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1. **Introduction.** Let X and Y be differentiable manifolds and $\mathfrak{A}$ a space of mappings from X to Y. A common problem in differential topology is to approximate a mapping in $\mathfrak{A}$ by another in $\mathfrak{A}$ which is transversal to a given submanifold $W \subset Y$. Thus if $\mathfrak{A}_{X,W}$ is the subspace of mappings transversal to W it is important to know if $\mathfrak{A}_{X,W}$ is dense in $\mathfrak{A}$. Some famous examples are the Whitney immersion and embedding theorems [**8**] and the Thom transversality theorem [**4**; **7**]. In the next section we give sufficient conditions for density in case $\mathfrak{A}$ is a Banach manifold. The proof of the density theorem is indicated in the third section, and in the final section the Thom transversality theorem is obtained as a corollary.

2. **Density theorems.** Throughout this section X will be a manifold with boundary, Y and Z manifolds, $W \subset Y$ a submanifold (W, Y, Z without boundary) all of class C^r, $r \geqq 1$, and modelled on Banach spaces (see [**3**] for definitions).

2.1. DEFINITION. A C^r mapping $f: X \to Y$ is *transversal to W at a point* $x \in X$ iff either $f(x) \notin W$, or $f(x) = w \in W$ and there exists a neighborhood U of $x \in X$ and a local chart (V, ψ) at $w \in Y$ such that

$$\psi: V \to \boldsymbol{E} \times \boldsymbol{F}: V \cap W \to \boldsymbol{E} \times 0,$$

$\pi_1 \circ \psi$ is a diffeomorphism of $V \cap W$ onto an open set of $\boldsymbol{E}$, and $\pi_2 \circ \psi \circ f | U$ is a submersion [**3**, p. 20], where $\pi_1: \boldsymbol{E} \times \boldsymbol{F} \to \boldsymbol{E}$ and

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π_2: $\boldsymbol{E}\times\boldsymbol{F}\to\boldsymbol{F}$ are the projections. The mapping f is *transversal to W on a subset $K\subset X$, $f|K\pitchfork W$*, iff f is transversal to W at every point $x\in K$; and f is *transversal to W, $f\pitchfork W$*, iff $f|X\pitchfork W$.

For some basic properties of transversality see Lang [**3**, p. 22]. Suppose $f: X\to Y$ and $g: Z\to Y$ are C^r mappings, and let $\Delta\subset Y\times Y$ denote the diagonal.

2.2. Definition. The mappings $f: X\to Y$ and $g: Z\to Y$ are *transversal at points $x\in X$ and $z\in Z$* iff the product $f\times g: X\times Z\to Y\times Y$ is transversal to Δ at (x, z) in the sense of Definition 2.1. The mappings are *transversal on sets $K\subset X$ and $M\subset Z$, $f|K\pitchfork g|M$*, iff

$$f\times g|\, K\times M \pitchfork \Delta,$$

and they are *transversal, $f\pitchfork g$*, iff $f\times g\pitchfork\Delta$.

2.3. Definition. A *C^r manifold of mappings* from X to Y is a set $\mathcal{A}$ of C^r mappings from X to Y which is a C^r manifold such that the evaluation mapping

$$\text{ev}: \mathcal{A}\times X\to Y: (f, x)\to f(x)$$

is of class C^r.

For example it is known that if X is compact then the set $\mathcal{C}^r(X, Y)$ of all C^r mappings from X to Y has a natural structure of C^r manifold of mappings [**1**, p. 31; **2**; **5**].

If $K\subset X$ is any subset, and $\mathcal{A}$ a manifold of mappings from X to Y, let $\mathcal{A}_{K,W}\subset\mathcal{A}$ be the subspace of mappings which are transversal to W on the set K. The following is easy to prove.

