_1, \cdots, Y_q) = \nabla t(\varphi^1, \cdots, \varphi^p; X, Y_1, \cdots, Y_q).$$

We may also consider that for each X, ∇_X is an endomorphism of the $\mathfrak{I}(p, q)$ taking t into $\nabla_X t$. using these identifications we shall define the covariant differential in almost the classical way.

Definition 4.1. An R-linear homomorphism $\nabla: \mathfrak{I}(p, q) \to \mathfrak{I}(p, q+1)$, for all (p, q), is a *piecewise C^∞ covariant differential*, or *linear connection*, if and only if

(a) $\nabla_X f = X(f)$ for all $X \in \mathfrak{X}$, $f \in \mathfrak{F}$, or equivalently, $\nabla f = df$, the ordinary differential of f,
(b) ∇_X is a $\otimes$-derivation for all $X \in \mathfrak{X}$ and (p, q),
(c) ∇ commutes with contraction, that is $\nabla \circ C^r_s = C^{r+1}_s \circ \nabla$ for all (p, q) and permissible r and s.

A linear connection if of *class* C^m if and only if

$$\nabla : \mathfrak{I}^{m+1}(p, q) \to \mathfrak{I}^m(p, q+1)$$

for all m and (p, q).

Theorem 4.2. $\nabla\delta = 0$.

Proof. This follows directly from 4.1(c). For $\delta(\phi, Y) = \phi(Y)$, so differentiating both sides by ∇_Z, we have by 4.1(b)

$$\nabla_Z\, \delta(\phi, Y) + \delta(\nabla_Z\phi, Y) + \delta(\phi, \nabla_Z Y) = \nabla_Z\{\phi(Y)\}.$$

From 4.1(c), the second and third terms on the left comprise exactly the right hand side, so $\nabla_Z\delta \equiv 0$ for all Z, or $\nabla\delta = 0$.

We shall now express the covariant differential in canonical components. Suppose (U, x) is any admissible chart, $\{X_i\}$ and $\{\theta^i\}$ dual bases for $\mathfrak{X} \mid U$ and $*\mathfrak{X} \mid U$ respectively. Write

$$\nabla X_j = \Gamma^k_{ij} X_k \otimes \theta^i, \tag{9}$$

where we have used the summation convention for repeated indices with range

$1, \cdots, n$. It is easily verified that $\delta = X_i \otimes \theta^i$ in U, so taking the differential of both sides, we obtain, with 4.1(b), 4.2, and (9) above.

$$X_i \otimes \nabla \theta^i = -\Gamma^k_{ji} X_k \otimes \theta^j \otimes \theta^i,$$

Applying both sides to (θ^r, X_s, X_t), we have

$$\nabla_{X_s} \theta^r(X_t) = -\Gamma^r_{st}$$

or

$$\nabla \theta^r = -\Gamma^r_{st} \theta^s \otimes \theta^t. \tag{10}$$

Now, if

$$T = t^{i \cdots j}_{r \cdots s} X_i \otimes \cdots \otimes X_j \otimes \theta^r \otimes \cdots \otimes \theta^s,$$

taking the differential of both sides by means of 4.1(a) and (b), (9), and (10), we get

$$\nabla T = \nabla_k t^{i \cdots j}_{r \cdots s} X_i \otimes \cdots \otimes X_j \otimes \theta^k \otimes \theta^r \otimes \cdots \otimes \theta^s, \tag{11}$$

where

$$\nabla_k t^{i \cdots j}_{r \cdots s} = X_k t^{i \cdots j}_{r \cdots s} + \Gamma^i_{km} t^{m \cdots j}_{r \cdots s} + \cdots + \Gamma^j_{km} t^{i \cdots m}_{r \cdots s} - \Gamma^m_{kr} t^{i \cdots j}_{m \cdots s} - \cdots - \Gamma^m_{ks} t^{i \cdots j}_{r \cdots m}. \tag{12}$$

The functions Γ^k_{ij}, $i, j, k = 1, \cdots, n$, are called the *canonical components* of the connection, and the equations (11) and (12) show that the connection is uniquely determined by its components. In other words, if it is known how the connection acts on vectors, then it is known how it acts on tensors of all types. This is essentially the content of a uniqueness theorem of WILLMORE [7].

A repetition of an earlier argument based on the equations (11) and (12) proves the following result.

Theorem 4.3. *A linear connection is of class C^r if and only if the manifold admits a C^{r+1} coordinate system in which all of the canonical components of the connection are C^r functions.*

Definition 4.4. The *torsion* of a linear connection ∇ is a function $\mathbf{T} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ defined by

$$\mathbf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

It is well known that $\mathbf{T}$ is $\mathfrak{F}$-bilinear, so that the function

$$T: *\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{F}$$

defined by $T(\phi, X, Y) = \phi\{\mathbf{T}(X, Y)\}$ is a tensor, called the *torsion tensor* of ∇. If $T \equiv 0$, ∇ is said to be *torsion-free* or *symmetric*.

Theorem 4.5. *The torsion tensor of a linear connection of class C^r is of class C^r.*

Proof. The canonical basis $\{X_i\}$ is a commuting basis, $[X_i, X_j] = 0$, so obviously the canonical components of T are

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} .$$

Thus the proof follows immediately from 4.3.

Definition 4.6. The *curvature* of a linear connection ∇ is a function

$$\mathbf{R}: \mathfrak{X} \times \mathfrak{X} \to \mathrm{End}\,(\mathfrak{T}(p, q))$$

defined by

$$\mathbf{R}(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} .$$

It is well known that $\mathbf{R}$ is $\mathfrak{F}$-bilinear, so the function

$$R: *\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{F}$$

defined by $R(\phi, X, Y, Z) = \phi\{R(X, Y)Z\}$ is a tensor, called the *curvature tensor*.

The following properties of the curvature are immediately evident:

Theorem 4.7. (a) *The curvature tensor of a C^r linear connection is of class C^{r-1}.* (b) $\mathbf{R}(X, Y)$ *is a $\otimes$-derivation mapping $\mathfrak{F}$ and δ into zeros.*

The classical symmetry conditions for the curvature have been found in this formulation by NOMIZU ([8], p. 61).

Theorem 4.8. *Let $\mathfrak{S}$ indicate cyclic summing, that is*

$$\mathfrak{S}(X, Y) = (X, Y) + (Y, X).$$

First identity: $\mathfrak{S}\{\mathbf{R}(X, Y)\} = 0$

Second identity: $\mathfrak{S}\{\mathbf{R}(X, Y)Z\} = \mathfrak{S}\{\mathbf{T}(\mathbf{T}(X, Y), Z\} + \mathfrak{S}\{(\nabla_X\mathbf{T}(Y, Z)\}$

Bianchi's identity: $\mathfrak{S}\{(\nabla_Z\mathbf{R})(X, Y)\} + \mathfrak{S}\{\mathbf{R}(\mathbf{T}(X, Y), Z)\} = 0.$

Concerning the existence of a linear connection, the following remark has been attributed to HANO-OZEKI ([8], p. 48).

Theorem 4.9. *A connected manifold may be given a C^∞ linear connection only if it is second countable.*

The converse of this is true, as is well known and will be discussed in the next section.

5. Metric.

Definition 5.1. A tensor of type (0, 2), $\mathbf{g}$, induces two functions, $\mathbf{g}_L$, $\mathbf{g}_R$: $\mathfrak{X} \to *\mathfrak{X}$ by

$$\mathbf{g}_L X(Y) = \mathbf{g}(X, Y) \quad \text{and} \quad \mathbf{g}_R X(Y) = \mathbf{g}(Y, X).$$

$\mathbf{g}$ is symmetric if $\mathbf{g_L} = \mathbf{g_R}$, non-degenerate if $\mathbf{g_L}$ amd $\mathbf{g_R}$ both have kernel zero. Every point in the manifold has a neighborhood, V, in which a non-degenerate $\mathbf{g}$ may be written

$$\mathbf{g} = \sum_{i=1}^{n} \sigma_i \theta^i \otimes \theta^i, \qquad \sigma_i = \pm 1,$$

where the forms θ^i are non-zero and linearly independent in V. The symbol $\sigma(\mathbf{g}) = (\sigma_1, \cdots, \sigma_n)$ is the *signature* of $\mathbf{g}$, obviously a constant on any connected open set in which $\mathbf{g}$ is continuous. $\mathbf{g}$ is *positive definite* if its signature is $(+, \cdots, +)$ everywhere, *normal hyperbolic* if its signature is $(+, -, \cdots, -)$ everywhere. If $\mathbf{g}$ is non-degenerate, the conjugate of $\mathbf{g}$ is the tensor of type $(2, 0)$ defined by $*\mathbf{g}(\theta, \phi) = \theta(\mathbf{g}_L^{-1}\phi)$.

Definition 5.2. A *piecewise C^∞ metric* is a symmetric tensor of type $(0, 2)$, $\mathbf{g}$. It is *of class C^r* if it is a C^r tensor, a *pseudo-Riemannian metric* if $\mathbf{g}$ non-degenerate, *Lorentz metric* if $\mathbf{g}$ is normal hyperbolic, a *Riemannian metric* if $\mathbf{g}$ is positive definite. For a pseudo-Riemannian metric, $\mathbf{g}$ and $*\mathbf{g}$ are called the *covariant metric tensor* and the *contravariant metric tensor*, respectively.

It is well known that a Riemannian metric induces a topological metric on the manifold. Concerning the existence of a metric the following remarks are known ([6], p. 37, 9). (Recall that on a connected HAUSDORFF manifold, second countable, paracompact, and separable-metric are equivalent.)

Theorem 5.3. (a) *A C^r-manifold, $r \geqq 1$, may be given a C^{r-1} Riemannian metric if and only if it is paracompact.*
(b) *A compact C^r-manifold, $r \geqq 1$, may be given a C^{r-1}* LORENTZ *metric if and only if all its components have* EULER-POINCARÉ *characteristic zero.*
(c) *A non-compact C^r-manifold, $r \geqq 1$, may be given a C^{r-1}* LORENTZ *metric if and only if it is paracompact.*

In these remarks "piecewise C^∞", which we ordinarily understand by tacit agreement, is unnecessary.

Definition 5.4. Let a manifold have a fixed metric $\mathbf{g}$. Then for any linear connection ∇, we say that $\nabla\mathbf{g}$ is the *gradient* of ∇ (with respect to $\mathbf{g}$), and ∇ is *metrical* (with respect to $\mathbf{g}$) if its gradient vanishes. If ∇ is metrical and symmetric (torsion-free), it is *compatible* (with $\mathbf{g}$).

Theorem 5.5. *Let $\mathbf{g}$ be a C^r pseudo-Riemannian metric, S a C^{r-1} $(0, 3)$ tensor satisfying*

$$S(X, Y, Z) - S(X, Z, Y) = 0$$

and T a C^{r-1} $(1, 2)$-tensor satisfying

$$T(\varphi, X, Y) + T(\varphi, Y, X) = 0$$

for all $X, Y, Z \in \mathfrak{X}$ and $\varphi \in \mathfrak{X}^*$. Then there exists a unique C^{r-1} linear connection having gradient S and torsion T.

Proof. Suppose $\mathbf{g}$, S, and T as above, and that ∇ is a linear connection having gradient S and torsion T. We shall express the action of ∇ on vectors in a construction showing both existence and uniqueness. By the definition of gradient, we may write

$$(13)\qquad -2S'(X, Y, Z) \equiv S(X, Y, Z) + S(Y, Z, X) - S(Z, X, Y) = \nabla_X \mathbf{g}(Y, Z) + \nabla_Y \mathbf{g}(Z, X) - \nabla_Z \mathbf{g}(X, Y).$$

We now write $X \cdot Y$ for $\mathbf{g}(X, Y)$. Then because of the defining properties of ∇, 4.1(a), (b) and (c), each term on the right of (13) may be expanded, for example,

$$(14)\qquad \nabla_X \mathbf{g}(Y, Z) = X(Y \cdot Z) - \nabla_X Y \cdot Z - Y \cdot \nabla_X Z.$$

Also, as we assume ∇ has torsion tensor T, we have by definition

$$(15)\qquad \nabla_Y X = \nabla_X Y - [X, Y] - \mathbf{T}(X, Y),$$

where $\varphi(\mathbf{T}(X, Y)) = T(\varphi, X, Y)$. If we now expand (13) as indicated in (14) and simplify by means of (15), the following is obtained,

$$(16)\qquad \nabla_X Y \cdot Z = \{X, Y \mid Z\} + T'(X, Y, Z) + S'(X, Y, Z),$$

where we have used the abbreviations

$$(17)\qquad 2\{X, Y \mid Z\} = X(Y \cdot Z) + Y(Z \cdot X) - Z(X \cdot Y) - X \cdot [Y, Z] + Y \cdot [Z, X] + Z \cdot [X, Y],$$

$$(18)\qquad 2T'(X, Y, Z) = \mathbf{T}(X, Y) \cdot Z - \mathbf{T}(Y, Z) \cdot X + \mathbf{T}(Z, X) \cdot Y.$$

As $\mathbf{g}$ is pseudo-Riemannian (non-degenerate) we may "solve" (16) for the vector $\nabla_X Y$, obtaining

$$(19)\qquad \nabla_X Y = \{X, Y \mid + \mathbf{T}'(X, Y) + \mathbf{S}'(X, Y),$$

where the vectors on the right have obvious meanings, especially,

$$(20)\qquad \varphi(\{X, Y \mid) = \{X, Y \mid \mathbf{g}_{\mathrm{L}}^{-1}(\varphi)\}.$$

We have already remarked that ∇ is determined uniquely by its action on vectors according to equations (11) and (12). Thus (17), (18) and (19) show that ∇ is uniquely determined by $\mathbf{g}$, S, T.

