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Corrected

LECTURES OF SMALE ON DIFFERENTIAL TOPOLOGY

by

R. Abraham

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C O N T E N T S

0. Introduction. These notes developed from the introductory lectures of a course given by Prof. Stephen Smale at Columbia University 1962-1963. As an introduction to the course, Prof. Smale remarked: "Recent events in differential topology indicate a change of direction is taking place, away from manifolds and toward differentiable mappings and analysis. In this course the new direction will be followed, with global calculus of variations as the main goal."

The first nine sections of these notes follow the lectures closely, and rely heavily on two basic books, by Dieudonne [2] and Lang [3]. Sections 10 and 11 are adapted from a letter of R. Palais [4], replacing the equivalent formulation given in the lectures. The remaining sections follow lectures and a paper by the author [1].

Thanks are due to Profs. Smale and Lang for many valuable suggestions, and to Prof. Palais for permission to use his letter.

CHAPTER I

REVIEW OF DIFFERENTIAL CALCULUS

This chapter presents the basic facts and notations for the derivative of a continuous mapping of Banach spaces. For details and proofs, see Dieudonne [2, Ch. V, VI, VII and VIII]. Herein, linear spaces are real.

1. Definition of derivative

Let A be a set, and $\mathbb{F}$ a normed linear space. Then $\mathcal{B}(A, \mathbb{F})$ will denote the normed linear space of bounded functions from A to $\mathbb{F}$ with the norm, $\|u\| = \sup_{x \in A} \|u(x)\|$. Similarly if A is a topological space $\mathcal{C}^0(A, \mathbb{F})$ will denote the normed linear space of bounded continuous mappings from A to $\mathbb{F}$ with the same norm.

1.1. Theorem: *If $\mathbb{F}$ is a Banach space, then so are $\mathcal{B}(A, \mathbb{F})$ and $\mathcal{C}^0(A, \mathbb{F})$.*

1.2. Definition: Let $\mathbb{E}^i$ and $\mathbb{F}$ be Banach spaces, $i = 1,\ldots,n$, and $f : \prod_{i=1}^{n} \mathbb{E}^i \to \mathbb{F}$. Then f is *bounded* iff there exists a positive real M such that for all $(x^1,\ldots,x^n) \in \prod \mathbb{E}^i$,

$$\|f(x^1,\ldots,x^n)\| \leq M \prod_{i=1}^{n} \|x^i\| .$$

<u>1.3. Theorem</u>: <u>Let</u> $f : \prod_{i=1}^{n} \mathbb{E}^{i} \to \mathbb{F}$ <u>be an n-linear function. Then</u> f <u>is continuous iff it is bounded.</u>

<u>1.4. Corollary</u>: <u>The linear space</u> $L_n(\mathbb{E};\mathbb{F})$ <u>of continuous n-linear mappings from</u> $\prod^{n} \mathbb{E}$ <u>to</u> $\mathbb{F}$ <u>with the norm,</u>

$$\|u\| = \sup_{\|x\| = 1} \|u(x)\| ,$$

<u>is a Banach space.</u>

Let $\mathbb{E}$, $\mathbb{F}$ be Banach spaces, $U \subset \mathbb{E}$ an open set, $f,g : U \to \mathbb{F}$ continuous mappings, and $u_o \in U$.

<u>1.5. Definition</u>: The mappings f and g are <u>tangent</u> at u_o iff the function

$$Q(u) = \frac{\|f(u) - g(u)\|}{\|u - u_o\|} , \quad u \neq u_o$$

$$Q(u_o) = 0$$

is continuous at u_o.

<u>1.6. Lemma</u>: <u>There exists at most one continuous linear mapping</u> $L : \mathbb{E} \to \mathbb{F}$ <u>such that the affine mapping</u>

$$a[f(u_o),L] : \mathbb{E} \to \mathbb{F} : u \to F(u_o) + L(u - u_o)$$

<u>is tangent to</u> f <u>at</u> u_o.

1.7 Definition: A continuous mapping $f : U \to \mathbb{F}$ is differentiable at u_o iff there exists a continuous linear mapping $L : \mathbb{E} \to \mathbb{F}$ such that $a[f(u_o),L]$ is tangent to f at u_o. If f is differentiable at u_o, the unique linear mapping $L = Df(u_o)$ is the derivative of f at u_o. If f is differentiable at u for all $u \in U$, then f is differentiable on U, and the function

$$Df : U \to \mathbb{L}(\mathbb{E}, \mathbb{F}) : u \to Df(u)$$

is the derivative of f.

2. Spaces of differentiable mappings

2.1 Lemma: If $\mathbb{E}$ and $\mathbb{F}$ are Banach spaces, then $\mathbb{L}(\mathbb{E}, \mathbb{L}_{n-1}(\mathbb{E}, \mathbb{F}))$ and $\mathbb{L}_n(\mathbb{E}; \mathbb{F})$ are canonically isomorphic Banach spaces.

Hereafter, these spaces will be identified by this isomorphism.

2.2 Definition: If $f : U \to \mathbb{F}$ is a continuous mapping differentiable on U, and $D^i f \equiv D(D^{i-1}f)$ is continuous and differentiable on U for $i = 1,\dots,r-1$, then the function

$$D^r f : U \to \mathbb{L}(\mathbb{E}, \mathbb{F})$$

is the r^{th} derivative of f. A mapping f is of class C^r iff its i^{th} derivative is defined, continuous, and bounded over U for $i = 0,\dots,r$. If $A \subset \mathbb{E}$ is an arbitrary subset and $f : A \to \mathbb{F}$, then f is of class C^r iff there exists an open set U of $\mathbb{E}$ containing A and a C^r mapping $g : U \to \mathbb{F}$ such that $g|A = f$. Let $\mathcal{C}^r(A, \mathbb{F})$ denote the normed linear space of C^r mappings from A to $\mathbb{F}$ with the C^r sup norm:

$$\|f\|^r = \sup_{u \in A} \sum_{i=0}^{r} \|D^i f(u)\|$$

where $D^0 f = f$ and $\|D^i f(u)\| = \sup\limits_{\|v\| = 1} \|D^i f(u)\cdot v\|$ for $i > 0$.

2.3 Theorem: If U is an open set of a Banach space, then $\mathcal{C}^r(U, \mathbb{F})$ is a Banach space, for $0 \leq r \leq \infty$.

2.4 Theorem: If U is an open set of a Banach space, $f \in \mathcal{C}^r(U, \mathbb{F})$, and $u \in U$, then $D^r f(u)$ is a symmetric r-linear mapping.

2.5 Theorem: If U is an open set of a Banach space $\mathbb{E}$, the derivative mapping $D : \mathcal{C}^r(U, \mathbb{F}) \to \mathcal{C}^{r-1}(U, L(\mathbb{E}, \mathbb{F}))$ is a continuous linear mapping.

3. Composition of mappings

Let $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ be Banach spaces, $U \subset \mathbb{D}$ an open set, and

$$B : \mathbb{E} \times \mathbb{F} \to \mathbb{G} : (e,f) \to e\cdot f$$

a continuous bilinear mapping. Then a bilinear mapping is induced on the set of functions:

$$B^U : \mathbb{E}^U \times \mathbb{F}^U \to \mathbb{G}^U : (\alpha, \beta) \to \alpha\cdot\beta,$$

where $\qquad \alpha\cdot\beta(u) = \alpha(u)\cdot\beta(u).$

3.1 Leibnitz Formula: If $\alpha \in \mathcal{C}^1(U, \mathbb{E})$ and $\beta \in \mathcal{C}^1(U, \mathbb{F})$, then $\alpha\cdot\beta \in \mathcal{C}^1(U, \mathbb{G})$, and

$$D(\alpha\cdot\beta) = D\alpha\cdot\beta + \alpha\cdot D\beta\ .$$

The proof consists of a standard argument on the difference quotient and is omitted. The formula is easily extended to multilinear mappings.