2.4. Openness theorem. *If $K\subset X$ is a compact set, $W\subset Y$ a closed submanifold, and $\mathcal{A}\subset\mathcal{C}^r(X, Y)$ a C^r manifold of mappings, then the subset $\mathcal{A}_{K,W}$ is open in $\mathcal{A}$.*

Recall that a *residual set* in a topological space is a countable intersection of open dense sets, a *Baire space* is one in which every residual set is dense, and by the Baire category theorem every Banach manifold is a Baire space.

2.5. Density theorem. *Let X be an n-manifold with boundary, $K\subset X$ any subset, Y a Banach manifold (without boundary), and $W\subset Y$ a closed submanifold (without boundary) of finite codimension q, all of class C^r. Let $\mathcal{A}\subset\mathcal{C}^r(X, Y)$ be a C^r manifold of mappings. If the evaluation map of $\mathcal{A}$*

$$\text{ev}: \mathcal{A}\times X\to Y: (f, x)\to f(x)$$

is transversal to W on K and $r>\max(n-q, 0)$, then $\mathcal{A}_{K,W}\subset\mathcal{A}$ is residual.

This theorem is the main result, and may be generalized in several ways. For example suppose X, Y and Z are finite dimensional, $\mathfrak{A}\subset\mathcal{C}^r(X, Y)$ and $\mathfrak{B}\subset\mathcal{C}^r(Z, Y)$ are C^r manifolds of mappings, and $\text{ev}_{\mathfrak{A}}$ and $\text{ev}_{\mathfrak{B}}$ are the respective evaluation mappings. If $K\subset X$ and $M\subset Z$, let $\mathfrak{A}\times\mathfrak{B}_{K\times M}=\{(f, g)\in\mathfrak{A}\times\mathfrak{B}: f|K\pitchfork g|M\}$. Then the following are immediate.

2.6. Corollary. *If* $\text{ev}_{\mathfrak{A}}|\mathfrak{A}\times K \pitchfork \text{ev}_{\mathfrak{B}}|\mathfrak{B}\times M$ *and* $r>\max(\dim X+\dim Z-\dim Y, 0)$, *then* $\mathfrak{A}\times\mathfrak{B}_{K\times M}\subset\mathfrak{A}\times\mathfrak{B}$ *is residual.*

Let $\mathfrak{B}=\{g\}$, a single map, and

$$\mathfrak{A}_{K,g} = \{f\in\mathfrak{A}: f|K\pitchfork g|Z\}.$$

2.7. Corollary. *If* $\text{ev}_{\mathfrak{A}}|\mathfrak{A}\times K\pitchfork g$ *and*

$$r > \max(\dim X + \dim Z - \dim Y, 0),$$

then $\mathfrak{A}_{K,g}\subset\mathfrak{A}$ *is residual.*

If in 2.7 g is an embedding and W its image, then 2.5 is obtained with the condition "W closed" deleted.

Note the symmetry of 2.2 and 2.6. Both may be extended to n-tuples of mappings having a common target, and the symmetry can be completed by allowing all sources to be manifolds with boundary, making only trivial modifications.

3. Proof of the density theorem. The Density Theorem 2.5 is proved from the following lemma by an easy point set argument.

Density lemma. *If X has finite dimension n, $W\subset Y$ is a closed submanifold having finite codimension q, $\mathfrak{A}\subset\mathcal{C}^r(X, Y)$ is a C^r manifold of mappings with $r>\max(n-q, 0)$, and the evaluation mapping is transversal to W at a point $(f, x)\in\mathfrak{A}\times X$, then there exists a neighborhood $\mathfrak{U}$ of $f\in\mathfrak{A}$ and a neighborhood V of $x\in X$ such that $\mathfrak{U}_{V,W}\subset\mathfrak{U}$ is dense.*

If $f(x)\notin W$ the lemma is trivial. If $f(x)=w\in W$ the proof is immediate from these three propositions.