Finally, it is evident from (16), (17) and (18) that if $\mathbf{g}$ is C^r, S and T both C^{r-1}, then $\nabla_Y X$ is C^{r-1} for all $Y \in \mathfrak{X}^{r-1}$ and $X \in \mathfrak{X}^r$. Thus ∇ is of class C^{r-1}, according to 4.1 and 4.3. This completes the proof.

Note that as any linear connection has unique gradient and torsion, the set $\mathfrak{D}$ of all linear connections is in one-one correspondence with the set of pairs (S, T) of tensors having the symmetries mentioned in 5.5. Further, each choice of a pseudo-Riemannian metric induces a unique correspondence. This gives an idea of the size of $\mathfrak{D}$. The existence of a unique $\nabla \in \mathfrak{D}$ for each g corresponding to the pair (0, 0) provides the fundamental theorem of pseudo-Riemannian geometry:

Corollary 5.6. *A C^r pseudo-Riemannian metric induces a unique compatible connection of class C^{r-1}.*

In this case the induced connection is the classical *pseudo-Riemannian* or *Levi-Civita connection*, and equation (19) is the classical relation between the CHRISTOFFEL symbols of the first and second kinds. We thus consider $\{X, Y \mid Z\}$ a generalization of the CHRISTOFFEL symbols.

Concerning the existence of a linear connection, which we have discussed in the previous section, it appears from 5.3(a) and 5.6 that a paracompact manifold may be given a linear connection of class C^∞. As a second countable manifold is paracompact, we may extend 4.9 to an equivalence.

Theorem 5.7. *A connected C^r manifold ($r \geqq 1$) admits a C^{r-1} linear connection if and only if it is paracompact.*

6. Pseudo-Riemannian geometry.

Definition 6.1. By a *piecewise differentiable pseudo-Riemannian n-space* we mean a triple $(M, \mathbf{g}, \nabla)$ such that M is a paracompact piecewise differentiable n-manifold, $\mathbf{g}$ a pseudo-Riemannian (non-degenerate) metric of class (C^1, C^∞), and ∇ its unique continuous pseudo-Riemannian connection. The isomorphism $\mathbf{g}_L$ connecting $\mathfrak{X}$ and $*\mathfrak{X}$ induces isomorphisms between all the $\mathfrak{I}(p, q)$ having $p + q$ fixed, for which we use the following notation.

$$\#_i : \mathfrak{I}(p, q) \to \mathfrak{I}(p + 1, q - 1), \qquad 1 \leqq i \leqq q,$$

defined by

$$\#_i T(\phi^1, \cdots, \phi^{p+1}; Y_1, \cdots, Y_{q-1}) = T(\phi^1, \cdots, \phi^p; Y_1, \cdots, Y_{i-1}, \mathbf{g}_L^{-1}\phi^{p+1}, Y_i, \cdots),$$

$$b_i : \mathfrak{I}(p, q) \to \mathfrak{I}(p - 1, q + 1), \qquad 1 \leqq i \leqq p,$$

defined by

$$b_i T(\phi^1, \cdots, \phi^{p-1}; Y_0, \cdots, Y_q) = T(\phi^1, \cdots, \phi^{i-1}, \mathbf{g}_L Y_0, \phi^i, \cdots; Y, \cdots, Y_q).$$

The isomorphism $\#_i$ corresponds to the operation of *raising the i^{th} covariant index to the last contravariant place* in classical tensor calculus. The isomorphism b_i corresponds to *lowering the i^{th} contravariant index to the first covariant place.* These induce the *covariant contraction operators*

$$C^{rs} = C^r_{p+1} \circ \#_s : \mathfrak{I}(p, q + 2) \to \mathfrak{I}(p, q)$$

and the *contravariant contraction operators*

$$C_{rs} = C^1_r \circ b_s : \mathfrak{I}(p + 2, q) \to T(p, q),$$

(see 3.8). It is immediately evident that $C_{rs} = C_{sr}$, $C^{rs} = C^{sr}$. The metric induces a *magnitude*, $\sqrt{\mathbf{g}}$ by $\sqrt{\mathbf{g}}(X) = \sqrt{|\mathbf{g}(X, X)|}$, if X is a vector. A vector X is *positively oriented*, *null*, or *negatively oriented* at a point p of M according as $\mathbf{g}(X, X)(p)$ is positive, zero, or negative. Two vectors X, Y, are orthogonal at p if and only if $\mathbf{g}(X, Y)(p) = 0$. A vector X is *autoparallel* in an open set U of M if and only if $\nabla_X X \mid U \equiv 0$. A *geodesic* is an integral curve of an autoparallel vector. The covariant curvature tensor ${}^bR$ is called the *Riemann tensor*. The contracted curvature tensor

$$\mathbf{r} = C_1^1 R$$

is called the *Ricci tensor*. The *scalar curvature* is the function

$$r = C^{12}\mathbf{r}.$$

The *Einstein tensor* is defined by

$$\mathbf{G} = \mathbf{r} - \tfrac{1}{2}(r + \lambda)\mathbf{g}$$

where λ is an fixed real number, called the *cosmological constant*.

Theorem 6.2. *The curvature satisfies the following symmetry conditions.*

First identity: $\mathfrak{S}\{\mathbf{R}(X, Y)\} = 0$,

Second identity: $\mathfrak{S}\{\mathbf{R}(X, Y)Z\} = 0$,

Third identity: ${}^bR(V, X, Y, Z) + {}^bR(Z, X, Y, V) = 0$,

Fourth identity: ${}^bR(V, X, Y, Z) - {}^bR(Y, Z, V, X) = 0$,

Fifth identity: ${}^bR(V, X, Y, Z) - {}^bR(X, V, Z, Y) = 0$,

BIANCHI *identity*: $\mathfrak{S}\{(\nabla_Z \mathbf{R})(X, Y)\} = 0$.

Proof. The first, second, and BIANCHI identities follow directly from 4.8 as $\mathbf{T} \equiv 0$. For the third, note $\mathbf{R}(X, Y)\{\theta(Z)\} = 0$ by 4.8(b), expand by 4.8(b), set $V = \#\theta$. For the fourth, note that the second may be written in the form

$$(21) \qquad {}^bR(V, X, Y, Z) + {}^bR(V, Y, Z, X) + {}^bR(V, Z, X, Y) = 0.$$

Writing (21) four times for the even permutations of (V, X, Y, Z) and simplifying by means of the third identity, the fourth is obtained. The fifth is obtained directly from the first, third, and fourth.

Theorem 6.3. *The* RICCI *and* EINSTEIN *tensors of a pseudo-Riemannian space are symmetric.*

Proof. We have defined $\mathbf{r} = C_1^1 R$, which is evidently the same as $C^{12}({}^bR)$. Define another tensor by ${}'R(V, X, Y, Z) = {}^bR(X, V, Y, Z)$, that is by interchanging the first and second "slots". Obviously $C^{12}\,{}'R = C^{21}\,{}'R$, but $C^{12} = C^{21}$, so $\mathbf{r} = C^{12}\,{}'R$. Now note that the fifth identity of 6.2 may be written

$$(22) \qquad {}^bR(-, -, Y, Z) - {}'R(-, -, Y, Z) = 0.$$

Contracting (22) with C^{12}, have

$$\mathfrak{r}(X, Z) = \mathfrak{r}(Z, X), \tag{23}$$

so $\mathfrak{r}$ is symmetric. As $\mathbf{G} = \mathfrak{r} - \frac{1}{2}(\lambda + r)\mathbf{g}$, obviously $\mathbf{G}$ is symmetric also.

Definition 6.4. The *divergence* is a function

$$\boldsymbol{\delta}: \mathfrak{I}(p, q + 1) \to \mathfrak{I}(p, q)$$

by $\boldsymbol{\delta} = C^{12} \circ \nabla$. A tensor in a kernel of $\boldsymbol{\delta}$ is said to be *conservative.*

Theorem 6.5. *The* EINSTEIN *tensor of a pseudo-Riemannian space is conservative.*

Proof. We define a new tensor by

$$S(U, V, X, Y, Z) = \nabla_U{}^bR(V, X, Y, Z) - \nabla_X{}^bR(Z, Y, U, V) - \nabla_Y{}^bR(V, X, U, Z). \tag{24}$$

From the first, third, and BIANCHI identities of 6.2 it follows that $S \equiv 0$. But from (24) it is evident that

$$C^{23} \circ C^{24}(S) = dr - 2\,\delta\mathfrak{r} = 0. \tag{25}$$

Note as $\mathbf{G} = \mathfrak{r} - \frac{1}{2}(r + \lambda)\mathbf{g}$ that $\delta\mathbf{G} = \delta\mathfrak{r} - \frac{1}{2}dr$, thus $\delta\mathbf{G} = 0$.

7. The rigging of hypersurfaces.

Definition 7.1. A (C^m, C^∞) *submanifold* of a (C^r, C^∞)-n-manifold M is a pair (i, Σ) such that

(a) Σ is a (C^m, C^∞)-s-manifold, $s \leqq n$, $r \leqq m$,
(b) i is a differentiable one-to-one function of Σ into M of class (C^m, C^∞), that is if (U, x) is an admissible chart on M, then $x \circ i$ is of class (C^m, C^∞) on $i^{-1}(U \cap i\Sigma)$,
(c) the map $i_*: \mathfrak{X}_\Sigma \to \mathfrak{X}_{i\Sigma}$ defined by $i_*X(f) = X(f \circ i)$ is univalent.

Further, (i, Σ) is *regularly imbedded* if and only if, in addition,
(d) i is a homeomorphism.

A *hypersurface* is a regularly imbedded submanifold of dimension $s = n - 1$. It is well known ([10], p. 40) that all of the geometric structures on Σ and $S = i\Sigma$ are equivalent if i is a regular imbedding. Hereafter we consider hypersurface to mean the $(n - 1)$-dimensional image S. In the next section we will relate the geometric structures of a hypersurface S in a piecewise differentiable manifold M to the structures on M. The central notion is that of extending the structures on a hypersurface by means of a "rigging". (See [11], p. 77 for the classical analogue.) *We consider in this section only continuous functions and tensors.*

Definition 7.2. If S is a hypersurface of M, denote the piecewise differentiable functions of S by $\mathfrak{F}_s$, those of M by $\mathfrak{F}_m$, and similarly for the vectors and tensors of all orders. Indicate the ordinary restriction of continuous functions from M

to S by π, so $\pi : \mathfrak{F}_m \to \mathfrak{F}_s$. A map $\epsilon : \mathfrak{F}_s \to \mathfrak{F}_m$ is called an *extension* if $\pi \circ \epsilon$ is the identity on $\mathfrak{F}_s$. If an extension exists, the continuous tensors of all types may be brought back and forth freely between M and S as follows. Define

$$\epsilon_0^1 : \mathfrak{X}_s \to \mathfrak{X}_m \quad \text{by} \quad \epsilon_0^1 X_s(f_m) = \epsilon\{X_s(\pi f_m)\},$$

$$\pi_0^1 : \mathfrak{X}_m \to \mathfrak{X}_s \quad \text{by} \quad \pi_0^1 X_m(f_s) = \pi\{X_m(\epsilon f_s)\},$$

$$\epsilon_1^0 : *\mathfrak{X}_s \to *\mathfrak{X}_m \quad \text{by} \quad \epsilon_0^1 \theta_s(X_m) = \epsilon\{\theta_s(\pi_0^1 X_m)\},$$

$$\pi_1^0 : *\mathfrak{X}_m \to *\mathfrak{X}_s \quad \text{by} \quad \pi_1^0 \theta_m(X_s) = \pi\{\theta_m(\epsilon_0^1 X_s)\},$$

$$\epsilon_q^p : \mathfrak{I}_s(p, q) \to \mathfrak{I}_m(p, q) \quad \text{by} \quad \epsilon_q^p = (\epsilon_0^1)^{\otimes p} \otimes (\epsilon_1^0)^{\otimes q},$$

$$\pi_q^p : \mathfrak{I}_m(p, q) \to \mathfrak{I}_s(p, q) \quad \text{by} \quad \pi_q^p = (\pi_0^1)^{\otimes p} \otimes (\pi_1^0)^{\otimes q}.$$

It is easily verified that ϵ_0^1 is a LIE algebra isomorphism. In general we write π and ϵ without indices, for they are clear from context. Then π is the *restriction* for tensors, ϵ the *extension*. It is clear that the maps $\pi \circ \epsilon$ are identities for all the modules. The same is not true for $\epsilon \circ \pi$.

Definition 7.3. Suppose (i, Σ) is a hypersurface of M and (j, Σ') is a regularly imbedded submanifold, Σ' any fiber space having base space Σ, fiber $I = \{x : -1 < x < 1\}$, and projection λ, with Σ identified with the zero cross section, and $j \mid \Sigma = i$. Then (j, Σ') constitutes a *thickening* of (i, Σ). For every point p of Σ denote by $C_{i(p)}$ the curve $j(\lambda^{-1}(p))$. The set $\mathfrak{C}$ of curves C_q such that $q \epsilon S = i\Sigma$ is a *normal congruence of curves* on S.