3.2 Notations: If $\alpha : U \to L(\mathbb{E}, \mathbb{F})$ and $\beta : U \to L(\mathbb{F}, \mathbb{G})$, the compositional product of α and β is the function

$$\beta\cdot\alpha : U \to L(\mathbb{E}, \mathbb{G}) : u \to \beta(u) \circ \alpha(u).$$

If $\mathbb{A}$ and $\mathbb{B}$ are Banach spaces and $\alpha : U \to \mathbb{A}$, $\beta : U \to \mathbb{B}$, let $\alpha \oplus \beta$ denote the direct sum of α and β,

$$\alpha \oplus \beta : U \to \mathbb{A} \oplus \mathbb{B} : u \to \alpha(u) \oplus \beta(u).$$

Note that the compositional product is induced by the continuous bilinear mapping

$$\circ : L(\mathbb{F}, \mathbb{G}) \times L(\mathbb{E}, \mathbb{F}) \to L(\mathbb{E}, \mathbb{G}) : (T_2, T_1) \to T_2 \circ T_1$$

while the direct sum is induced by the continuous bilinear mapping

$$\oplus : \mathbb{A} \times \mathbb{B} \to \mathbb{A} \oplus \mathbb{B} : (a,b) \to a \oplus b.$$

Thus Leibnitz' formula applies in each case.

Let $U \subset \mathbb{E}$ and $V \subset \mathbb{F}$ be open sets of Banach spaces and $\mathbb{G}$ a Banach space.

Let p and k be positive integers, $p \geq k$, and $(i_1, \ldots, i_k)$ a k-tuple of positive integers satisfying

(i) $i_1 + \ldots + i_k = p$, and

(ii) $1 \leq i_1 \leq \ldots \leq i_k$.

Then let the positive integer $\sigma_k^p(i_1, \ldots, i_k)$ be defined recursively by the rules

(iii) $\sigma_1^p = 1$, and

(iv) $$\sigma_k^p(i_1,\ldots,i_k) = \delta_{i_1}^1 \, \sigma_{k-1}^{p-1}(i_2,\ldots,i_k) + \sum_{\ell=1}^{k} \sigma_k^{p-1}(i_1,\ldots,i_\ell + 1,\ldots,i_k),$$

where $\delta_{i_1}^1 = 1$ if $i_1 = 1$, $\delta_{i_1}^1 = 0$ otherwise.

3.3 Composite Mapping Formula: *If* $f \in \mathcal{C}^p(U,V)$ *and* $g \in \mathcal{C}^p(V,\mathbb{G})$, *then* $g \circ f \in \mathcal{C}^p(U,\mathbb{G})$ *and*

$$D^p(g \circ f) = \sum_{k=1}^{p} (D^k g \circ f)\cdot P_k^p f,$$

where

$$P_k^p f = \sum_{\substack{i_1+\ldots+i_k=p \\ 1 \le i_1 \le \ldots \le i_k}} \sigma_k^p(i_1,\ldots,i_k) D^{i_1} f \oplus \ldots \oplus D^{i_k} f.$$

The proof, a straightforward induction on p which uses Leibnitz' formula for the compositional product and the direct sum, is omitted.

If $f : U \to V$ and $g : V \to \mathbb{G}$, let $\alpha_f(g)$ denote the composite $g \circ f : U \to \mathbb{G}$.

3.4 Theorem: *If* $f : U \to V$ *is a class* C^r, *then* $\alpha_f : \mathcal{C}^r(U,G) \to \mathcal{C}^r(U,\mathbb{G})$ *is a continuous linear mapping*.

Proof. The linearity of α_f is obvious, and we prove continuity by induction. If $r = 0$, we have $\|\alpha_f(g)\|_0 \le \|g\|_0$, so α_f is bounded. For the induction we have

$$\|\alpha_f(g)\|_r \leq \|\alpha_f(g)\|_{r-1} + \|D^r(g \circ f)\| .$$

Suppose $\|g\|_r = 1$. Then the first term on the right is bounded by hypothesis and the second term has a bound which may be computed from the composition mapping formula, 3.4. Thus α_f is continuous in the C^r topology.

If $g : U \times V \to \mathbb{G}$, and $(u,v) \in U \times V$, we may consider the partial mappings $g_u = g|\{u\} \times V$ and $g_v = g|U \times \{v\}$, and the partial derivatives $D_1^p g(u,v) \equiv D^p g_u(v)$ and $D_2^s g(u,v) = D^s g_v(u)$.

<u>3.5 Definition:</u> A mapping $g : U \times V \to G$ <u>is of class (C_1^r , C_2^s)</u> iff the mappings

$$D_1^p g : U \times V \to L_s^p(\mathbb{E},\mathbb{G})$$

and

$$D_2^q g : U \times V \to L_s^q(\mathbb{F},\mathbb{G})$$

are defined, continuous and bounded over $U \times V$ for $0 \leq p \leq r$ and $0 \leq q \leq s$.

<u>3.6 Theorem:</u> <u>A mapping</u> $g : U \times V \to G$ <u>is of class $C^{\overset{1}{r}}$ iff it is of class $(C_1^{\overset{1}{r}}, C_2^{\overset{1}{r}})$, and in this case</u>

$$Dg(u,v)\cdot(e,f) = D_1 g(u,v)\cdot e + D_2 g(u,v)\cdot f.$$

For the proof, see Dieudonne [2 p. 167].

It is clear that in the above the open sets U and V may be replaced by arbitrary subsets A and B.

If $g : A \times B \to G$ and $f : A \to B$, let $\Omega_g(f)$ denote the composite $g \circ \text{graph}(f) : A \to \mathbb{G}$.

<u>3.7 Theorem:</u> <u>If</u> $K \subset \mathbb{E}$ <u>is compact</u>, $V \subset \mathbb{F}$ <u>open, and</u> $g : K \times V \to \mathbb{G}$ <u>is of class</u> (C_1^r, C_2^{r+s}), $0 \leq s \leq r$, <u>then</u> $\Omega_g : \mathcal{C}^r(K,V) \to \mathcal{C}^r(K,\mathbb{G}) : f \to \Omega_g(f)$ <u>is of class</u> C^s.

Proof. First we take $s = 0$ and proceed by induction on r. For $r = 0$ the proof follows at once from the continuity of g and the compactness of K. The induction is evident from the inequality

$$\|\Omega_g(f) - \Omega_g(f_o)\|_r \leq \|\Omega_g(f) - \Omega_g(f_o)\|_{r-1} + \|D(D^{r-1}\Omega_g(f)) - D(D^{r-1}\Omega_g(f_o))|$$

and the continuity of D, 2.5. For arbitrary $s \leq r$, a simple computation shows that $D^s\Omega_g = \Omega_{D^s g}$, so by the case above, $r = s = 0$, $D^s\Omega_g$ is continuous.

This theorem admits a simple specialization which is dual to 3.5. If $g : B \to \mathbb{G}$ and $f : A \to B$, let ω_g denote the composition $g \circ f : A \to G$.

<u>3.8 Corollary:</u> <u>If</u> $K \subset \mathbb{E}$ <u>is compact</u>, $V \subset \mathbb{F}$ <u>open, and</u> $g \in \mathcal{C}^{r+s}(V,G)$, $0 \leq s \leq r$, <u>then</u> $\omega_g : \mathcal{C}^r(K,V) \to \mathcal{C}^r(K,\mathbb{G})$ <u>is of class</u> C^s.

CHAPTER II

BANACH MANIFOLDS AND BUNDLES

This chapter contains the basic definitions and theorems of differentiable manifolds modelled on Banach spaces. For details and missing proofs see Lang [3, Ch. II, III].

4. Differentiable manifolds and mappings

Let S be a Hausdorff space.