Proposition A. *If the evaluation map is transversal to W at (f, x) and $f(x)=w\in W$, there are neighborhoods $\mathfrak{U}$ of $f\in\mathfrak{A}$ and V of $x\in X$ such that every point $g\in\mathfrak{U}$ is contained in a p-dimensional submanifold $\sum_g^p$, $0\leq p\leq q$, such that* $\text{ev}|\sum_g^p\times V\pitchfork W$.

The proof of this proposition is prosaic, relying on techniques which have become standard since the publication of Lang's book [3].

For the second proposition suppose $\sum^p$ is a p-dimensional submanifold of $\mathfrak{A}$, V an open set of X, and $\xi = \mathrm{ev} \mid \sum^p \times V$ is transversal to W. Then $W' = \xi^{-1}(W)$ is a submanifold of codimension q of $\sum^p \times V$. Let $\sigma: W' \to \sum^p$ denote the restriction to W' of the projection $\sum^p \times V \to \sum^p$.

PROPOSITION B. *If σ is transversal to a point $f \in \sum^p$, then f is transversal to W on V.*

The proof of this proposition is a straightforward interpretation of the definitions.

The third proposition is a well-known theorem of Sard [6]. Recall that if $f: X \to Y$ is any C^1 mapping, a point $y \in Y$ is a *critical value* of f iff it is false that $f \pitchfork \{y\}$. Let Ω_f be the set of all critical values of f.

PROPOSITION C (SARD). *If $f: R^s \to R^t$ is of class C^r with $r > \max(s-t, 0)$, then $\Omega_f \subset R^t$ has outer measure zero.*

4. The Thom transversality theorem. Let X be a C^r manifold with boundary, Y a C^r manifold (without boundary), $\pi^k: J^k(X, Y) \to X \times Y$ the k-jet bundle of C^k maps from X to Y, $0 \leq k \leq r$, and

$$j^k: \mathcal{C}^r(X, Y) \to \mathcal{C}^{r-k}(X, J^k(X, Y))$$

the k-jet extension, where $\mathcal{C}^r(X, Y)$ has the C^r topology of compact convergence (a Baire space). Let W be a C^{r-k} manifold (without boundary), $F \in \mathcal{C}^{r-k}(W, J^k(X, Y))$, and

$$\mathcal{C}^r_F(X, Y) = \{f \in \mathcal{C}^r(X, Y): j^k f \pitchfork F\}.$$

Finally, suppose W, X and Y are finite dimensional.

JET TRANSVERSALITY THEOREM (THOM). *If*

$$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\},$$

then the subspace $\mathcal{C}^r_F(X, Y) \subset \mathcal{C}^r(X, Y)$ is residual for every C^{r-k} mapping $F: W \to J^k(X, Y)$.

PROOF. First suppose X is compact. Then it is known that $\mathfrak{A} = j^k[\mathcal{C}^r(X, Y)]$ has a natural structure of C^{r-k} manifold of mappings compatible with the topology of compact convergence [1]. Furthermore a standard computation using a local chart and a C^r characteristic function shows that the differential of the evaluation mapping

$$\mathrm{ev}: \mathfrak{A} \times X \to J^k(X, Y): (j^k f, x) \to j^k f(x)$$

is surjective at any point $(j^k f, x) \in \mathfrak{A} \times X$, so the evaluation map is transversal to any mapping $F: W \to J^k(X, Y)$. Thus if

$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\}$, the Openness and Density Theorems 2.4 and 2.7 imply that $\mathfrak{A}_{X,F} \subset \mathfrak{A}$ is open and dense. Now suppose X is not compact. Then, using a countable covering of X by compact manifolds with boundary, a simple point set argument, and the proof above, $\mathfrak{A}_{X,F} \subset \mathfrak{A}$ is seen to be residual. But $j^k: \mathcal{C}^r(X, Y) \to \mathfrak{A}$ is a homeomorphism so $\mathcal{C}^r_F(X, Y) = j^{k^{-1}}(\mathfrak{A}_{X,F})$ is residual in $\mathcal{C}^r(X, Y)$.

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