Theorem 7.4. *A normal congruence of curves on a hypersurface induces an extension.*

Proof. The projection λ in Σ' induces a retract of the thickening, $r : j(\Sigma') \to S$ by $r = i \circ \lambda \circ j^{-1}$. Define a map ϵ on $\mathfrak{F}_s$ by $\epsilon f = f \circ r$ in $j(\Sigma')$, $\epsilon f = 0$ in $M - j(\Sigma')$. We must show first that ϵ is an extension, that is ϵ maps $\mathfrak{F}_s \to \mathfrak{F}_m$. It is evident that the boundary of $j(\Sigma')$ consists of hypersurfaces, so the obvious discontinuity of ϵf occurs across hypersurfaces. We must show that ϵf is piecewise C^∞. By hypothesis i, j are (C^1, C^∞) diffeomorphisms, so if (U, x^α), $\alpha = 1, \cdots, n-1$ is an admissible chart on Σ, then $(iU \times I, x^\alpha \circ i^{-1}, t \circ j^{-1})$ is an admissible chart on M, where t us a coordinate in the fiber I. If f is any function of $\mathfrak{F}_s$, with induced function $f_{iU}^* = (f_1, \cdots, f_{n-1})$, clearly ϵf has induced function $(\epsilon f)_{iU\times(-1,1)}^* = (f_1, \cdots, f_{n-1}, 0)$. Thus f piecewise C^∞ implies ϵf piecewise C^∞.

Definition 7.5. If S is a hypersurface, with a normal congruence of curves, the extensions and restrictions given by 7.2 and the induced extension of the normal congruence constitute a *rigging* of S, the curves of the congruence are the *rigging curves*, which generalize the rigging lines of the Euclidean case. If X is a vector on M, we denote by $\eta_0^1 X$ the vector obtained by assigning to every point p of $S' = j\Sigma'$ the component of X which is tangent to the rigging curve through p, and letting η_0^1 be the identity on the compliment of S' in M. Thus $\eta_0^1 X$ is piecewise continuous, with discontinuities across the boundary of S'. We call the map $\eta_0^1 : \mathfrak{X}_m \to \mathfrak{X}_m$ thus defined the *normal projection*. We extend the normal projection to forms as follows: $\eta_1^0\theta(X) = \theta(\eta_0^1 X)$, that is, η_1^0 is the adjoint of η_0^1. We shall need also the *tangential projections*: $\tau_0^1 = \delta - \eta_0^1$, $\tau_1^0 = \delta - \eta_1^0$. Evidently we may define projections for tensors of arbitrary type by forming tensor products of the projections already defined. We generally indicate these by τ and η, the order being understood from context. We say a tensor T is *tangential* if $\eta T = 0$, *normal* if $\tau T = 0$. To return to the notion of rigging, we say that a rigging of a hypersurface in a manifold with connection is a *geodesic rigging* if its rigging curves are geodesic with respect to the given connection. If S is a rigged hypersurface, and the manifold admits a coordinate system, A, in which S is covered by patches (U, x^j), $j = 0, 1, \cdots, n - 1$, such that (1) the patches $(U \cap S, \pi x^\alpha)$, $\alpha = 1, \cdots, n - 1$, constitute an admissible coordinate system on S, and (2) the coordinate lines of x^0 are rigging curves, we say that S has an *admissible rigging*, and A is a *conforming coordinate system*. A coordinate chart of A with the properties (1) and (2) is a *conforming chart* for S.

Concerning the existence of a rigging for a given hypersurface, we shall show that every hypersurface in a second countable manifold may be rigged.

Theorem 7.6. *For every hypersurface in a piecewise differentiable manifold with continuous connection there exists an admissible geodesic rigging. If the hypersurface is C^2 imbedded, it admits a conforming coordinate system of class C^2.*

Proof. We shall (**a**) construct a thickening by a normal congruence of geodesics, which by 7.4 induces a geodesic rigging, then (**b**) construct a conforming coordinate system, the conforming charts of which are the skew-Gaussian coordinates of classical differential geometry, and show that it is admissible, and finally (**c**) show that if S is C^2 imbedded, then the conforming coordinate system is of class C^2.

(**a**) As the manifold admits a continuous connection, it is paracompact (5.7) and thus admits a C^∞ Riemannian metric (5.3a). The idea of the construction is to thicken the hypersurface with a subset of the orthogonal compliment of $\mathfrak{X}_S$ in $\mathfrak{X}_M$ with respect to an auxiliary metric, by means of the exponential map. It is necessary to find a function k on S, with values between zero and one, such that on the subset of orthogonal vectors of length less than k, the exponential map is a diffeomorphism.

Let $\mathbf{g}$ be any C^∞ Riemannian metric. If p is any point in the image $S = i\Sigma$ of a hypersurface (i, Σ), the tangent space to S at p, $\mathfrak{X}_S \mid p$, may be considered an $(n - 1)$-dimensional linear subspace of the tangent space to the manifold M at p, $\mathfrak{X}_M \mid p$. Let L_p denote the orthogonal compliment of this subspace with respect to the auxiliary metric, $\mathbf{g}$. As $\mathbf{g}$ is positive definite by hypothesis, L_p does not lie in $\mathfrak{X}_S \mid p$, so $\mathfrak{X}_M \mid p = \mathfrak{X}_S \mid p \oplus L_p$. As i is by hypothesis a C^1 imbedding, and $\mathbf{g}$ is C^∞, it is not hard to see that $\{L_p : p \in S\}$ constitutes a C^1 line element field on S, which is thus locally orientable.

curve through p, and letting η_0^1 be the identity on the compliment of S' in M. Thus $\eta_0^1 X$ is piecewise continuous, with discontinuities across the boundary of S'. We call the map $\eta_0^1 : \mathfrak{X}_m \to \mathfrak{X}_m$ thus defined the *normal projection*. We extend the normal projection to forms as follows: $\eta_1^0\theta(X) = \theta(\eta_0^1 X)$, that is, η_1^0 is the adjoint of η_0^1. We shall need also the *tangential projections*: $\tau_0^1 = \delta - \eta_0^1$, $\tau_1^0 = \delta - \eta_1^0$. Evidently we may define projections for tensors of arbitrary type by forming tensor products of the projections already defined. We generally indicate these by τ and η, the order being understood from context. We say a tensor T is *tangential* if $\eta T = 0$, *normal* if $\tau T = 0$. To return to the notion of rigging, we say that a rigging of a hypersurface in a manifold with connection is a *geodesic rigging* if its rigging curves are geodesic with respect to the given connection. If S is a rigged hypersurface, and the manifold admits a coordinate system, A, in which S is covered by patches (U, x^j), $j = 0, 1, \cdots, n-1$, such that (1) the patches $(U \cap S, \pi x^\alpha)$, $\alpha = 1, \cdots, n-1$, constitute an admissible coordinate system on S, and (2) the coordinate lines of x^0 are rigging curves, we say that S has an *admissible rigging*, and A is a *conforming coordinate system.* A coordinate chart of A with the properties (1) and (2) is a *conforming chart* for S.

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Proof. We shall (**a**) construct a thickening by a normal congruence of geodesics, which by 7.4 induces a geodesic rigging, then (**b**) construct a conforming coordinate system, the conforming charts of which are the skew-Gaussian coordinates of classical differential geometry, and show that it is admissible, and finally (**c**) show that if S is C^2 imbedded, then the conforming coordinate system is of class C^2.

(**a**) As the manifold admits a continuous connection, it is paracompact (5.7) and thus admits a C^∞ Riemannian metric (5.3a). The idea of the construction is to thicken the hypersurface with a subset of the orthogonal compliment of $\mathfrak{X}_S$ in $\mathfrak{X}_M$ with respect to an auxiliary metric, by means of the exponential map. It is necessary to find a function k on S, with values between zero and one, such that on the subset of orthogonal vectors of length less than k, the exponential map is a diffeomorphism.

Let $\mathbf{g}$ be any C^∞ Riemannian metric. If p is any point in the image $S = i\Sigma$ of a hypersurface (i, Σ), the tangent space to S at p, $\mathfrak{X}_S \mid p$, may be considered an $(n-1)$-dimensional linear subspace of the tangent space to the manifold M at p, $\mathfrak{X}_M \mid p$. Let L_p denote the orthogonal compliment of this subspace with respect to the auxiliary metric, $\mathbf{g}$. As $\mathbf{g}$ is positive definite by hypothesis, L_p does not lie in $\mathfrak{X}_S \mid p$, so $\mathfrak{X}_M \mid p = \mathfrak{X}_S \mid p \oplus L_p$. As i is by hypothesis a C^1 imbedding, and $\mathbf{g}$ is C^∞, it is not hard to see that $\{L_p : p \in S\}$ constitutes a C^1 line element field on S, which is thus locally orientable.

We wish now to demonstrate the existence of a countable covering of S by open tubular neighborhoods T_i having the properties:

(1) $\{L_p : p \in T_i\}$ is orientable,

(2) every point p of T_i can be connected to a point q of $T_i \cap S$ by a unique geodesic arc pq tangent to L_q at q: C_q, (T_i has *property P*),

(3) $$\text{if} \quad p, q \in T_i, \quad p \neq q, \quad \text{then} \quad C_q \cap C_p \cap T_i = 0,$$

(4) there is an admissible coordinate system on M, A, containing (T_i, x_i), with

$$x_i(p) = (x_i^0(p), \cdots, x_i^{n-1}(p)),$$

such that $x_i^0 \mid T_i \cap S = 0$, and $\{(T_i \cap S, x_i \mid S)\}$ is an admissible coordinate system on S,

(5) $\{T_i \cap S\}$ is a locally finite covering of S, $x_i(T_i \cap S)$ is the open $(n-1)$-ball of radius 3, $C^{n-1}(3)$, and the sets $x_i^{-1}(C^{n-1}(1))$ cover S.

As the normal distribution is locally orientable, every $p \in S$ has a neighborhood O_p satisfying (1). It is well known that every point $p \in S$ has a normal neighborhood N_p ([8], p. 64) and a regular neighborhood R_p [12]. We take $R_p \subset N_p$. Further, for every $p \in S$ there is a unique geodesic arc tangent to L_p, denote its maximal extension by c_p. If

$$C_p = \bigcup_{q \in R_p} c_p ,$$

let $V_p = R_p \cap C_p$. Then obviously V_p satisfies (2). It is easy to see that V_p may be restricted to an open neighborhood $V_p' \subset V_p$ of p in which (3) is satisfied, as $V_p \subset N_p$, $\{L_p\}$ is of class C^1, and S is C^1 imbedded by assumption. Thus the neighborhoods $W_p = V_p' \cap O_p$ satisfy (1), (2), and (3). It is well known that M admits a coordinate system with the property (4) ([10], p. 40), and W_p may be chosen small with respect to this coordinate system, then further restricted such that its image under one of the coordinate charts (U_p, x_p) of this system has the property $x_p(W_p' \cap S) = C^{n-1}(3)$. Let W_p' indicate this final restriction. Then as S is second countable there is a countable set of points p_i such that $\{T_i = W_{p_i}'\}$ has the property (5) ([6], p. 15). Then $\{(T_i, x_{p_i})\}$ has the properties (1) to (5).

We shall now construct from $\{T_i\}$ a normal congruence of geodesics. The idea is to chop off the ends of the tubes T_i in such a way that $U_{i=1}^{\infty} T_i$ has property P, and glue the resulting cylinders together with a partition of unity.

In every T_i choose a vector field X_i, of unit length with respect to the auxiliary metric, which generates $\{L_p : p \in T_i \cap S\}$, let t_i be the unique canonical parameter on the geodesics tangent to $X_i(p)$ at $p \in S$. For any k, $0 < k \leqq 1$, let

$$T_i(k) = \{p : p \in T_i , |t_i(p)| < k\}.$$

Obviously $T_i(k)$ is an open subset of T_i, and $T_i(k) \cap S = T_i \cap S$. We assert that there exists a sequence $k_1, \cdots, k_i, \cdots, 0 < k_i \leqq 1$, such that the union $\bigcup_{i=1}^{\infty} T_i(k_i)$ has the property P. In fact, it is evident that for any k_1, $0 < k_1 \leqq 1$,

$T_1(k_1)$ has the property P. Assume $U = \bigcup_{i=1}^{n-1} T_i(k_i)$ has the property P. If $\bar{U} \cap S$ contains any point p not in $\bigcup_{i=1}^{n-1} \overline{T_i \cap S}$, the k_i, $i = 1, \cdots, n-1$ may be reduced so that $p \not\in \bar{U}$, so we may suppose that $S - U \cap S$ and U may be separated by open sets. Thus if $T_n \cap S$ disjoint from U, there is a k_n, $0 < k_n \leqq 1$ such that $T_n(k_n) \cap U = 0$, so $U \cup T_n(k_n)$ has the property P. On the other hand if $T_n \cap S$ meets U, $T_n(k_{n-1}) \cap U$ has the property P because the definition of geodesic is the same in every neighborhood T_i, and the argument above for the disjoint case applies to $(T_n - T_n \cap U)^0 \cap S$. Thus we have k_n, $0 < k_n \leqq 1$, such that $\bigcup_{i=1}^{n} T_i(k_i)$ has the property P, and induction on n proves our assertion.