4.1 Definition: A chart on S is a pair (U,φ) where U is an open set of S, and $\varphi : U \to \mathbb{E}$ a homeomorphism onto an open set of a Banach space $\mathbb{E}$. A C^r atlas on S is a collection $\mathcal{a} = (U_i,\varphi_i)$ of charts on S such that $\{U_i\}$ covers S and for all i, j,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a C^r diffeomorphism. Two C^r atlases are C^r equivalent iff their union is a C^r atlas. A C^r- equivalence class of C^r atlases is a C^r-structure on S. A C^r-manifold is a pair $X = (S,\mathcal{a})$ where $\mathcal{a}$ is a C^r-structure on S.

Note that if X is a connected C^r manifold then all defining Banach spaces are equivalent by a continuous linear isomorphism, $\mathbb{E}_i \approx \mathbb{E}$, and X is said to be a C^r- $\mathbb{E}$-manifold, or to be modelled on $\mathbb{E}$.

4.2 Definition: Let X and Y be C^r manifolds and $f : X \to Y$. Then f is a C^r mapping iff there exists an atlas $\{(U_i,\varphi_i)\}$ of X and an atlas $\{(V_j,\psi_j)\}$ of Y such that for all i, j,

$$f'_{ji} = \psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1} V_j) \longrightarrow \psi_j V_j$$

is a C^r mapping in the sense of 2.2. Also, f is a <u>C^r diffeomorphism</u> if f is a C^r map with a C^r inverse.

Hereafter we tacitly assume $r \geq 1$ whenever C^r is written.

<u>4.3 Definition</u>: Let Y be a C^r manifold and X a subset of Y. Then X is a <u>submanifold</u> of Y iff for every $x \in X$ there is a chart (U,φ) of Y, $p \in U$, and closed complementary subspaces $\mathbb{E}_1$ and $\mathbb{E}_2$ of $\mathbb{E}$, the target of φ, such that

$$\varphi(U \cap X) = \varphi(U) \cap (\mathbb{E}_1 \times 0).$$

Note that if $\{(U_i,\varphi_i)\}$ is a covering of X by charts of Y having the property above, then $\{(U_i \cap X, \varphi|U_i \cap X)\}$ is a C^r atlas for X, hence:

<u>4.4 Theorem</u>: <u>If</u> $X \subset Y$ <u>is a submanifold of Y</u>, <u>then</u> X <u>has a C^r structure such that the inclusion</u> $i : X \to Y$ <u>is a C^r mapping</u>.

Hereafter submanifold means the C^r manifold with such a C^r structure.

Let X,Y be C^r manifolds and $f : X \to Y$ a C^r mapping.

<u>4.5 Definition</u>: The mapping f is an <u>immersion</u> iff it is locally a diffeomorphism onto a submanifold of Y, an <u>embedding</u> iff it is a diffeomorphism onto a submanifold of Y, a <u>submersion</u> iff it is "locally equivalent to a projection": for every $x \in X$ there is a chart (U,φ)

at x and a chart (V,ψ) at $f(x)$ such that $\varphi : U \to U_1 \times U_2$, a product of open sets of Banach spaces, and

$$\psi \circ f \circ \varphi^{-1} : U_1 \times U_2 \to \psi(V)$$

is a projection.

Now let X be a C^r manifold, $x \varepsilon X$, (U,φ) a chart at x. We consider triples (U,φ,u) such that $u \varepsilon \mathbb{E}$, where $\varphi : U \to \mathbb{E}$ and $\varphi(x) = 0$.

4.6 Definition: Two triples (U,φ,u) and (V,ψ,v) are equivalent iff $D(\psi \circ \varphi^{-1})(\varphi x)(u) = v$. An equivalence class of such triples is the tangent space to X at x, $T_x(X)$.

Note that any chart (U,φ) at x induces a bijection between $T_x(X)$ and a Banach space. The Banach space structure thereby induced on $T_x(X)$ is unique up to a continuous linear isomorphism, so hereafter $T_x(X)$ will denote this Banach space.

Let X,Y be C^r manifolds and $f : X \to Y$ a C^r mapping. Then relative to charts (U,φ) at $x \varepsilon X$ and (V,ψ) at $f(x) \varepsilon Y$ the derivative $D(\psi \circ f \circ \varphi^{-1})(\varphi x)$ induces a map $T_x f : T_x(X) \to T_{f(x)}(Y)$ which is independent of the charts used. The following theorem is the first main result in this section, and a proof based on the contraction mapping lemma may be found in Lang [3].

Recall that closed linear subspace of a Banach space splits if it has a closed complement. An injection of one Banach space into another splits if its image is closed and has a closed complement.

4.7 Implicit Function Theorem: Let X and Y be C^r manifolds, $f : X \to Y$ a C^r mapping. Then

(a) if $T_x f$ is injective and splits there exists a neighborhood U of $x \in X$ such that $f|U$ is an embedding.

(b) if $T_x(f)$ is surjective and its kernel splits, there exists a neighborhood U of $x \in X$ such that $f|U$ is a submersion.

Part (b) has an important consequence, which follows directly from definitions 4.3 and 4.5.

4.8 Corollary: Let $f : X \to Y$ be a C^r submersion and $y \in Y$. Then $f^{-1}(y)$ is a submanifold of X.

For example, if Y is a Hilbert space and $X \subset Y$ is the unit sphere, then X is a submanifold. For if $f : Y \to \mathbb{R} : y \to (y,y)$, then $Df(y_1)(y_2) = \langle y_1, y_2 \rangle$, so $f|Y \setminus 0$ is a submersion by 4.7 (b) and $X = f^{-1}(1)$ is a submanifold by 4.8.

5. Banach bundles

The next goal is to define Banach bundles and show that the union of all tangent spaces of a manifold is such a bundle. Throughout, "bundle" shall mean "Banach bundle", also called "vector bundle" by Lang [3].

The following two notions will be used to define bundles. A local bundle is a triple $(U \times \mathbb{E}, U, \pi)$, where U is an open set of a Banach space, $\mathbb{E}$ is a Banach space, and $\pi : U \times E \to U$ is the projection on the first factor. Let $(U \times \mathbb{E}, U, \pi)$ and $(V \times \mathbb{F}, V, \rho)$ be two local bundles. A C^r local bundle map is a pair (f,f_B) of C^r mappings such that

(i)

$$\begin{tikzcd} U \times \mathbb{E} \arrow[r, "f"] \arrow[d, "\pi"] & V \times \mathbb{F} \arrow[d, "\rho"] \\ U \arrow[r, "f_B"] & V \end{tikzcd}$$

commutes,

(ii) $f|\pi^{-1}(u)$ is a continuous linear mapping for all $u \in U$,

and (iii) $f^* : U \to L(\mathbb{E}, \mathbb{F}) : u \to f|\pi^{-1}(u)$ is a C^r mapping.

5.1 Definition: A C^r Banach bundle is a triple $\pi = (X,X_B,\pi)$, in which X and X_B are C^r manifolds and $\pi : X \to X_B$ a C^r mapping, such that there exists an atlas $\{(U_i,\varphi_i)\}$ of X_B and a collection of mappings $\{\psi_i\}$ satisfying:

(i) $\psi_i : \pi^{-1}U_i \to \varphi_i U_i \times E_i$, a local bundle,

(ii) $\{(\pi^{-1}U_i,\psi_i)\}$ is an atlas of X, and

(iii) for all i,j, $(\psi_j \circ \psi_i^{-1},\varphi_j \circ \varphi_i^{-1})$ is a C^r local bundle map.

The charts $(\pi^{-1}U_i,\psi_i)$ are local bundle charts. Then X is called the total space, X_B the base and $\pi^{-1}(x)$ for $x \in X_B$ is the fiber over x. Let $\pi : X \to X_B$ and $\rho : Y \to Y_B$ be C^r bundles. A C^r bundle map from π to ρ is a pair (f,f_B) of C^r mappings such that the diagram commutes, and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \arrow[d, "\rho"] \\ X_B \arrow[r, "f_B"] & Y_B \end{tikzcd}$$

such that the mappings of local bundles induced by f through local bundle charts are C^r local bundle maps.