We now smooth the ends of the cylinders $T_i(k_i)$ so that their union will have hypersurfaces for boundaries. Let $\{\varphi_i\}$ be a partition of unity for the coordinate system $\{(T_i \cap S, X_i \mid S)\}$ of S provided by (4) above. Thus $\varphi_i = 1/\varphi$ on $X_i^{-1}(C^{n-1}(1))$, and is zero on $S - X_i^{-1}(C^{n-1}(2))$, where φ is the normalizing factor such that $\Sigma\varphi_i = 1$, if the sum at any point p of S is taken over the $T_i \cap S$ containing p. Define $k : S \to R$ by $k = \Sigma\varphi_i k_i$, where the sum has the obvious meaning. It is clear that $k \in \mathfrak{F}_S^1$, that is k is of class C^1. Now let

$$T = \left\{p : p \in \bigcup_{n=1}^{\infty} T_i(k_i), \, |t(p)| < k(q)\right\},$$

where q is the unique point of S such that $c_q(+t(p)) = p$. We have put no index on the coordinate function t along the curves as it is unique up to sign. Now it is readily verified that T comprises a normal congruence of geodesics, in the sense of 7.3. In fact if Σ' is the bundle B_1 of orthogonal vectors with length less than one, k' the map $(X, p) \to (X/k, p)$ of B_1 into itself, and $j = \exp \circ k'$, then (j, Σ') is the thickening of (i, Σ) which gives rise to the congruence of geodesics constructed above. The function k is constructed so that j is a diffeomorphism.

(b) In the preceding, the function t_i may be used as a coordinate function in the coordinate chart $T_i(k_i)$, for it is easily seen that if

$$'x_i : T_i(k_i) \to (-k_i, k_i) \times C^{n-1}(3) \subset R^n$$

by

$$'x_i(p) = (t_i(p), x_i^1(q), \cdots, x_i^{n-1}(q)),$$

where q is the "foot" of the normal geodesic through p, and $x_1^1, \cdots, x_i^{n-1}$ are the given coordinate functions of (4), then $'x_i$ is a homeomorphism. These coordinates, $'x_i$, are the skew-Gaussian coordinates of classical differential geometry. Let us now write simply T_i for $T_i(k_i)$. We must show that the coordinate system A', consisting of A of (4) with $\{(T_i, x_i)\}$ added, is admissible on M, for A' is a conforming coordinate system for the rigging defined in (a), in the sense of 7.5. For A' to be admissible it suffices to show that the transformation of coordinates $'x_i \to x_i$ in T_i is of class (C^1, C^∞), or in other words that $x_i^0 \circ {'x_i^{-1}}$ is a (C^1, C^∞) function on $x_i(T_i)$. As everything considered is piecewise C^∞, we need only show $\partial x_i^0/\partial t$ is continuous across S. But $\partial/\partial t$ generates $\{L_p\}$ in T_i, and $\{L_p\}$ is a C^1 distribution, so $\partial/\partial t$ is a local C^1 vector, x_i^0 is a local C^1 function, so $\partial x_i^0/\partial t$ is a local C^0 function in T_i. Thus A' is admissible on M, and the rigging induced in (a) is an admissible geodesic rigging.

(c) We show now that if S is C^2 imbedded, A' is of class C^2. If S is C^2 imbedded, the coordinate system A of (4) above may be taken to be of class C^2, as is well known ([10], p. 40). Thus to show A' is C^2, it suffices to show the transformation $'x_i \to x_i$ is of class (C^2, C^∞) in T_i. The idea of this argument is originally due to LICHNEROWICZ ([3], p. 61). In the coordinates x_i, we see that

$$\frac{\partial}{\partial t_i} = \frac{\partial x_i^0}{\partial t_i}\frac{\partial}{\partial x_i^0},$$

and obviously $\partial/\partial t_i$ is a geodesic field, so we have

$$\nabla_{\partial/\partial t_i}\frac{\partial}{\partial t_i} = 0. \tag{26}$$

In the coordinates x_i, this equation becomes

$$\frac{\partial x_i^0}{\partial t_i}\left\{\frac{\partial^2 x_i^\alpha}{\partial t_i^2} + \Gamma_{00}^\alpha \frac{\partial x^0}{\partial t_i}\right\} = 0, \qquad \alpha = 0, \cdots, n-1, \tag{27}$$

so evidently

$$\frac{\partial^2 x_i^0}{\partial t_i^2} = -\Gamma_{00}^0 \frac{\partial x_n^0}{\partial t_i}. \tag{28}$$

As the connection is (C^0, C^∞) and $\partial x^0/\partial t$ is (C^0, C^∞) by (b), we see that $\partial^2 x^0/\partial t^2$ is (C^0, C^∞) in T_i, and thus A' is of class (C^2, C^∞). This completes the proof of 7.6.

We have obviously the following consequence:

Corollary 7.7. *Every hypersurface in a paracompact manifold may be given an admissible rigging.*

8. Gauss-Codazzi equations for hypersurfaces. We suppose now that S is a hypersurface in a manifold M with connection ∇, and that S is rigged, with restrictions π, extensions ϵ, tangential projections τ, and normal projections η. As in the last section, *we suppose that all tensors are continuous*. There is a naturally induced connection on S defined by $'\nabla = \pi \circ \nabla \circ \epsilon$. Thus we have a curvature $\mathbf{R}$ of ∇ on M, and a surface curvature $'\mathbf{R}$ of $'\nabla$ on S. In this section we shall relate $'\mathbf{R}$ and $\pi\mathbf{R}$ (GAUSS' equation), and express the mixed normal and tangential components of the tensor $\mathbf{R}(X, Y)$ in terms of the imbedding curvatures of S (CODAZZI equations). (For the classical analogue, see [11], pp. 235, 253.)

Definition 8.1. The *rigging curvature* of S in M is a function

$$\mathbf{H} : \mathfrak{X}_M \to \mathrm{End}\,(\mathfrak{X}_M)$$

defined by

$$\mathbf{H}: X \to \mathbf{H}_X = \nabla_{\tau X}\tau_0^1 .$$

As $\mathbf{H}$ is thus $\mathfrak{F}$-linear, it defines a tensor by $H(\theta, X, Y) = \theta\mathbf{H}_X Y$, which is essentially the *curvature tensor of Schouten and Struik* of the classical theory ([11], p. 256). The *rigging normal curvature* is a function

$$\mathbf{K} : \mathfrak{X}_M \to \text{End}\ (\mathfrak{X}_M)$$

defined by

$$\mathbf{K}: X \to \mathbf{K}_X = \nabla_{\eta X}\tau_0^1 .$$

Theorem 8.2. *The rigging curvatures have the properties.*

(a) $$\nabla_X \tau_0^1 = \mathbf{H}_X + \mathbf{K}_X ,$$

(b) $$\mathbf{H}_X + \mathbf{K}_X : \eta\mathfrak{X}_M \to \tau\mathfrak{X}_M ,$$

and

(c) $$\mathbf{H}_X + \mathbf{K}_X : \tau\mathfrak{X}_M \to \eta\mathfrak{X}_M .$$

Proof. The identity (a) is obvious because of the definitions, 8.1. Note that

$$\nabla_Y(\tau X) = \tau(\nabla_Y X) + \mathbf{H}_Y(X) + \mathbf{K}_Y(X) \tag{29}$$

from (a). If X is normal, equation (29) becomes

$$\mathbf{H}_Y(X) + \mathbf{K}_Y(X) = -\tau(\nabla_Y X), \tag{30}$$

which shows that $\mathbf{H}_Y + \mathbf{K}_Y$ maps every normal vector into a tangential one proving (b). Further if X is tangential, equation (29) reads

$$\mathbf{H}_Y(X) + \mathbf{K}_Y(X) = \eta(\nabla_Y X) \tag{31}$$

which proves (c).

One sees from (31) that, if Y is tangential, then

$$\mathbf{H}_Y X = \eta(\nabla_Y X), \tag{32}$$

as $\mathbf{K}_Y$ will be zero. Thus if A, B are vectors on the rigged hypersurface S, and ϵ is the given extension, we have

$$\nabla_{\epsilon A}\epsilon B = \tau\nabla_{\epsilon A}\epsilon B + \mathbf{H}_{\epsilon A}\epsilon B \tag{33}$$

because ϵA and ϵB are tangential vectors on M. This equation carries the geometric significance of the rigging curvature $\mathbf{H}$.

We have introduced above the induced connection $'\nabla = \pi \circ \nabla \circ \epsilon$. We shall now evaluate the second derivative $('\nabla)^2$. We remark to begin with that, for any vectors X of $\mathfrak{X}_m$ and A of $\mathfrak{X}_s$,

$$\pi\nabla_{\epsilon A}(\epsilon \circ \pi X) = \pi\nabla_{\epsilon A}(\tau X). \tag{34}$$

This is to be anticipated, as the vectors $\epsilon \circ \pi X$ and τX agree on S. A proof is easily given by evaluating both sides in a conforming coordinate chart, and we will omit it. Now consider three vectors A, B and C of $\mathfrak{X}_s$. From the definition of the induced connection we have

$$'\nabla_A\ '\nabla_B C = \pi\{\nabla_{\epsilon A}(\epsilon \circ \pi\nabla_{\epsilon B}\epsilon C)\}. \tag{35}$$

We invoke the identity (34), with $\nabla_{\epsilon B}\epsilon C$ replacing X, and expand, obtaining

$$(36) \qquad '\nabla_A \, '\nabla_B C = \pi\{\nabla_{\epsilon A}\nabla_{\epsilon B}\epsilon C + \mathbf{H}_{\epsilon A}(\nabla_{\epsilon B}\epsilon C)\}.$$

In the second term on the right hand side, the vector $\nabla_{\epsilon B}\epsilon C$ may be replaced by its normal part, for the tangential part is switched to a normal vector by $\mathbf{H}_{\epsilon A}$ according to 8.2(c), which is then mapped to the zero of $\mathfrak{X}_*$ by π. But ϵC is tangential, so (32) provides

$$(37) \qquad \eta\nabla_{\epsilon B}\epsilon C = \mathbf{H}_{\epsilon B}\epsilon C.$$

Thus we have for the second derivative

$$(38) \qquad '\nabla_A \, '\nabla_B C = \pi\{\nabla_{\epsilon A}\nabla_{\epsilon B}\epsilon C - \mathbf{H}_{\epsilon A}\mathbf{H}_{\epsilon B}\epsilon C\},$$

or equivalently

$$(39) \qquad '\nabla^2 = \pi \circ (\nabla^2 - \mathbf{H}^2) \circ \epsilon,$$

which is the analogue of $'\nabla = \pi \circ \nabla \circ \epsilon$.

Substitution of the second derivative formula (38) into the definition of the curvature tensor $'R$ of the hypersurface, which is determined by the induced connection, proves the following theorem.

Theorem 8.3. *The* GAUSS *equation holds for the rigged hypersurface S,*

$$'R = \pi R + \pi\Omega,$$

where the tensor Ω is defined by

$$\Omega(\theta, X, Y, Z) = \theta\{\mathbf{H}_X\mathbf{H}_Y Z - \mathbf{H}_Y\mathbf{H}_X Z\}.$$

This theorem gains strength by the remark, which we shall not prove, that $\pi\Omega$ is independent of the rigging.

We have already remarked that $\mathbf{R}(X, Y)$ is a $\otimes$-derivation, and maps the KRONECKER delta to zero. Suppose Z is a tangential vector, so $\eta Z = 0$. Then evidently

$$(40) \qquad \mathbf{R}(X, Y)(\eta Z) = \mathbf{R}(X, Y)\eta(Z) + \eta\{\mathbf{R}(X, Y)Z\} = 0.$$

But $\eta = \delta - \tau$, so we may write (40) as

$$(41) \qquad \eta\{\mathbf{R}(X, Y)Z\} = \mathbf{R}(X, Y)\tau(Z).$$

Similarly, if Z is normal, $\tau Z = 0$, and we obtain

$$(42) \qquad \tau\{\mathbf{R}(X, Y)Z) = -\mathbf{R}(X, Y)\tau(Z).$$

Moreover, $\mathbf{R}(X, Y)\tau$ is readily evaluated by means of 8.2(a), and thus (41) and (42) imply the following.

Theorem 8.4. *The* CODAZZI *equations hold for the rigged hypersurface S, that is, for all X, Y, in $\mathfrak{X}_m$,*

(a) $\eta \circ \mathbf{R}(X, Y) = \nabla_X \mathbf{H}_Y - \nabla_Y \mathbf{H}_X + \nabla_X \mathbf{K}_Y - \nabla_Y \mathbf{K}_X - \mathbf{H}_{[X,Y]} - \mathbf{K}_{[X,Y]}$,

on $\tau\mathfrak{X}_m$, and

(b) $\tau \circ \mathbf{R}(X, Y) = -\nabla_X \mathbf{H}_Y + \nabla_Y \mathbf{H}_X - \nabla_X \mathbf{K}_Y + \nabla_Y \mathbf{K}_X + \mathbf{H}_{[X,Y]} + \mathbf{K}_{[X,Y]}$

on $\eta\mathfrak{X}_m$.

One sees that the CODAZZI equations are identities in M, whereas the GAUSS equation relates objects on S only. We shall now remedy this difficulty by extending the GAUSS equation.

Definition 8.5. Suppose $S = i(\Sigma)$ is a rigged hypersurface with normal congruence (j, Σ'). We define the *intrinsic curvature of S and its parallel hypersurfaces*, $R' \in \mathfrak{I}_m(1, 3)$, as follows. Suppose $p \in S' = j(\Sigma')$. Then p has a neighborhood U sufficiently small so that $j^{-1}(U)$ is contained in a product neighborhood $V \times I$ of Σ', for some neighborhood V of $\lambda(j^{-1}(p))$ in Σ. Let Σ_p signify the cross-section of $V \times I$ containing $j^{-1}(p)$, and $j_p = j \mid \Sigma_p$. Then $S_p = j_p(\Sigma_p)$ is a hypersurface of M containing p, and $(j, V \times I)$ is a rigging, that is S_p is rigged by the normal congruence already defined for S. One says that S_p is the local hypersurface through p *parallel to* S. We denote the projection onto S_p of this common rigging by π_p. The geometrical understanding of π_p stems from the following. If for any vector X of M we assign the vector $\pi_p X$ to points p of S' and the zero vector elsewhere, the vector of M which is obtained is just τX. We assume M has a connection, ∇. Then each parallel hypersurface S_p has an induced connection $'\nabla_p$, and hence an intrinsic curvature tensor $'R_p$. Now in effect, we define the tensor R' by assigning $'R_p$ to each $p \in S'$, zero elsewhere. To be precise, we let

$$R'(\theta, X, Y, Z)(p) = \begin{cases} 'R_p(\pi_p\theta, \pi_p X, \pi_p Y, \pi_p \theta)(p), & \text{for } p \in S', \\ 0, & \text{for } p \in M - S'. \end{cases}$$

One readily sees that the definition is coherent, that is independent of the neighborhoods U and V chosen at each point, so R' is a tensor of $\mathfrak{I}_m(1, 3)$, with acceptable discontinuities at the boundary of S'.

In terms of the intrinsic curvature, it is immediately seen that 8.3 takes the following form.

Corollary 8.6. *The* GAUSS *equation holds for S and its parallel hypersurfaces,*

$$\tau R = R' - \tau\Omega.$$

We may consider that the GAUSS and CODAZZI equations, 8.4 and 8.6, give a partial decomposition of the tensor R into its mixed normal and tangential components: $\tau^4 R$, $\tau\eta\tau^2 R$, *etc.*, expressed in terms of the intrinsic curvature R', the SCHOUTEN-STRUIK tensor H, and K and the first covariant derivatives of the latter. In fact, 8.4 and 8.6, together with the symmetry identities 4.8, provide twelve of the sixteen possible components. In the case that M is a pseudo-Riemannian manifold, all sixteen components may be obtained from 8.4(a) and 8.6, together with the symmetry identities 6.2. We turn now to the pseudo-Riemannian case.

9. Pseudo-Riemannian hypersurfaces. Suppose S is a rigged hypersurface in a pseudo-Riemannian space $(M, \mathbf{g}, \nabla)$. Then in addition to the induced connection $'\nabla = \pi \circ \nabla \circ \epsilon$ there is an induced metric on S, $'\mathbf{g} = \pi\mathbf{g}$, which may be pseudo-Riemannian or not. If it is, it induces a unique compatible connection, say $''\nabla$, and then $(S, '\mathbf{g}, ''\nabla)$ is a pseudo-Riemannian space. The important hypersurfaces of relativity theory, for example, are of this sort, and it is convenient that in this case the rigging may be chosen such that $'\nabla$ and $''\nabla$ are identical.

Definition 9.1. If S is rigged in $(M, \mathbf{g}, \nabla)$ and $\pi\mathbf{g}$ is pseudo-Riemannian, that is non-degenerate, then S is a *pseudo-Riemannian hypersurface*. If the rigging lines of S are orthogonal to S with respect to $\mathbf{g}$, the rigging is said to be *normal*. The conforming coordinates of a normal geodesic rigging are *normal geodesic* or *Gaussian coordinates*.

Theorem 9.2. *Every pseudo-Riemannian hypersurface may be given a unique normal geodesic rigging. This rigging has the properties:*

(a) *the Gaussian coordinate system is of class* C^2,
(b) *for all* $X, Y \in \mathfrak{X}_m$, $\mathbf{g}(\tau X, \eta Y) = 0$,
(c) $'\nabla = ''\nabla$,
(d) $\mathbf{K} = 0$.

Proof. (a) If $\pi\mathbf{g}$ is nondegenerate, then no null direction of $\mathbf{g}$ is tangent to S, so $\mathbf{g}$ may be used to construct the normal line element field on S in the proof of 7.5. Thus S has a unique normal geodesic rigging, and the property (a) follows directly from 7.5.

(b) Look at a local conforming patch (U, x) of the Gaussian system, where

$$x(p) = (x^0(p), \cdots, x^{n-1}(p)),$$

and $S \cap U$ is the slice $x^0 = 0$. Let $\alpha = 1, \cdots, n-1$. Then in U it is easily verified that if

$$X = X^i\,\partial_i\,, \qquad Y = Y^i\,\partial_i\,, \qquad i = 0, \cdots, n-1,$$

then $\tau X = X^\alpha\,\partial_\alpha$ and $\eta Y = Y^0\,\partial_0$, so $\mathbf{g}(\tau X, \eta Y) = X^\alpha Y^0 g_{\alpha 0}$, where $\mathbf{g} = g_{ij}\,dx^i \otimes dx^j$. But in local Gaussian coordinates $g_{\alpha 0} = 0$ is a classical theorem, expressing the fact that the rigging lines (coordinate lines of x^0) are the orthogonal trajectories of the "geodesically parallel" hypersurfaces defined by x^0 constant ([13], p. 69).

(c) As $''\nabla$ is the unique compatible connection of $'\mathbf{g}$, it will be sufficient to show that $'\nabla$ and $'\mathbf{g}$ are compatible. It is easily seen that $'T = \pi T = 0$, so we need only show that $'\nabla'\mathbf{g} = 0$. Note by equation (34) that

$$'\nabla'\mathbf{g} = \pi \circ \nabla(\epsilon \circ \pi\mathbf{g}) = \pi \circ \nabla(\tau_2^0\mathbf{g}) = \pi\{\nabla\tau_2^0(\mathbf{g})\}, \tag{43}$$

the last equality being due to the fact that ∇ and $\mathbf{g}$ are compatible, so $\nabla\mathbf{g} = 0$. But a direct computation shows that

$$\nabla_Z\tau_2^0(\mathbf{g})(X, Y) = \mathbf{g}(\mathbf{H}_Z X, Y) + \mathbf{g}(X, \mathbf{H}_Z Y). \tag{44}$$

By definition,

$$'\nabla'\mathbf{g}(A, B, C) = \pi_0^0\{'\nabla_{\epsilon A}'\mathbf{g}(\epsilon B, \epsilon C)\},$$

so from (43) and (44) we have

$$'\nabla'\mathbf{g}(A, B, C) = \pi_0^0\{\mathbf{g}(\mathbf{H}_{\epsilon A}\epsilon B, \epsilon C) + \mathbf{g}(\epsilon B, \mathbf{H}_{\epsilon A}\epsilon C)\}. \tag{45}$$

As $\mathbf{H}_{\epsilon A}\epsilon B$ is normal by 8.2(c), the right side of (45) vanishes by (b) above, and $'\nabla'\mathbf{g} = 0$, which proves (c).

(d) Let U be a tubular neighborhood of the thickened hypersurface in which the normal line element field of S is orientable, and let N be a unit vector of M tangent to the rigging lines in U. Then clearly N is geodesic, that is $\nabla_N N = 0$. Further, if X is a vector on M, we have $\eta X = fN$ in U, so $\eta_0^1 = \sigma^b N \otimes N$ in U, where $\sigma = \mathbf{g}(N, N) = \pm 1$, the orientation indicator of S. Thus

$$K_X = -\sigma f\nabla_N(^b N \otimes N) = 0,$$

which proves (d).

In the event that the tubular neighborhood U in the proof above may be taken to be the entire thickening of S, then we would have $\tau_0^1 = \delta - \sigma^b N \otimes N$ globally, and the Gauss and Codazzi equations might be put in a more familiar form.

Definition 9.3. We say S is *pseudo-orientable* if its normal bundle in M is trivial, that is if it can be thickened by $S \times I$. If S is pseudo-orientable and pseudo-Riemannian, let N indicate a specific unit normal vector, tangent to the rigging lines throughout the thickening. For such a hypersurface we define the *second fundamental form of S and its parallel hypersurfaces*, $\mathbf{h}$, by

$$\mathbf{h}(X, Y) = -^b N(\mathbf{H}_X Y),$$

or equivalently,

$$\mathbf{h}(X, Y) = -H(^b N, X, Y),$$

where H is the Schouten-Struik tensor.

Similarly, define the *second normal form of S and its parallel hypersurfaces*, $\mathbf{k}$, by

$$\mathbf{k}(X, Y) = -^b N(\mathbf{K}_X Y).$$

This form vanishes for a geodesic rigging (9.2d).

In terms of $\mathbf{h}$ and $\mathbf{k}$ the equations of section 8 take a more familiar form.

Theorem 9.4. *Let S be a pseudo-orientable pseudo-Riemannian hypersurface with a normal rigging in $(M, \mathbf{g}, \nabla)$, a pseudo-Riemannian space. Let N be a specific unit normal vector, with $\sigma = \mathbf{g}(N, N)$. Then:*

(a) $$\tau_0^1 = \delta - \sigma^b N \otimes N,$$

(b) $$\mathbf{h}_L X = \nabla_{\tau X}{}^b N,$$

(c) $$\mathbf{h} = \nabla^b N - \sigma^b N \otimes \nabla_N{}^b N,$$

(d) $$\mathbf{H}_X = -\sigma\{\mathbf{h}_L X \otimes N + {}^b N \otimes {}^{\#}\mathbf{h}_L X\},$$

(e) $$\mathbf{k} = \sigma^b N \otimes \nabla_N{}^b N,$$

(f) $$\mathbf{K}_X = \sigma\{\mathbf{k}_L X \otimes N + {}^b N \otimes {}^{\#}(\mathbf{k}_L X)\},$$

(g) $$\mathbf{k}_L X = \nabla_{\eta X}{}^b N.$$

Proof. (a) Suppose X is a normal vector, then $X = fN$ for some function f, and ${}^b N(X) = f\sigma$, so

$$X = \sigma^b N(X)N = \sigma N \otimes N(X).$$

If X is any vector, ηX is normal, so

$$\eta X = \sigma^b N \otimes N(\eta X) = \sigma^b N \otimes N(X)$$

so $\eta = \sigma^b N \otimes N$. As $\tau = \delta - \eta$, this proves (a).

(b) By definition, $\mathbf{H}_X = \nabla_{\tau x}\tau_0^1$. Differentiating (a), we have

(46) $$\mathbf{H}_X = -\sigma\{\nabla_{\tau x}{}^b N \otimes N + {}^b N \otimes \nabla_{\tau x} N\}.$$

As $\mathbf{g}(N, N) = 0$, we see that $\nabla_Y {}^b N(N) + {}^b N(\nabla_Y N) = 0$, or

(47) $${}^b N(\nabla_Y N) = 0$$

for all vectors Y. Substituting $\mathbf{H}_X$ from (46) into the definition of $\mathbf{h}$ (9.3), and simplifying by means of (47), we obtain

$$\mathbf{h}(X, Y) = \nabla_{\tau x} N(Y),$$

which proves (b).

(c) As $\tau X = X - \sigma^b N \otimes N(X)$, and ∇ is $\mathfrak{F}$-linear in the lower argument, have from (47)

(48) $$\mathbf{h}(X, Y) = \{\nabla_X{}^b N - \sigma^b N(X)\nabla_N{}^b N\}(Y)$$

from which (c) follows. Note that this expression for the second fundamental form agrees with the classical one ([14], p. 188).

(d) Substitution of (b) in (45) provides (d).

(e) From (a), $\mathbf{K}_X = {}^b N(X)\nabla_N({}^b N \otimes N)$, or

(49) $$\mathbf{K}_X = {}^b N(X)\{\nabla_N{}^b N \otimes N + {}^b N \otimes \nabla_N N\}.$$

Substituting (49) in the definition of $\mathbf{k}$ (9.3) and simplifing with (46), we obtain

(50) $$\mathbf{k}(X, Y) = \sigma^b N \otimes \nabla_N{}^b N(X, Y),$$

which proves (e). Note $\nabla^b N = \mathbf{h} + \mathbf{k}$.

(f) Note from (e) that $\mathbf{k}_L X = \sigma^b N(X) \nabla_N{}^b N$, so (49) may be written

(51) $$\mathbf{K}_X = \sigma\{\mathbf{k}_L X \otimes N + {}^b N \otimes (\mathbf{k}_L X)\},$$

proving (f).

(g) This follows at once from (f).

From these identities follow the classical properties of the second fundamental form.

Corollary 9.5. *The second fundamental form of a pseudo-orientable pseudo-Riemannian hypersurface S and its parallel hypersurfaces is tangential and symmetric, that is*
(a) $\tau\mathbf{h} = \mathbf{h}$ *or equivalently,* $\mathbf{h}(X, Y) = \mathbf{h}(\tau X, \tau Y)$, *and*
(b) $\mathbf{h}(X, Y) = \mathbf{h}(Y, X)$.

We omit the proof, which utilizes the facts that N is a unit vector, ∇ is torsion free, and the bracket product of two tangential vectors is tangential.

We shall now transcribe the Gauss equation, 8.6, and the Codazzi equation, 8.4(a), to their classical forms in terms of the covariant curvature, induced curvature, and second fundamental forms, in the case of a pseudo-orientable pseudo-Riemannian hypersurface.

First, we shall evaluate the tensor $\tau\Omega$ appearing in the Gauss equation. Suppose V, X, Y and Z are tangential vectors on M. From 9.4(d) we have

(52) $$\mathbf{H}_X \mathbf{H}_Y Z = \sigma \mathbf{h}(Y, Z)^{*}(\mathbf{h}_L X).$$

Substituting (52) into Ω of 8.3, we have

(53) $$\Omega({}^b V, X, Y, Z) = \sigma\{\mathbf{h}(Y, Z)\mathbf{h}(X, V) - \mathbf{h}(X, Z)\mathbf{h}(Y, V)\}.$$

Thus, if we define a tangential tensor Ω' by

(54) $$\Omega'(V, X, Y, Z) = \mathbf{h}(V, X)\mathbf{h}(Y, Z) - \mathbf{h}(V, Y)\mathbf{h}(X, Z),$$

we see from 9.5 and (53) that $\tau^b\Omega = \sigma\Omega'$, so the Gauss equation, 8.6, may be written

(55) $$\tau^b R = {}^b R' - \sigma\Omega',$$

which agrees with the classical formula.