Now let X be a manifold, $T(X) = \bigcup_{x \in X} T_x(X)$, and $\tau : T(X) \to X : T_x(X) \to x$.

<u>5.2 Theorem</u>: <u>If</u> X <u>is a</u> C^{r+1} <u>manifold, then</u> $\tau : T(X) \to X$ <u>has a</u> C^r <u>bundle structure</u>.

The next goal is the space of cross-sections of a bundle. Let $\pi : X \to X_B$ be a C^r bundle.

<u>5.3 Definition</u>: A <u>section</u> of π is a C^r mapping $\gamma : X_B \to X$ such that $\pi \circ \gamma$ is the identity of X_B.

Let $\Gamma(\pi)$ denote the vector space of all sections of π with the linear structure induced by that on the fibers. (Observe that if $x \in X_B$, $\gamma_1, \gamma_2 \in \Gamma(\pi)$, then $\gamma_1(x)$ and $\gamma_2(x)$ belong to the same fiber by the section property.)

We shall now define a norm for $\Gamma(\pi)$ in the case in which X_B is compact. In this case a <u>norm atlas</u> of π is a finite set of 4-tuples $\mathcal{a} = \{(K_i, U_i, \varphi_i, \psi_k)\}_{i=1}^{k}$ such that $\{(U_i, \varphi_i)\}$ is an atlas of X_B, $\{(\pi^{-1}U_i, \psi_i)\}$ is a local bundle atlas of X with $\psi_i : \pi^{-1}U_i \to V_i \times \mathbb{E}_i$, $K_i \subset U_i$ are compact, and $\{K_i\}$ is a covering of X_B. If $\gamma \in \Gamma(\pi)$, let γ_i denote the induced local bundle section:

$$\gamma_i = \psi_i \circ \gamma \circ \varphi_i^{-1} \Big| \varphi_i K_i : \varphi_i K_i \to \varphi_i K_i \cap \mathbb{E}_i.$$

Let $\rho_i : V_i \times \mathbb{E}_i \to \mathbb{E}_i$ be the projection on the second factor. As π and γ are of class C^r, we have $\rho_i \circ \gamma_i \in \mathcal{C}^r(\varphi_i . K_i, \mathbb{E}_i)$ which has the C^r norm $\| \ \|_i$. Now define

$$^i\| \gamma \|_{\mathcal{a}} = \| \rho_i \circ \gamma_i \|_i$$

and $$\|\gamma\|_{\mathcal{A}} = \sup_{i \leq k} {}^{i}\|\gamma\| .$$

<u>5.4 Theorem: If $\mathcal{A}$ and $\mathcal{A}'$ are norm atlases for</u> $\pi : X \to X_B$ <u>and</u> X_B <u>is compact, then</u>

(i) $\|\ \|_{\mathcal{A}}$ <u>is a norm for</u> $\Gamma(\pi)$

(ii) $\|\ \|_{\mathcal{A}}$ <u>is equivalent to</u> $\|\ \|_{\mathcal{A}'}$, <u>and</u>

(iii) $\{\Gamma(\pi), \|\ \|_{\mathcal{A}}\}$ <u>is a Banach space</u>.

Proof. (i) As ${}^{i}\|\ \|_{\mathcal{A}}$ is a semi-norm on $\Gamma(\pi)$ and the supremum of a finite set of semi-norms is a semi-norm, $\|\ \|_{\mathcal{A}}$ is a semi-norm. But $\|\gamma\|_{\mathcal{A}} = 0$ implies $\gamma = 0$, so $\|\ \|_{\mathcal{A}}$ is a norm.

(ii) Let $\mathcal{A} = \{(K_i, U_i, \varphi_i, \psi_i)\}_{i=1}^{k}$ and $\mathcal{A}' = \{(K_j', U_j', \varphi_j', \psi_j')\}_{j=1}^{k'}$ be norm atlases for π. Let $\{(K_\alpha, U_\alpha)\}_{\alpha=1}^{N}$ be a common refinement of $\{(K_i, U_i)\}$ and $\{(K_j', U_j')\}$, $\mathcal{B} = \{(K_\alpha, U_\alpha, \varphi_\alpha, \psi_\alpha)\}_{\alpha=1}^{N}$, and $\mathcal{B}' = \{(K_\alpha, U_\alpha, \varphi_\alpha', \psi_\alpha')\}_{\alpha=1}^{N}$ where $U_\alpha \subset U_i$ implies $\varphi_\alpha = \varphi_i | U_\alpha$ and $\psi_\alpha = \psi_i | \pi^{-1} U_\alpha$, and similarly for φ_α' and ψ_α' . It is clear that $\|\ \|_{\mathcal{A}} = \|\ \|_{\mathcal{B}}$ and $\|\ \|_{\mathcal{A}'} = \|\ \|_{\mathcal{B}'}$, while the new norms are computed over the same coverings. Thus $\|\ \|_{\mathcal{A}}$ and $\|\ \|_{\mathcal{A}'}$ are equivalent iff ${}^{\alpha}\|\ \|_{\mathcal{B}}$ and ${}^{\alpha}\|\ \|_{\mathcal{B}'}$ are equivalent for each α.

Let $\pi_\alpha : \varphi_\alpha K_\alpha \times \mathbb{E}_\alpha \to \varphi_\alpha K_\alpha$ and $\pi_\alpha' : \varphi_\alpha' K_\alpha \times \mathbb{E}_\alpha' \to \varphi_\alpha' K_\alpha$ denote the local bundles. Then if $\gamma \in \Gamma(\pi)$, we have the induced local sections $\gamma_\alpha \in \Gamma(\pi_\alpha)$ and $\gamma_\alpha' \in \Gamma(\pi_\alpha')$. Also $\{\Gamma(\pi_\alpha), {}^{\alpha}\|\ \|_{\mathcal{B}}\}$ and $\{\Gamma(\pi_\alpha'), {}^{\alpha}\|\ \|_{\mathcal{B}'}\}$ are Banach spaces. But γ_α and γ_α' are related by the equation

$$\gamma_\alpha = F_\alpha(\gamma_\alpha') \equiv \psi_\alpha \circ (\psi_\alpha')^{-1} \circ \gamma_\alpha' \circ \varphi_\alpha' \circ (\varphi_\alpha)^{-1},$$

and it is clear from 3.4 and 3.7 that $F_\alpha : \Gamma(\pi'_\alpha) \to \Gamma(\pi_\alpha)$ is a continuour linear isomorphism.

(iii) Completeness follows at once from (i) and (ii).

Next we consider a C^r mapping $f : X \to Y$ into the base of a C^r bundle $\pi : F \to Y$. Let $F^*(F) \subset X \times F$ be the set of pairs (x,y) such $f(x) = \pi(y)$, $f^*(\pi) : f^*(F) \to X$ and $\pi^*(f) : F^*(F) \to F$ the projections into the first and second features, resp.

<u>5.5 Theorem</u>: <u>With</u> f <u>and</u> π <u>as above</u>, $f^*(\pi)$ <u>is a C^r bundle, the pair</u> $(f,\pi^*(f))$ <u>is a C^r bundle map, and any bundle map into</u> π <u>factors uniquely through</u> $f^*(\pi)$.

6. Exact sequences

Let π and ρ be bundles having a common base X, and let (f_B, f) and (g_B, g) be bundle maps from π to ρ with $f_B = g_B$ the identity map of X.

<u>6.1 Definition</u>: The sequence $0 \longrightarrow \pi \xrightarrow{f} \rho$ is <u>exact</u> iff for every $x \in X$, $f|\pi^{-1}(x)$ is a splitting injection. The sequence $\pi \xrightarrow{g} \rho \longrightarrow 0$ is <u>exact</u> iff for every $x \in X$, $f|\pi^{-1}(x)$ is a splitting surjection.