Now note that for arbitrary vectors V, X, Y, Z we have for $\eta_1^0 \otimes \tau_3^0({}^b R)$, which we write as $\eta\tau^{3b}R$ for simplicity,

(56) $$\eta\tau^{3b}R(V, X, Y, Z) = {}^b V\{\eta\mathbf{R}(\tau X, \tau Y)\tau Z\}.$$

Substituting 8.4(a) in the right side of (56), and expressing the rigging curvature in terms of the second fundamental form according to 9.4(d), we obtain

$$(57)\qquad \eta\tau^{3b}R(V, X, Y, Z) = -\sigma^{b}N(V)\{\nabla_{\tau X}\mathbf{h}(\tau Y, \tau Z) - \nabla_{\tau Y}\mathbf{h}(\tau X, \tau Z)\}.$$

Thus if we define a tensor ω by

$$(58)\qquad \omega(X, Y, Z) = \nabla_X\mathbf{h}(Y, Z) - \nabla_Y\mathbf{h}(X, Z),$$

then (57) becomes

$$(59)\qquad \eta\tau^{3b}R = -\sigma^{b}N \otimes \tau\omega,$$

which agrees with the classical CODAZZI equations for a hypersurface.

Finally, we consider $\eta^2\tau^{2b}R$, for which we have by definition

$$(60)\qquad \eta^2\tau^{2b}R(V, X, Y, Z) = {}^{b}V\{\eta R(\tau X, \tau Y)\tau Z\}.$$

Rather than express the right hand side in terms of the CODAZZI equation 8.4(a) we shall shorten the computation by using the equivalent equation (41). Substituting 9.4(a) in equation (41), we have

$$(61)\qquad \eta\{\mathbf{R}(\eta X, \tau X)\tau Z\} = -\sigma\{\mathbf{R}(\eta X, \tau Y)^{b}N(\tau Z)\}N,$$

so (60) may be written

$$(62)\qquad \eta^2\tau^{2b}R(V, X, Y, Z) = -{}^{b}N(V)^{b}N(X)\mathbf{R}(N, \tau Y)^{b}N(\tau Z).$$

Supposing A and B are arbitrary tangential vectors, we shall compute $\mathbf{R}(N, A)^{b}N(B)$ directly;

$$(63)\qquad \mathbf{R}(N, A)^{b}N(B) = \nabla_N\nabla_A{}^{b}N(B) - \nabla_A\nabla_N{}^{b}N(B) - \nabla_{[N,A]}{}^{b}N(B).$$

According to 9.4(b) and (g) we replace $\nabla_A\ {}^{b}N$ by $\mathbf{h}_L A$, *etc.*, obtaining after some simplification (note $\nabla_N A - [N, A] = \nabla_A N$, *etc.*)

$$(64)\quad \mathbf{R}(N, A)^{b}N(B) = \nabla_N\mathbf{h}(A, B) - \nabla_A\mathbf{k}(N, B) + \mathbf{h}({}^{\#}\mathbf{h}A, B) - \mathbf{k}(\nabla_N A, B).$$

But ${}^{b}N(A) = 0$, so by 9.4(e) we have

$$(65)\qquad \mathbf{k}(\nabla_N A, B) = \sigma^{b}N(\nabla_N A)\nabla_N{}^{b}N(B) = -\sigma\nabla_N{}^{b}N(A)\nabla_N{}^{b}N(B)$$
$$= -\sigma\mathbf{k}(N, A)\mathbf{k}(N, B).$$

Substituting (65) in the last term on the right side of (64), and the result, with $A = \tau Y$, $B = \tau Z$ in (62), we finally obtain

$$(66)\qquad \eta^2\tau^{2b}R(V, X, Y, Z) = -{}^{b}N(V)^{b}N(X)\{\nabla_N\mathbf{h}(\tau Y, \tau Z) - \nabla_{\tau Y}\mathbf{k}(N, \tau Z)$$
$$+ \mathbf{h}({}^{\#}\mathbf{h}_L\tau Y, \tau Z) + \sigma\mathbf{k}(N, \tau Y)\mathbf{k}(N, \tau Z)\}.$$

We summarize these results as follows.

Theorem 9.6. *Let S be a rigged pseudo-orientable pseudo-Riemannian hypersurface with unit normal vector N. Then*

$$(a)\qquad \tau^{4b}R = {}^{b}R' - \sigma\Omega',$$

where $\Omega'(V, X, Y, Z) = \mathbf{h}(V, X)\mathbf{h}(Y, Z) - \mathbf{h}(V, Y)\mathbf{h}(X, Z)$,

(b) $$\eta\tau^{3b}R = -\sigma^{b}N \otimes \tau\omega,$$

where $\omega(V, X, Y) = \nabla_V\mathbf{h}(X, Y) - \nabla_X\mathbf{h}(V, Y)$, *and*

(c) $$\eta^2\tau^{2b}R = -{}^bN \otimes {}^bN \otimes \tau\gamma,$$

where

$$\gamma(X, Y) = \nabla_N\mathbf{h}(X, Y) - \nabla_X\mathbf{k}(N, Y) + \mathbf{h}\cdot\mathbf{h}(X, Y) + \sigma\mathbf{k}_LN \otimes \mathbf{k}_LN(X, Y)$$

and $\mathbf{h}\cdot\mathbf{h} = C_{13}\mathbf{h} \otimes \mathbf{h}$. *Moreover, these identities constitute a complete decomposition of the covariant curvature tensor in terms of tangential, normal, and mixed components.*

The last statement is a consequence of the symmetry identities 6.2, from which one sees that any of the sixteen possible components, $\tau\eta\tau\eta^bR$, *etc.*, are either zero or obtainable from one of the components of 9.6. Note that (a) and (b) are the classical GAUSS and CODAZZI equations of the pseudo-Riemannian case.

Because of the definitions of the RICCI tensor, scalar curvature, and EINSTEIN tensor in terms of the curvature tensor (6.1), the decomposition of the RIEMANN tensor (9.6) with respect to a rigged hypersurface enjoins decompositions of these associated tensors. For simplicity we shall express these decompositions only for a hypersurface with normal geodesic rigging.

Definition 9.7. Let S be a rigged hypersurface in a pseudo-Riemanian space. The *intrinsic Ricci tensor of S and its parallel hypersurfaces*, $\mathbf{r}'$, is defined as follows:

$$\mathbf{r}'(X, Y)(p) = \mathbf{r}'_p(\pi_pX, \pi_pY)(p),$$

where $\mathbf{r}'_p$ is the RICCI tensor of (S_p, π_pg), considered as a pseudo-Riemannian space (see 8.5). Similarly, we may define the *intrinsic scalar curvature*, r', and the *intrinsic Einstein tensor*, $\mathbf{G}'$.

Theorem 9.8. *If S is a pseudo-orientable hypersurface with normal geodesic rigging, a complete decomposition of the* RICCI *tensor is:*

(a) $$\eta^2\mathbf{r} = -(N\mathbf{h} + \mathbf{h} : \mathbf{h})^bN \otimes {}^bN,$$

(b) $$\eta\tau\mathbf{r} = {}^bN \otimes \tau(\delta\mathbf{h} - dh),$$

(c) $$\tau^2\mathbf{r} = \mathbf{r}' - \sigma(\nabla_N\mathbf{h} - h\mathbf{h}),$$

where $\mathbf{a} : \mathbf{b} = C^{12} \circ C^{24}\, \mathbf{a} \otimes \mathbf{b}$, *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are arbitrary covariant tensors of order 2, and* $h = \mathbf{g} : \mathbf{h}$.

The scalar curvature and intrinsic scalar curvature are related by

(d) $$r = r' - \sigma(2Nh + \mathbf{h} : \mathbf{h} - h^2).$$

A complete decomposition for the EINSTEIN *tensor is*

(e) $$\eta^2\mathbf{G} = -\tfrac{1}{2}\{\mathbf{h} : \mathbf{h} + h^2 + \sigma(r' + \lambda)\}{}^bN \otimes {}^bN,$$

(f) $$\eta\tau\mathbf{G} = {}^bN \otimes \tau(\boldsymbol{\delta}\mathbf{h} - dh),$$

(g) $$\tau^2\mathbf{G} = \mathbf{G}' - \sigma\{\nabla_N\mathbf{h} - Nh\mathbf{g} - h\mathbf{h} + (h^2 - \mathbf{h} : \mathbf{h})\mathbf{g}\}.$$

Proof. As the definitions of the tensors involved in the identities above contain several contractions, it is easiest to perform the computations on the components of the tensors in the standard kernel-index notation. We choose a conforming coordinate chart of the rigging for this purpose, that is, we carry out the computations in Gaussian coordinates. In these coordinates, we let Latin indices take values $0, 1, \cdots, n-1$, and Greek indices take values $1, \cdots, n-1$, where x^0 is the arc length along the normal geodesics. We choose for the unit normal vector $N = \partial/\partial x^0$. From 9.2(b), we have $g_{\alpha 0} = 0$, $g^{0\alpha} = 0$, $g_{00} = \sigma$, $g^{00} = \sigma$. Clearly ${}^bN = \sigma\, dx^0$. From 9.5(a) we have $h_{0i} = h_{i0} = 0$. It is easy to see from the definitions that

$$r'_{0i} = 0, \qquad r'_{\alpha\beta} = g^{\gamma\delta}R'_{\gamma\delta\alpha\beta}, \qquad r' = g^{\alpha\beta}r'_{\alpha\beta}, \qquad G'_{0i} = 0,$$

and

$$G'_{\alpha\beta} = R'_{\alpha\beta} - \tfrac{1}{2}(r' + \lambda)g_{\alpha\beta}.$$

Also we see that $h = g^{\alpha\beta}h_{\alpha\beta}$, $\mathbf{h} : \mathbf{h} = h^{\alpha\beta}h_{\alpha\beta}$, and

(67) $$r_{jk} = \sigma R_{00jk} + g^{\alpha\beta}R_{\alpha\beta jk}.$$

Moreover, in Gaussian coordinates the identities of 9.6 take the form

(68) $$R_{\alpha\beta\gamma\delta} = R'_{\alpha\beta\gamma\delta} - \sigma(h_{\alpha\beta}h_{\gamma\delta} - h_{\alpha\gamma}h_{\beta\delta}),$$

(69) $$R_{0\beta\gamma\delta} = -\nabla_\beta h_{\gamma\delta} + \nabla_\gamma h_{\beta\delta},$$

(70) $$R_{00\gamma\delta} = -\nabla_0 h_{\gamma\delta} - h_{\beta\gamma}h^\beta_\delta,$$

(71) $$R_{000j} = 0.$$

From (70) we see that $g^{\alpha\beta}R_{\alpha\beta00} = -\nabla_0 h - \mathbf{h} : \mathbf{h}$, so taking $j = k = 0$ in (67) and noting (71), we find

(72) $$R_{00} = -\nabla_0 h - \mathbf{h} : \mathbf{h}.$$

Similar computations provide the other components

(73) $$R_{0\beta} = -\nabla_\beta h + \nabla_\gamma h^\gamma_\beta,$$

(74) $$R_{\gamma\delta} = R'_{\gamma\delta} - \sigma\{\nabla_0 h_{\gamma\delta} - hh_{\gamma\delta}\},$$

(75) $$r = r' - \sigma\{2\nabla_0 h + \mathbf{h} : \mathbf{h} - h^2\}.$$

Combining these, we obtain

(76) $$G_{00} = -\tfrac{1}{2}\{h : h + h^2 + \sigma(r' + \lambda)\},$$

(77) $$G_{0\beta} = -\nabla_\beta h + \nabla_\gamma h^\gamma_\beta,$$

(78) $$G_{\gamma\delta} = G'_{\gamma\delta} - \sigma\{\nabla_0 h_{\gamma\delta} - (\nabla_0 h)g_{\gamma\delta} - hh_{\gamma\delta} + (h^2 - \mathbf{h} : \mathbf{h})g_{\gamma\delta}\}.$$

The equations (72) to (78) are the component equivalents of the identities (a) to (g), which establishes the identities. The completeness of the decompositions follows from the symmetry of the RICCI and EINSTEIN tensors, 6.3. For example,

$$\tau\eta\mathbf{r}(X, Y) = \mathbf{r}(\tau X, \eta Y) = \mathbf{r}(\eta Y, \tau X) = \eta\tau\mathbf{r}(Y, X), \tag{79}$$

and thus $\tau\eta\mathbf{r}$ is expressed in terms of $\eta\tau\mathbf{r}$. This completes the proof.

The identities of 9.6 and 9.8 express all of the normal, tangential and mixed components of the tensors ${}^bR$, $\mathbf{r}$ and $\mathbf{G}$ in terms of the associated intrinsic tensors and the second fundamental form of S and its parallel hypersurfaces.

10. Discontinuity hypersurfaces. We have pointed out that the ordinary restriction π_0^0 for functions applies only to those functions which are continuous across the hypersurface. In this section we extend the notion of rigging to functions and tensors with discontinuities.

Definition 10.1. A pseudo-orientable hypersurface has two "sides" in the ordinary sense, let us label them "+" and "−". More precisely, any thickening S' is divided into two disjoint components by S, $S' - S = S^+ \cup S^-$. If S belongs to the discontinuity set of a function f, then the hypotheses of 2.5 ensure that f is C^∞ on the half spaces $S^+ \cup S$ and $S^- \cup S$. Let $f^\pm$ be the function on S defined by the uniform limit of $f \mid S^\pm$, and $\pi_\pm : f \to f^\pm$. These extend to two complete *one-sided riggings*: (π_+ , ϵ) and $\pi_- , \epsilon)$, if ϵ is an extension, for all functions and tensors, whether continuous across S or not. We denote the *jump of f* by $[f] = f^+ - f^-$, which is a function on S.

Obviously, if a tensor is continuous, π_+ and π_- agree for it, so the ideas of the previous section are preserved. Moreover, discontinuous tensors may be decomposed into normal and tangential components separately in the two half-spaces, so the decomposition theorems apply to pseudo-orientable discontinuity hypersurfaces in this two-sided sense. In the remainder of this section we shall assume that S is pseudo-orientable in M with riggings (π_+ , ϵ) and (π_- , ϵ).

Theorem 10.2. *If X is a continuous tangential vector, then*

$$[Xf] = \pi X[f].$$

Proof. If (U, x^0, x^α), $\alpha = 1, \cdots, n - 1$, is a conforming coordinate chart with $(U \cap S, x^\alpha \mid S)$ a coordinate chart on S, then $X = X^\alpha\, \partial_\alpha$, and it suffices to show

$$(\partial_\alpha f)^+ - (\partial_\alpha f)^- = \partial_\alpha(f^+ - f^-), \qquad \alpha = 1, \cdots, n - 1. \tag{80}$$

But this interchange of limits is permissible because of the uniformity hypothesis 2.1(b).

It appears that if f is continuous, and X as in 10.2, then Xf is continuous. Recall that the differential (or gradient) of f, df, is a covariant vector defined by $df(X) = Xf$. Thus if X is continuous $df(\tau X) = df(X)$ is continuous, so df is continuous. This generalizes a classical result of HADAMARD ([15], p. 83).

Corollary 10.3. *The tangential component of the gradient of a continuous function is continuous across S.*

If M has a continuous connection, this important property of the ordinary differential carries over to the covariant differential.

Theorem 10.4. *If ∇ is continuous, then the tangential component of the covariant differential of a continuous tensor is continuous across S. That is, if X is a continuous tangential vector, and T is a continuous tensor, then $\nabla_X T$ is continuous across S.*

Proof. In conforming coordinates we have, by equation (12), section 4,

$$\nabla_X T = X^a(\partial_a t_{r \cdots s}^{i \cdots j} + \Gamma_{am}^i t_{r \cdots s}^{m \cdots j} + \cdots - \Gamma_{as}^m t_{r \cdots m}^{i \cdots j})\, \partial_i \otimes \cdots \otimes dx^s. \tag{81}$$

Here X^a, $t_{r \cdots s}^{i \cdots j}$, and Γ_{jk}^i are continuous functions by hypothesis, and $\partial_a t_{r \cdots s}^{i \cdots j}$ are continuous by 10.3, so $\nabla_X T$ has continuous canonical components, and is thus a continuous tensor.

Theorem 10.5. *If M has a continuous connection ∇, then the* SCHOUTEN-STRUIK *tensor of S is continuous. If M is a pseudo-Riemannian space and S a pseudo-Riemannian hypersurface, then its second fundamental form is continuous.*

Proof. It is obvious from the definition that τ_0^1 is continuous, that is if X is a continuous vector, so is $\tau_0^1 X$. For a continuous form φ and continuous vectors X, Y, we have

$$H(\varphi, X, Y) = \varphi\{\nabla_{\tau X}\tau(Y)\},$$

which is continuous by 10.4. Thus H is a continuous tensor. From the definition, 9.3, we have

$$\mathbf{h}(X, Y) = -H({}^bN, X, Y)$$

when S is pseudo-Riemannian with unit normal N. As N is continuous, $\mathbf{h}$ is continuous.

Recall that in a pseudo-Riemannian space we assume that the metric is C^1 (6.1), so its connection is C^0 (5.6), and its Riemann tensor may thus have discontinuities (4.7a). Now if a discontinuity hypersurface is pseudo-Riemannian, its second fundamental form $\mathbf{h}$ is continuous, and so also its tangential covariant derivatives $\nabla_{\tau X}h$, according to 10.4. We may now look at the decompositions 9.6 and 9.8 for ${}^bR$, $\mathbf{r}$, and $\mathbf{G}$, which apply separately to each side of S, and identify at once continuous and discontinuous terms. It is obvious that in a pseudo-Riemannian space, the intrinsic quantities ${}^bR'$, $\mathbf{r}'$, $\mathbf{G}'$ are continuous across S. Thus from inspection, we find the following result:

Theorem 10.6. *If S is a pseudo-Riemannian discontinuity hypersurface of the Riemann tensor of a pseudo-Riemannian space, and has a normal geodesic rigging, then the following components are continuous across S: $\tau^b R$, $\eta\tau^{3b}R$, $\eta\tau\mathbf{r}$, $\eta^2\mathbf{G}$, and $\eta\tau\mathbf{G}$.*

The last two generalize the O'BRIEN-SYNGE *continuity condition* of general relativity [1].

11. The space-time of general relativity.

Definition 11.1. By a *space-time* we mean a pseudo-Riemannian space of four dimensions with a LORENTZ metric. Recalling earlier definitions, this consists of the following:

(a) a paracompact 4-manifold, M_4, with piecewise differentiable structures, and piecewise C^∞ tensors, $\mathfrak{J}(p, q)$, which may have uniform bounded jump discontinuities across hypersurfaces,

(b) a (C^1, C^∞) metric, $\mathbf{g} \in \mathfrak{J}(0, 2)$ with signature $(+, -, -, -)$, and symmetry condition: $\mathbf{g}(X, Y) = \mathbf{g}(Y, X)$, and

(c) the unique (C^0, C^∞) linear connection ∇ such that $\nabla \mathbf{g} = 0$ and $\nabla_X Y = \nabla_Y X + [X, Y]$, with its curvature tensor $R \in \mathfrak{J}(1, 3)$, RICCI tensor, $\mathbf{r}$, and EINSTEIN tensor $\mathbf{G}$, which may have discontinuities. A vector which is positively-oriented at a point $p(\mathbf{g}(X, X)(p) > 0)$ is said to be *time-like at* p, a negatively-oriented one is *space-like at* p. A hypersurface S in M_4 is *space-like* (*time-like, null*) at a point $p \in S$ according as a vector N orthogonal to S at p is time-like (space-like, null) at p. The *indicatrix of* S is a map $\sigma : S \to \{1, 0, -1\}$ defined by assigning to p the sign of $\mathbf{g}(N, N)(p)$.

We recall that the components of a C^1 tensor are local C^1 functions in a chart of a C^2 coordinate system, but in an arbitrary chart their first partial derivatives may have artificial discontinuities. Thus the *Schwarzschild continuity hypothesis* is a property of space-time as defined in 11.1 (see [3], p. 61).

One makes a physical model in space-time (at least in the point of view of deductive mathematical science) by postulating some scalars and tensors to which are attached "vulgar names" suggesting their physical interpretation.

Definition 11.2. A *physical scheme* in space-time is a set of named piecewise C^∞ tensors (*physical parameters*) and relating equations. We suppose every scheme to contain a *density function* ρ, a *velocity vector* $\mathbf{u}$, which is unit time-like on the support of $\rho(\rho \neq 0)$ and zero elsewhere, and a tensor T of type $(0, 2)$ called the *energy-momentum tensor*. The simplest scheme, in which $\rho = 0$ everywhere, is called the *scheme of empty space*, for which we take $T = 0$. The next simplest scheme, in which ρ and $\mathbf{u}$ are the only parameters, is the *scheme of pure material*. The trajectories (maximal integral curves) of $\mathbf{u}$ are called *world-lines*, or occasionally, elementary particles. The energy-momentum tensor is

$$T = \rho\, {}^b\mathbf{u} \otimes {}^b\mathbf{u}.$$

The *scheme of perfect fluid* contains a third parameter, p, the *pressure scalar*, and has for energy momentum tensor

$$T = \rho\, {}^b\mathbf{u} \otimes {}^b\mathbf{u} + p({}^b\mathbf{u} \otimes {}^b\mathbf{u} - \mathbf{g}).$$

The trajectories of $\mathbf{u}$ are sometimes called *streamlines*.

Definition 11.3. An *observer* (or laboratory) in space-time is a normal congruence of curves (j, Σ') where $\Sigma' = E_3 \times R$, E_3 being the unit cube centered at the origin in R^3, We further suppose that (1) the density is non-zero in $O = j(\Sigma')$, (2) the sub-manifolds $S_t = j(E_3 \times t)$, called the *spatial sections* of the observer, are space-like, and (3) the curves $j(p \times R)$, $p \in E_3$, are world-lines, and are orthogonal to the spatial sections. It is evident that if (x, y, z) are coordinates in E_3 , then (O, x) is an admissible chart in M_4 , where

$$x = (x \circ j^{-1}, y \circ j^{-1}, z \circ j^{-1}, t \circ j^{-1}).$$

This is called the *rest chart* of the observer O. A *test particle* is obtained from an observer by shrinking E_3 to a point.

Obviously the S_t are parallel hypersurfaces in the sense of definition 8.5, if we consider the rigging induced by the normal congruence of world-lines in a natural way. We denote the tangential projection by $\mathbf{s}$ instead of τ, it is the spatial projection in O. The intrinsic connection of the spatial sections, $\mathbf{F} = \mathbf{s} \circ \nabla$, is called the *Fermi connection* (even though it is not a connection in our sense.) If a vector X satisfies $\mathbf{F}_{\mathbf{u}} X = 0$ in O, then X is *irrotational* with respect to O, or *Fermi propagated* along the world lines. If the measuring base vectors of O, $\partial/\partial x$ *etc.*, are irrotational, then O is an *irrotational observer*.

The justification for this definition of irrotationality is the fact that according to it the axis of a gyroscope is irrotational. It is known that if $\mathbf{u}$ is geodesic $(\nabla_{\mathbf{u}} \mathbf{u} = 0)$. Then $\mathbf{F} = \nabla$ [16].

We have now precise descriptions of most of the elementary notions of general relativity. Another important and useful idea is that of propagation speed. Suppose S is a hypersurface in M_4 which intersects an observer O. The intersection is evidently a hypersurface of O. For some range of t the spatial sections S_t will meet S, and in general the intersection $S_t \cap S$ will be a surface (two dimensional sub-manifold of O). We may interpret S_t as the instantaneous laboratory of O at the time t, and $S_t \cap S$ as the instantaneous "appearance" of S in the laboratory S_t . Then at a later time t' , $S_{t'} \cap S$ "appears" to have moved, if we identify S_t and $S_{t'}$ by the rest coordinates with E_3 . In fact if λ is the projection $E_3 \times R \to E_3$ defined by $\lambda(x, y, z, t) = (x, y, z)$, then $\lambda \circ j^{-1}(S_t \cap S)$ represents a moving surface in E_3 .

Definition 11.4. If (i, Σ) is a hypersurface in M_4 and $O = j(E_3 \times R)$ is an observer meeting $S = i\Sigma$, then *the speed of propagation of S with respect to O*, U_S , is the ordinary 3-speed of the moving surface $\lambda \circ j^{-1}(S_t \cap S)$ in E_3 , measured by the metric of O, that is, $j_*(\mathbf{g})$.

Theorem 11.5. *If N is any non-zero normal vector of the hypersurface $S \cap O$, and $\mathbf{u}$ the velocity vector of an observer O, then the speed of propagation of S with respect to O at a point p in $S \cap O$ is given by*

$$U_S^2 = \frac{\mathbf{g}(\mathbf{u}, N)^2}{\mathbf{g}(\mathbf{u}, N)^2 - \mathbf{g}(N, N)}.$$

Proof. We may take p to be the image of the origin of E_3 at a time $t = t_0$, without loss of generality. Let p' be the image of the origin at $t = t_1 = t_0 + \epsilon$. We choose a point p'' in $S_{t'} \cap S$ such that the plane of $x(p)$, $x(p')$, $x(p'')$ in $E_3 \times R$ contains the image of N at p'', $N(x^i)(p'') = N^i(p'')$. Then the motion $p' \to p''$ may be interpreted as the apparent motion of a point of the section $S_t \cap S$ seen by the observer as he moves from p to p'. These motions are sides of a curvilinear triangle in O, and a rectilinear triangle in $E_3 \times R$. We shall compute the speed of the motion

$$U_S = \lim_{p' \to p} \left| \frac{d(p', p'')}{d(p, p')} \right| \tag{82}$$

at p in $E_3 \times R$, where we may indicate displacements by vectors bound to the origin $(0, 0, 0, t_0)$. We write $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and let Latin indices take values 0, 1, 2 and 3. Then the vertices of the triangle are related by

$$x^i(p') = x^i(p) + \epsilon u^i, \qquad u^i u_i = 1, \tag{83}$$

$$x^i(p'') = x^i(p') + \eta v^i, \qquad v^i v_i = -1, \qquad u^i v_i = 0, \tag{84}$$

$$x^i(p'') = x^i(p) + \mu t^i, \qquad t^i t_i = \pm 1, \ \text{ or } \ 0, \tag{85}$$

and equation (82) becomes

$$U_S = \lim_{p' \to p} \left| \frac{\eta}{\epsilon} \right|, \tag{86}$$

representing the slope of the curve $x^i(p''(t))$ in the (p, p', p'') plane at p.