<u>6.2 Theorem</u>: (i) <u>If</u> $0 \longrightarrow \pi \xrightarrow{f} \rho$ <u>is exact, there exist local bundle charts</u> $(\pi^{-1}U,\varphi,\psi)$ <u>and</u> $(\rho^{-1}U,\varphi',\psi')$ at $x \in U \subset X$, <u>with</u> $\psi : \pi^{-1} U \to \varphi U \times \mathbb{E}$ <u>and</u> $\psi' : \rho^{-1}U \to \varphi'U \times \mathbb{E} \times \mathbb{F}$, <u>such that the induced local bundle map</u> f' <u>is a continuous linear isomorphism of</u> $\varphi U \times \mathbb{E}$ <u>onto</u> $\varphi'U \times \mathbb{E} \times \underline{0}$.

(ii) If $\pi \xrightarrow{g} \rho \longrightarrow 0$ is exact, there exist local bundle charts $(\pi^{-1}U, \varphi, \psi)$ and $(\rho^{-1}U, \varphi', \psi')$ at $x \in U \subset X$, with $\psi : \pi^{-1}U \to \varphi U \times \mathbb{E} \times \mathbb{F}$ and $\psi' : \rho^{-1}U \to \varphi' U \times \mathbb{E}$, such that the induced local bundle map f' is a continuous linear projection with kernel $\varphi U \times \mathbb{E} \times \underline{0}$.

If $0 \longrightarrow \pi \xrightarrow{f} \rho$ is exact, let $D = \bigcup_{x \in X} \rho^{-1}x \,/\, f(\pi^{-1}x)$ and coker $(f) \equiv \rho/\pi : D \to X : \rho^{-1}x \,/\, f(\pi^{-1}x) \to x$. If $\pi \xrightarrow{g} \rho \longrightarrow 0$ is exact, let $G = \bigcup_{x \in X} \ker(f|\pi^{-1}x)$ and $\ker(g) : G \to X : \ker(f|\pi^{-1}x) \to x$.

6.3 Theorem: (i) If $0 \longrightarrow \pi \xrightarrow{f} \rho$ is exact, then coker (f) is a bundle.

(ii) If $\pi \xrightarrow{g} \rho \longrightarrow 0$ is exact, then ker (g) is a bundle.

6.4 Definition: A sequence $0 \longrightarrow \pi \xrightarrow{f} \rho \xrightarrow{g} \sigma \longrightarrow 0$ is exact iff $0 \longrightarrow \pi \xrightarrow{f} \rho$ and $\rho \xrightarrow{g} \sigma \longrightarrow 0$ are exact and $\rho \,/\, f(\pi) = \ker(g)$.

6.5 Theorem: (i) If $0 \longrightarrow \pi \xrightarrow{f} \rho$ is exact, then $0 \longrightarrow \pi \xrightarrow{f} \rho \xrightarrow{g} \rho/f(\pi) \longrightarrow 0$ is exact, where g is the natural projection.

(ii) If $\rho \xrightarrow{g} \sigma \longrightarrow 0$ is exact, then $0 \longrightarrow \ker(g) \xrightarrow{f} \rho \xrightarrow{g} \sigma \longrightarrow 0$ is exact, where f is the natural injection.

6.6 Definitions: If $0 \longrightarrow \pi \xrightarrow{f} \rho$ is exact, it is a split exact sequence iff there exists an exact sequence $\rho \xrightarrow{g} \pi \longrightarrow 0$ such that $g \circ f$ is the identity. Let $\pi : E \to X$ and $\rho : F \to X$ be bundles with $E \subset F$. Then π is a sub-bundle of ρ iff $0 \longrightarrow \pi \xrightarrow{i} \rho$ is an exact sequence where i, the inclusion map, is a bundle map. Also, if $\delta : X \to X \times X : x \to (x,x)$; $\pi : E \to X$ and $\rho : F \to X$ are bundles, then $\delta^*(\pi \times \rho)$ is the Whitney sum of π and ρ, denoted by $\pi \oplus \rho$, where $\pi \times \rho : E \times F \to X \times X$ is the cartesian product of π and ρ.

6.7 Theorem: If $0 \to \pi \to \rho \to \sigma \to 0$ is an exact sequence and $0 \to \pi \to \rho$ splits, then $\rho \approx \pi \oplus \rho$.

Proof: See Lang [3 p. 52].

For example, if $f : X \to Y$ is an immersion, we have $T(f) : \tau_x \to \tau_y$, which factors through $f^*\tau_y$. Let f_* denote the unique factor, so we have the exact sequence

$$0 \longrightarrow \tau_x \xrightarrow{T^*f} f^*\tau_y \longrightarrow \nu_f \longrightarrow 0$$

where $\nu_f = f^*\tau_y / t^*f(\tau_x)$ is the normal bundle of f. If $0 \longrightarrow \tau_x \xrightarrow{T^*f} f^*\tau_y$ splits, then $f^*\tau_y \approx \tau_x \oplus \nu_f$. However, the splitting may not exist.

7. Partitions of unity

Let S be a paracompact space and $\{U_i\}$ a locally finite covering of S.

7.1 Definition: An associated partition of unity of $\{U_i\}$ is a collection $\{f_i\}$ of real functions $f_i : U_i \to \mathbb{R}$ such that

(i) for all $s \in S$ we have $f_i(s) \geq 0$,

(ii) the support of f_i is contained in U_i, and

(iii) for all $s \in S$, $\sum f_i(s) = 1$.

An outstanding problem on differential manifolds is the existence of differentiable partitions of unity. However, the following is known (see Lang [3, p. 30]).

7.2 Theorem: On a paracompact C^r separable-Hilbert manifold every locally finite covering has an associated partition of unity of class C^r.

7.3 Theorem: If X admits partitions of unity, then every exact sequence $0 \to \pi \to \rho$ splits.

Proof: See Lang [3, p. 51].

Thus if $f : X \to Y$ is an immersion, and X admits a partition of unity, we have $f^*\tau_y \approx \tau_x \oplus \nu_f$. If X is a submanifold of Y we will write $N(X)$ for the total space of ν_i, $i : X \subset Y$, and consider $N(X)$ embedded in $f^*T(Y)$ by some fixed splitting. The embedded image is a sub-bundle of $f^*\tau_y$, called the complementary bundle.

The following is a typical application of partitions of unity for which we are indebted to John McAlpin.

7.4 Theorem: *Every second countable C^r manifold (without boundary) modelled on a separable Hilbert space can be C^r embedded onto a closed submanifold of separable Hilbert space.*

Proof. Let $\{U_i, \varphi_i)\}$ be a countable atlas of the manifold X, with $\varphi_i(U_i)$ the unit disk D of the model $\mathbb{H}$, such that the inverse images of the half-unit disk $\frac{1}{2}$ D cover X. Let $g : \mathbb{H} \to \mathbb{R}$ be a C^∞ function which is zero outside D and one on the closure of $\frac{1}{2}$ D. Let $V_i = \varphi_i^{-1}(\frac{1}{2} D)$. Define

$$\Psi_i : X \to \mathbb{H} \times \mathbb{R} : x \to \begin{cases} [g(\varphi_i(x)) \cdot \varphi_i(x), g(\varphi_i(x))] & \text{if } x \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\Psi_i | V_i$ is an embedding. Now let $\mathbb{H}_i = \mathbb{H} \times \mathbb{R}$, $i = 1,2,\ldots$, and let $\mathcal{H}$ denote the Hilbert sum of the H_i. (See Dieudonne [2, p. 117]) for definitions.) Define

$$\Psi_o \mid X \to \mathcal{H} : x \to \sum_{i=1} (2)^{-i/2} \psi_i(x).$$

As $\|\psi_i(x)\| \leq 2$ in $\mathbb{H} \times \mathbb{R}$, the sum above converges and the mapping is defined into the Hilbert sum. Also, $\|\Psi_o(x)\| \leq 2$ in $\mathcal{H}$, and $\Psi_o(X)$ does not contain the origin.