Let $\bar{N}^i$ be a vector in the plane of (p, p', p'') in $E^3 \times R$ orthogonal to t^i, $\bar{N}^i \bar{N}_i = N^i N_i(p)$, oriented such that in the limit $p'' \to p$, $\bar{N}^i \to N^i$. Obviously we may write

$$\bar{N}^i = L u^i + M v^i. \tag{87}$$

Forming the scalar product of this equation with itself, we find

$$N^i N_i(p) = L^2 - M^2. \tag{88}$$

Comparing equations (83), (84), and (85), we see

$$\mu t^i = \epsilon u^i + \eta v^i. \tag{89}$$

Forming the scalar product of (89) with the corresponding sides of (87), we have

$$\epsilon L - \eta M = 0 \tag{90}$$

as $\bar{N}^i$ and t^i are orthogonal by hypothesis. Solving (90) for η/ϵ, squaring, and replacing M^2 by means of (88), we have

$$\frac{\eta^2}{\epsilon^2} = \frac{L^2}{L^2 - N^i N_i(p)}. \tag{91}$$

Now $L = \bar{N}^i u_i$, so as p' approaches p, L approaches $\mathbf{g}(N, \mathbf{u})(p)$, which completes the proof.

This formula for the propagation speed of a hypersurface with respect to an observer is invariant, that is, independent of the units used by O, and homogeneous of degree zero in N, that is, independent of the length of N. One sees from the formula that the propagation speed falls about one as the indicatrix of S at p falls about zero.

Corollary 11.6. *The prapagation speed of a hypersurface with respect to an observer at a point p in their intersection is greater than, equal to, or less than one according as the hypersurface is space-like, null, or time-like at p, respectively.*

It is known that light waves *in vacuo*, that is the characteristic hypersurfaces of MAXWELL's equations in empty space, are null hypersurfaces ([3], p. 51). Thus the propagation speed of light *in vacuo* is one with respect to every observer. In order that signals may not propagate faster then light in a given scheme, we must provide that the discontinuity hypersurfaces of the physical parameters are never space-like. This suggests the following model for the universe of general relativity.

Definition 11.7. A *model for the relativistic universe* is a physical scheme in space-time in which the discontinuity hypersurfaces of the metric and the physical parameters are nowhere space-like, with a *field equation* $S = T$ where T is the energy-momentum tensor of the scheme and S is a conservative symmetric tensor of type (0, 2) of the space-time, which depends on the metric in a second order quasi-linear manner, that is the components of S in an arbitrary coordinate system are functions of the components of $\mathbf{g}$ and their first and second partial derivatives, and are linear in the latter.

By a well known theorem of CARTAN the hypotheses on S above imply that S is the EINSTEIN tensor except for an arbitrary constant multiplier, so the field equation may be written

$$G = -8\pi kT, \tag{92}$$

where k is a constant, the choice of which constitutes part of the model. We always assume that k is the *universal gravitational constant*, in appropriate units. Then (92) is the EINSTEIN field equation.

The model we have defined in 11.7 is essentially the one used by LICHNEROWICZ and other authors in France ([3], p. 25). It is more general in two important respects, the coordinate transformations and the metric are allowed to be (C^1, C^∞). The model of LICHNEROWICZ is a (C^2, C^4) manifold with (C^2, C^4) LORENTZ metric. We shall show that this generalization preserves an essential property of LICHNEROWICZ' model, *the principle of geodesics*, and admits an additional property, important in the theory of relativity, that *the model may represent* SCHWARZSCHILD's *solar system.*

Theorem 11.8. *A test particle in empty space is a time-like geodesic.*

Proof. Since a test particle is obtained from an observer by shrinking the "laboratory" to a point, we express first the equations of motion for an observer O in empty space. Inside O we have the energy momentum tensor of pure material,

$$T = \rho^{b}u \otimes {}^{b}u, \tag{93}$$

whereas outside O, $T = 0$. Thus across the boundary of O, which is composed of six hypersurfaces $(j, E_2 \times R)$, T has a jump discontinuity. Let S be a "side" of the boundary of O, $S = j(E_2 \times R)$, and N any vector orthogonal to S. By the generalized O'BRIEN-SYNGE continuity condition (10.6), and (92), the vector $T_L(N)$ must be continuous across S, and thus is zero on S, From (93), we see that $T_L(N) = \rho \mathbf{u} L$, where $L = \mathbf{g}(\mathbf{u}, N)$. Assuming that ρ does not go to zero as S is approached from within, we see that $L = 0$ on S, so $\mathbf{u}$ (inside) is tangent to S. We conclude that S is generated by integral curves of $\mathbf{u}$.

We have shown (6.5) that the EINSTEIN tensor is conservative, so by (92), the energy-momentum tensor is conservative, (div $\equiv \boldsymbol{\delta}$)

$$\operatorname{div} T = 0. \tag{94}$$

Taking the divergence of (93), we see (94) may be written in the form

$$\operatorname{div} (\rho^{b}\mathbf{u})^{b}\mathbf{u} + \rho \nabla_{\mathbf{u}} {}^{b}\mathbf{u} = 0. \tag{95}$$

Now $\mathbf{u}$ is a unit time-like vector, so ${}^{b}\mathbf{u}(\mathbf{u}) = 1$. Differentiating this expression, we find

$$\nabla_X {}^{b}\mathbf{u}(\mathbf{u}) + {}^{b}\mathbf{u}(\nabla_X \mathbf{u}) = 0 \tag{96}$$

for any vector X. As $\nabla_X \mathbf{g} = 0$, it is clear that $\nabla_X {}^{b}\mathbf{u}(\mathbf{u}) = {}^{b}\mathbf{u}(\nabla_X \mathbf{u})$, so we have for all X,

$$\nabla_X {}^{b}\mathbf{u}(\mathbf{u}) = 0. \tag{97}$$

Now from (95) and (97), we see that

$$\operatorname{div} T(\mathbf{u}) = \operatorname{div} (\rho^{b}\mathbf{u}) = 0. \tag{98}$$

Hence from (95) we have

$$\operatorname{div} T - \operatorname{div} T(\mathbf{u})^{b}\mathbf{u} = \rho \nabla_{\mathbf{u}} {}^{b}\mathbf{u} = 0. \tag{99}$$

We replace (94) by (98) and (99). The latter are generally called the *equation of conservation* and the *equation of motion*, respectively. It is obvious from (99), as ρ is non-zero in O, that $\mathbf{u}$ is autoparallel, and thus *the world-lines of the observer are geodesics* (6.1). Thus S is generated by geodesics, and shrinking the "laboratory" to a point yields a single geodesic. This geodesic is time-like in the sense that its tangent at each point is a time-like direction, as $\mathbf{u}$ is time-like by hypothesis. This completes the proof, more or less as originally given by LICHNEROWICZ ([3], p. 64).

The main experimental evidence for general relativity concerns the principle of geodesics in SCHWARZSCHILD's model for the solar system. We shall now show

how SCHWARZSCHILD's model arises as a special case of the model for the relativistic universe by the addition of symmetry assumptions.

Definition 11.9. Suppose a manifold M has metric $\mathbf{g}$, and T is a diffeomorphism of M onto a manifold M'. Then T induces a metric Tg on M', defined as follows. If X is any vector on M', define a vector $T^{-1}X$ on M by

$$T^{-1}X(f) = X(f \circ T^{-1}) \circ T.$$

Then define $T\mathbf{g}$ by

$$T\mathbf{g}(X, Y) = \mathbf{g}(T^{-1}X, T^{-1}Y) \circ T^{-1}.$$

If M' has a metric $\mathbf{g}'$ and $T\mathbf{g} = \mathbf{g}'$, then T is said to be *isometric*. An isometric diffeomorphism of $(M, \mathbf{g})$ onto itself is an *isometry* of M. Let $I(M)$ be the set of all isometries of M, H_p the subset of $I(M)$ of transformations leaving p, a point of M, fixed. It is easy to see that $I(M)$ and H_p are groups. They are the *isometry group* and the *isotropy group at p*, respectively. It is well known that they are also manifolds in the case that $\mathbf{g}$ is Riemannian, and that the dimension of $I(M)$ is at most $\frac{1}{2}n(n+1)$, where n is the dimension of M ([17], p. 239). Also, it is not hard to see that the maximal dimension of H_p is $\frac{1}{2}n(n-1)$, the dimension of the orthogonal group O_n. We say that a manifold is *spherically symmetric* if it has a point p for which H_p has the maximal dimension, p is the *center* of the manifold.

Definition 11.10. We say that a space-time V_4 is *static* if (1) it is homeomorphic to the topological product of a (C^∞, C^∞) 3-manifold, V_3, and the real line: $j : V_3 \times R \to V_4$, (2) j is a (C^∞, C^∞) diffeomorphism in the sense of 7.1, (3) the curves $j(p \times R)$, p fixed in V_3, are time-like and orthogonal to the hypersurfaces $j(V_3 \times t)$, t fixed in R. The curves are the *time-lines* of V_4, the hypersurfaces are its *spatial sections*. (Note that this does not imply that V_4 is stationary, that is, the geometry of the spatial sections need not be independent of "time". Also, $V_3 \times R$ may be considered a thickening of V_3, and a space-time with metric $j^{-1}\mathbf{g}$.) Let $\mathbf{g}'$ be the projection of $j^{-1}\mathbf{g}$ into V_3. (Note that $\mathbf{g}'$ is a Riemannian metric.) We say V_4 is a *centre-symmetric space-time* if it is static, $(V_3, \mathbf{g}')$ is spherically symmetric, and the isotropies of V_3 about its center, p_0, induce isometries in V_4. The curve $j(p_0 \times R)$ is the *central axis* of V_4. If S_2 is an invariant submanifold of V_3 under the action of H_{p_0}, $j(S_2 \times R)$ is a *central hypercylinder* of V_4.

Definition 11.11. *Schwarzschild's solar model* is a model for the relativistic universe in which the space-time is centro-symmetric with Eucildean 3-space R^3, or its one-point compactification $R^3 \cup \infty$, for its "space", V_3 (center at the origin), having a scheme of perfect fluid (constant density) in a region interior to a central hypercylinder S empty space elsewhere. Of course S may be called the *surface of the sun*. It is well known that the diffeomorphism $j : V_3 \times R \to V_4$ may be chosen so that the invariant surfaces of V_3 are the ordinary 2-spheres ([18], 19, p. 239). Let r denote the Euclidean radius in R_3, r_0 the radius of the sphere generating the surface of the sun. For SCHWARZSCHILD's model to be real it is necessary to assume

$$r_0^2 < \frac{6}{16\pi\rho - \lambda},$$

where ρ is the density of the sun and λ is the cosmological constant. It has been remarked that this limit on the size of the sun is very generous and leads to no conflict with astronomical observation.

The implication of the hypotheses above is that almost all of the centro-symmetric V_4 may be covered by three coordinate charts with coordinates $(t, r, \vartheta, \varphi)$, t is the natural coordinate in R, and (r, ϑ, φ) are ordinary (Euclidean) spherical polar coordinates in V_3 (only the central axis and the time-line at infinity, if there is one, are excluded), and in these charts the metric takes the form

$$(100) \qquad g = e^{\mu}\, dt \otimes dt - e^{\nu}\, dr \otimes dr - r^2\, d\theta \otimes d\theta - r^2 \sin^2 \theta\, d\varphi \otimes d\varphi,$$

where μ and ν are functions of r and t only ([18], [19], p. 244). This form was deduced from intuition by SCHWARZSCHILD, and led him to the famous solution, which we write here in the notation of TOLMAN ([19], p. 245, note $\lambda = -2\Lambda$).

Theorem 11.12. *In the spherical polar charts of* SCHWARZSCHILD's *solar model, the metric is necessarily (we write dt^2 for $dt \otimes dt$):*

$$g = (A - B\sqrt{1 - r^2/R^2})\, dt^2 - (1 - r^2/R^2)^{-1}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2 \theta\, d\varphi^2$$

for $r \leqq r_0$, (SCHWARZSCHILD's *interior solution), and*

$$g = \left(1 - \frac{2m}{r} + \frac{\lambda r^2}{6}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{\lambda r^2}{6}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2 \theta\, d\varphi^2$$

for $r \geqq r_0$, (SCHWARZSCHILD's *exterior solution), where*

$$R^2 = \left(\frac{8}{3}\pi\rho - \frac{1}{6}\lambda\right)^{-1}, \qquad m = \frac{4\pi}{3}\rho r_0^3,$$

$$A = \frac{6 + \lambda R^2}{4}\sqrt{1 - r_0^2/R^2}, \qquad B = \frac{2 + \lambda R^2}{4}.$$

Let us consider the discontinuities of this metric. It is easy to show that the assumption on the size of the sun we have included in the definition of SCHWARZSCHILD's solar model leads to the inequalities $r < R$ for the interior, and $r > 2m$ for the exterior. Also, the points $r = 0$ and $r = \infty$ lie outside the charts being used. This disposes of the obvious singularities of the SCHWARZSCHILD metric. In fact the first actual discontinuity occurs in the partial derivative $\partial_r g_{rr}$ across S, and is of the simple type allowable in our piecewise C^∞ space-time. It is well known that there are local coordinates in which the SCHWARZSCHILD metric is (C^1, C^∞), so the spherical polar charts can be completed to a C^1 coordinate system for V_4 in which the canonical components of $\mathbf{g}$ are C^1, so $\mathbf{g}$ is a C^1 metric [2].

This discussion shows that our model for the relativistic universe is sufficiently general to admit SCHWARZSCHILD's solar model as a special case.

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