Note that Ψ_o is an injective immersion, as every $x \varepsilon X$ has a neighborhood, in a fact a V_i for some i, such that $\psi_i | V_i$ and therefore $\Psi_o | V_i$ is an embedding. Let $r : \mathcal{H} - 0 \to \mathcal{H}$ denote inversion through the unit sphere. As Ψ_o is an injective immersion into the 2-disk less the origin, $r \circ \Psi_o = \Psi$ is an injective immersion into the complement of the 2-disk in $\mathcal{H}$. Note that the image $\Psi(X)$ is closed in $\mathcal{H}$. For if $\{\Psi(x_n)\}$ converges to a point $y \varepsilon \mathcal{H}$, then $\{\Psi_o(x_n)\}$ converges to a point $y_o \varepsilon \mathcal{H} - 0$. Thus for some i, $\{\Psi_i(x_n)\}$ converges to a point $y_i \varepsilon \mathbb{H}_i - 0$, and there exists an integer N such that $x_n \varepsilon U_i$ for $n \geq N$. Also $\{\varphi_i(x_n)\}$ converges in D. But $\varphi_i : U_i \to D$ is a diffeomorphism, so $\{x_n\}$ converges to a point $x \varepsilon U_i$. We conclude that $\Psi : X \to \mathcal{H}$ is a closed injective immersion of class C^r, and obviously $\mathcal{H}$ is a separable Hilbert space. But it is immediate that a closed injective immersion is an embedding, so the proof is complete.

8. Differential equations

We shall now review the fundamentals of ordinary differential equations. For details, see Dieudonne and Lang. Let X be a C^r manifold and $\Gamma(\tau)$ the space of cross-sections of the tangent bundle $\tau : T(X) \to X$.

8.1 Definition: An integral curve of $\gamma \in \Gamma(\tau)$ is a C^r curve $c : (-a,a) \to X : t \to x_t$, $a > 0$, such that for all $t \in (-a,a)$, $D_t c(1) = \gamma(x_t)$.

A cross-section γ is also called a first order differential equation, and an integral curve a solution of γ at x.

Observe that if (U,φ) is a chart on X with $\varphi : U \to V \subset \mathbb{E}$, then γ induces a map $\gamma' : V \to \mathbb{E}$, the local representation of γ, and if $c : (-a,a) \to X$ is a C^r curve, with $\varphi \circ c(t) = v_t$, then c is a solution at x_o iff

$$\frac{\partial v_t}{\partial t}(t) = \gamma'(v_t)$$

for all $t \in (-a,a)$.

8.2 Cauchy-Picard Theorem: If V is an open set of a Banach space $\mathbb{E}$, $\gamma' : V \to V \times \mathbb{E}$ a C^r cross-section, and $x_o \in V$, there exists an $\varepsilon > 0$ such that for all positive $a < \varepsilon$ there is a unique C^r solution $c : (-a,a) \to V$ of γ' at x_o.

A stronger theorem in this direction is the following.

8.3 Local existence and uniqueness Theorem: If $\gamma' : V \to V \times \mathbb{E}$ is a C^r cross-section and $x_o \in V$, there exists a neighborhood W of $x_o \in V$, an $\varepsilon > 0$, and a C^r map $\Phi : (-\varepsilon,\varepsilon) \times W \to V$ such that for all $w \in W$, $\Phi\,(-\varepsilon,\varepsilon) \times \{w\}$ is a solution of γ' at w.

In this situation we write $\varphi_t = \Phi|\{t\} \times W$, and $\{\varphi_t\}$ is a local group of diffeomorphisms of W into V. The above globalizes as follows.

8.4 Corollary: Let $\gamma \in \Gamma(\tau)$ and $\mathcal{D} \subset \mathbb{R} \in X$ be the set of all pairs (t_o,x_o) such that there exists an integral curve

$c : (-a,a) \to X : t \to x_t$ with $|t_o| < a$. Then $\mathfrak{D}$ is open, contains $\{0\} \times X$, and the map $\Phi : \mathfrak{D} \to X : (t_o, x_o) \to \varphi_{t_o}(x_o)$ is of class C^r.

Let $\varphi_t = \Phi|(\{t\} \times X) \cap \mathfrak{D}$.

8.5 Theorem: The set $\{\varphi_t\}$ is a one-parameter group of diffeomorphisms iff $\mathfrak{D} = \mathbb{R} \times X$.

8.6 Corollary: If X is compact then $\mathfrak{D} = \mathbb{R} \times X$.

We turn now to second order equations. Let τ_1 and τ_2 denote the tangent bundles of X and $T(X)$, resp. Note that $T\tau_1 : T(T X) \to T(X)$ is also a bundle.

8.7 Definition: A second order equation on X is a cross-section γ of τ_2 and $T\tau_1$. A base solution with initial conditions $v_{x_o} \in TX$ of a second order equation γ on X is a curve $c : (-a,a) \to X : t \to x_t$ such that $c' : (-a,a) \to TX : t \to T_t c(1)$ is an integral curve of γ and $c'(0) = v_{x_o}$.

If (U, φ, ψ_1) is a local bundle chart of τ_1 with $\psi_1 : \tau_1^{-1} U \to \varphi U \times \mathbb{E}$, and $(\tau_1^{-1} U, \psi_1, \psi_2)$ a local bundle chart of τ_2 with $\psi_2 : \tau_2^{-1}(\tau_1^{-1} U) \to (\varphi U \times \mathbb{E}) \times \mathbb{E} \times \mathbb{E}$, then a cross-section $\gamma \in \Gamma(\tau_2)$ induces a local cross-section $\gamma' : \varphi U \times \mathbb{E} \to (\varphi U \times \mathbb{E}) \times \mathbb{E} \times \mathbb{E}$. If $\gamma'(v,e^o) = ((v,e^o),e^1,e^2)$, see $T\tau_1' \circ \gamma'(v,e^o) = (v,e^1)$, so γ' is a second order equation only if $e^o = e^1$. Further if $c' : (-a,a) \to T(X) : t \to \psi_1^{-1}(v_t, e^o{}_t)$ is an integral curve of a second order equation γ' at $\psi_1^{-1}(v_o, e^o{}_o)$, then the projected curve $c = \tau_1 \circ c' : (-a,a) \to X : t \to \varphi^{-1}(v_t)$ is a base solution of γ at $\varphi^{-1}(v_o)$, and satisfies the equations:

$$\frac{dv_t}{dt}(t) = e_t^o$$

$$\frac{de_t^o}{dt}(t) = \gamma'(v_t, e_t^o)$$

or equivalently,

$$\frac{d^2 v_t}{dt^2}(t) = \gamma'(v_t, \frac{dv_t}{dt})$$

9. Sprays

If $\lambda \in \mathbb{R}$, let $h_\lambda : T(X) \to T(X) : v \to \lambda v$.

9.1 Definition: A spray on X is a second order equation $\gamma \in \Gamma(\tau_{T(X)})$ such that for all $v \in T(X)$ and $\lambda \in \mathbb{R}$.

$$\gamma_{\lambda v} = \lambda Th_\lambda(\gamma_v).$$

Note that if $\gamma' : U \times \mathbb{E} \to (U \times \mathbb{E}) \times (\mathbb{E} \times \mathbb{E})$ is a local representative of a second order equation, with

$$\gamma'(x,u) = (x,u;u,f(x,u))$$

then γ' is a spray on U iff

$$f(x,\lambda u) = \lambda^2(x,u).$$

Now let γ be a spray on X, and $\beta_v(t)$ a solution of γ at a point $v \in T(X)$. Let $\mathcal{O} \subset T(X)$ be the set of $v \in T(X)$ such that $\beta_v(1)$ is defined.

9.2 Definition: The exponential of γ is the map $\exp^\gamma : \mathcal{O} \to X : v \to \tau_X[\beta_v(1)]$. Also $\exp^\gamma_x = \exp^\gamma \mid T_x(X) \cap \mathcal{O}$ for all $x \in X$, and

$$\mathrm{Exp}^{\gamma} : \mathfrak{S} \to X \times X : v_x \to (x, \exp^{\gamma}(x)).$$

Note that if X is of class C^{r+2}, then Exp^{γ} is of class C^r.

9.3 Theorem: $D(\exp_x^{\gamma})(o_x)$ is the identity map of $T_x(X)$.

Proof. First see that

$$(1) \qquad \beta_{\lambda_v}(t) = \lambda\beta_v(\lambda t).$$

For $\lambda\beta_v(\lambda t)$ is a solution of γ at v, as

$$\frac{d}{dt}\,|\lambda\beta_v(\lambda t)| = \lambda Th_{\lambda}\,\gamma\left[\beta_v(\lambda t)\right] = \gamma\left[\beta_v(t)\right]$$

by the spray property, so (1) follows from the uniqueness of solutions. Next we have

$$(2) \qquad \tau_x\left[\beta_v(t)\right] = \tau_x\left[\beta_{tv}(1)\right],$$

from (1), by taking $t = 1$ and $\lambda = t$, and applying τ_x. Finally we see that

$$D(\exp_x^{\gamma})(o_x)(v) = \frac{d}{dt}\left[\tau_x\beta_{tv}(1)\right]_{t=o} = \frac{d}{dt}\left[\tau_x\beta_v(t)\right]_{t=o}$$

by (2), and thus

$$D(\exp_x^{\gamma})(o_x)(v) = T\tau_x(\gamma_v) = v.$$

9.4 Corollary: There exists a neighborhood U of $o_x \in T_x(X)$ such that $\exp_x^{\gamma} \mid U$ is a diffeomorphism onto a neighborhood of $x \in X$.

Let $\gamma_o \in \Gamma(\tau_x)$ denote the zero cross-section

9.5 Corollary: If X is C^{r+2} and paracompact, there exists a neighborhood U of $\mathrm{Im}(\gamma_o) \subset \mathfrak{S}$ such that $\mathrm{Exp}^{\gamma} \mid U$ is a C^r diffeomorphism of U onto a neighborhood of the diagonal in $X \times X$, and carries

$\mathrm{Im}(\gamma_o)$ *onto the diagonal.*

Proof: See Lang [3, p. 74].

Let $X \subset Y$ be a submanifold, and $f : X \to Y$ the inclusion map. Then $T(f) : T(X) \to T(Y)$, and $T(f)^* [T(T(Y))] = T(T(X))$. Thus if $\gamma \in \Gamma(\tau_{T(Y)})$, $\gamma \circ T(f)$ factors through $T(T(X))$. We denote by $\gamma \mid X \in \Gamma(\tau_{T(X)})$ the unique factor.

9.6 Definition: Let $X \subset Y$ be a submanifold, and $\gamma \in \Gamma(\tau_{T(Y)})$. Then γ is a *spray of the pair* (Y,X) iff γ and $\gamma|X$ are sprays.

9.7 Theorem: *Let* Y *be a* C^{r+2} *manifold admitting* C^{r+1} *partitions of unity, and* $X \subset Y$ *a closed submanifold. Then there exists a spray of the pair* (Y,X).

Proof: Let $\{U_i,\theta_i)\}$ be a locally finite atlas on Y with associated C^{r+1} partition of unity $\{g_i\}$, and $\{(V_i,\varphi_i)\}$ an associated local bundle chart on $T(Y)$, $V_i = \tau_y^{-1}(U_i)$. Then $\{G_i\}$ is an associated C^{r+1} partition of unity for $\{(V_i,\varphi_i)\}$, where $G_i = g_i \circ \tau_y \mid V_i$. Let $\{(W_i,\psi_i)\}$ be an atlas of $T(T(Y))$ associated to $\{(V_i,\varphi_i)\}$, so $W_i = \tau_{T(Y)}^{-1}(V_i)$. For each i let γ_i be the spray on U_i with local representative in $\{(W_i,\psi_i)\}$,

$$\gamma_i'(u,v) = (u,v;v,o),$$

and $\gamma = \sum G_i\gamma_i$. Clearly γ is a spray on Y. Now we assume that that the first atlas $\{(U_i,\Theta_i)\}$ has been chosen such that whenever $U_i \cap X \neq \emptyset$, then $\Theta_i : U_i \to \mathbb{E}_1 \times \mathbb{E}_2$, with $\Theta_i \mid U_i \cap V : U_i \cap V \to \mathbb{E}_1 \times 0$. As $X \subset Y$ is closed, this may always be done. Then it is clear that $\gamma \mid X$ is a spray on X, and thus γ is a spray of the pair (Y,X).

Now let $X \subset Y$ be a manifold, and γ a spray of the pair (Y,X). Then we have the diagram

$$\begin{tikzcd}
TX \arrow[r] & f^*(TY) \arrow[r] & TY \\
\mathcal{D}_X \arrow[u, phantom, "\cup" description] \arrow[r, "f^*\tau_X"] \arrow[d, "\mathrm{Exp}^\gamma | X"] & \mathcal{D}_{f^*} \arrow[u, phantom, "\cup" description] \arrow[r, "\tau_Y^* f"] \arrow[d, "f^*\mathrm{Exp}^\gamma"] & \mathcal{D}_Y \arrow[u, phantom, "\cup" description] \arrow[d, "\mathrm{Exp}^\gamma"] \\
X \times X \arrow[r] & X \times Y \arrow[r] & Y \times Y
\end{tikzcd}$$

where $\mathcal{D}_{f^*} = f^*(\mathcal{D}_Y)$ and $f^*\mathrm{Exp}^\gamma$ is defined by

$$f^*\mathrm{Exp}^\gamma(x, v_{f(x)}) = (x, \exp^\gamma(v_{f(x)})).$$

9.8 Theorem: *If* Y *admits partitions of unity, then* $f^*\mathrm{Exp}^\gamma$ *is a diffeomorphism of* D_{f^*} *onto a neighborhood of* $X \times X$ *in* $X \times Y$.

9.9 Definition: If $X \subset Y$ is submanifold, then a *tubular neighborhood* of (Y,X) is a pair (π, f) such that $\pi : E \to X$ is a bundle and f is a diffeomorphism from a neighborhood of the zero cross-section $\gamma_o \subset E$ onto a neighborhood of X in Y.

9.10 Theorem: *If* Y *admits partitions of unity and* X *is a closed submanifold of* Y, *then there exists a tubular neighborhood of* (Y,X).

Proof: As Y admits partitions of unity, so does X, so $f^*(\tau_y) \approx \tau_x \oplus \nu_f$, where $f : X \to Y$ is the inclusion map. Then if γ is a spray for (Y,X), $(\nu_f, f^*\,\mathrm{Exp}^\gamma)$ is a tubular neighborhood for (Y,X).

10. Vertical tangents

In this section the notion of partial derivative is extended to maps of Banach space bundles. The main goal is a global version of the composition theorem, 3.7. Let $\pi : E \to X$ be a C^r bundle. If $\tau_X : TX \to X$ is the tangent bundle of X, the mapping $T\pi : TE \to TX$ factors through the induced C^{r-1} bundle $\pi^*\tau_X : \pi^*TX \to E$ and we have an exact sequence

$$\tau_E \xrightarrow{\pi^*(T\pi)} \pi^*\tau_X \longrightarrow 0.$$

Let $VT(E) = \mathrm{Ker}\ (\pi^*(T\pi))$.

<u>10.1 Definition:</u> The C^{r-1} bundle $V\tau_E = \tau_E \big| VT(E) : VT(E) \to E$ is the <u>vertical tangent bundle</u> of E. The <u>s-th tangent bundle</u> is defined recursively $V\tau_E^s = \mathrm{Ker}\ \pi^*(T(V\tau_E^{s-1}))$, $s \leq r$, a C^{r-s} bundle.

Note that the vertical tangent bundle of E consists of the subspaces of tangent spaces T_eE tangent to the fiber through e, E_e, that is, $VT_eE = T_e(E_e) < T_eE$.

Let F be a C^r manifold and $f : E \to F$ a mapping. For each point $e \in E$ let f_e denote the restriction of f to the fiber through e, $f_e = f|E_e$. Suppose for each e, f_e is C^s. Then we may define the s-th vertical tangent of f, $VT^s(f)$, as the mapping

$$VT^s(f) : VT^s(E) \to T^s(G) : (v_e^1, \ldots, v_e^s) \to T^s f_e(v_e^1, \ldots, v_e^s).$$

10.2 Definition: A mapping $f : E \to F$ is vertically C^t iff $VT^t(f)$ is defined and continuous, $t \leq r$, and with respect to local bundle charts on E, the local representatives are of class C_2^t (second factor along fibers). It is of class (C^s, VC^t) iff it is C^s and vertically C^t, $0 \leq s \leq t \leq r$.

Let $\rho: F \to Y$ be another C^r bundle.

10.3 Definition: A mapping $f : E \to F$ is fiber-preserving iff there exists a mapping $f_B : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & F \arrow[d, "\rho"] \\ X \arrow[r, "f_B"] & Y \end{tikzcd}$$

Let G be a C^r manifold. Then the following is immediate.

10.4 Theorem: If $f : E \to F$ and $g : F \to G$ are of class (C^s, VC^t), and f is fiber preserving, then $g \circ f : E \to G$ is of class (C^s, VC^t), $0 \leq s \leq t \leq r$.

Let $\Gamma^s(\pi)$ and $\Gamma^s(\rho)$ denote the Banach spaces of C^s sections of π and ρ, $0 \leq s \leq r$, $X = Y$ compact. If $U \subset E$ is an open set such that $\pi|U : U \to X$ is surjective, let $\Gamma^s(U) \subset \Gamma^s(\pi)$ denote the open set of sections with image in U. If $f : U \to F$ is a C^s fiber-preserving mapping, let Ω_f denote the composition mapping induced by f,

$$\Omega_f : \Gamma^s(U) \to \Gamma^s(\rho) : \gamma \to f \circ \gamma.$$

The local composition theorem 3.7 may be globalized as follows (suggested by R. Palais [4]).

10.5 Omega lemma: If $f: U \to F$ is of class (C^s, VC^{s+t}), $0 \leq s \leq s + t$, and X compact, then Ω_f is of class C^t.

The proof is immediate from 3.8.

Let X and Y be compact C^r manifolds without boundary, and $\pi : E \to Y$ a C^r bundle. Then if $f \in \mathcal{C}^r(X,Y)$, we have an induced mapping $A_f : \Gamma^r(\pi) \to \Gamma^r(f^*\pi) : \gamma \to \gamma \circ f$. The following generalizes and follows easily from the local composition lemma 3.5.

10.6 Alpha lemma: If $f \in \mathcal{C}^r(X,Y)$, then $A_f : \Gamma^r(\pi) \to \Gamma^r(f^*\pi)$ is a continuous linear mapping.

11. Manifolds of mappings

This section presents one of the main results of the notes, the manifold structure of the set of C^r mappings $C^r(X,Y)$. This manifold and its submanifolds are the most important non-trivial examples of Banach manifolds in current use. The form of this presentation is largely due to R. Palais [4].

Let X be a compact C^r manifold, $r \geq 1$, and Y a C^{r+s+2} manifold admitting partitions of unity. We shall construct a C^s differential structure for $\mathcal{C}^r(X,Y)$. The construction is easily generalized in case X is a manifold with boundary (for definitions, see Lang [3, p. 30]).

If $\mathcal{S} : TY \to T^2Y$ is a C^{r+s} spray on Y, recall that there is a neighborhood $\mathcal{O}_\mathcal{S} \subset TY$ of the zero-section and a neighborhood $\mathcal{F}_\mathcal{S} \subset Y \times Y$ of the diagonal such that $\mathrm{Exp}^\mathcal{S} : \mathcal{O}_\mathcal{S} \to \mathcal{F}_\mathcal{S}$ is a C^{r+s} diffeomorphism, by 9.5.

If $f \in \mathcal{C}^r(X,Y)$, we have, as in section 9, the diffeomorphism $\mathcal{s}_f \equiv f^*\mathrm{Exp}^{\mathcal{s}}: f^*\mathcal{O}_{\mathcal{s}} \to \mathcal{S}_{f,\mathcal{s}}$ where $\mathcal{S}_{f,\mathcal{s}} \subset X \times Y$ is a neighborhood of the graph of f. If $U_{f,\mathcal{s}} \subset \mathcal{C}^r(X,Y)$ consists of maps g such that graph $(g) \subset \mathcal{S}_{f,\mathcal{s}}$, then $(U_{f,\mathcal{s}}, \Omega_{\mathcal{s}_f^{-1}})$ is evidently a chart for $\mathcal{C}^r(X,Y)$, where

$$\Omega_{\mathcal{s}_f^{-1}} : U_{f,\mathcal{s}} \to \Gamma^r(f^*\tau_Y) : g \to \mathcal{s}_f^{-1} \circ g.$$

We shall call such a chart <u>natural</u> for $\mathcal{C}^r(X,Y)$, and the collection of all natural charts the <u>natural atlas</u>.

<u>11.1 Theorem</u>: <u>If</u> X <u>is a compact</u> C^r <u>manifold and</u> Y <u>a</u> C^{r+s+2} <u>manifold, admitting partitions of unity, then the natural atlas of</u> $\mathcal{C}^r(X,Y)$ <u>is of class</u> C^s.

Proof: Let $(U_{f,\mathcal{s}}, \varphi_{f,\mathcal{s}})$ and $(U_{f',\mathcal{s}'}, \varphi_{f',\mathcal{s}'})$ be natural charts, and suppose $U_{f,\mathcal{s}} = \varphi_{f',\mathcal{s}'}$. It suffices to show that $\varphi_{f',\mathcal{s}'} \circ \varphi_{f,\mathcal{s}}^{-1}$ is a C^s diffeomorphism. But it is clear that

$$\varphi_{f',\mathcal{s}'} \circ \varphi_{f,\mathcal{s}}(\gamma) = \Omega_F(\gamma) \equiv F \circ \gamma,$$

where
$$F = \left[f'^* \ \mathrm{Exp}^{\mathcal{s}'}\right]^{-1} \circ f^* \ \mathrm{Exp}^{\mathcal{s}}.$$

But $\mathcal{s}$ and $\mathcal{s}'$ are C^{r+s} sprays, and f and f' are C^r, so it is evident that F is of class (C^r, VC^{r+s}). By the omega lemma 10.5, and the compactness of X, Ω_F is of class C^s. Clearly $(\Omega_F)^{-1} = \Omega_{F^{-1}}$, so Ω_F is a C^s diffeomorphism.

Hereafter $\mathcal{C}^r(X,Y)$ shall denote the C^s manifold determined by the natural atlas, if X is compact C^r and Y is C^{r+s+2} and admits partitions of unity.

Note: If $f \in \mathcal{C}^r(X,Y)$, the tangent space $T_f\mathcal{C}^r(X,Y)$ may be identified with $\Gamma^r(f^*TY)$.

Let X and Y be compact C^r manifolds, and Z a C^{r+s+2} manifold admitting partitions of unity. If $f \in \mathcal{C}^r(X,Y)$, we have the induced map

$$\alpha_f : \mathcal{C}^r(Y,Z) \to \mathcal{C}^r(X,Z) : g \to g \circ f.$$

11.2 Alpha Theorem: If $0 \leq s \leq r$, then α_f is of class C^s.

Proof: Let (U_g, φ_g) be a natural chart at g, so $\varphi_g : U_g \to \Gamma^r(\mathcal{O}_g) \subset \Gamma^r(g^*TZ)$. Then there is a natural chart $(U_{\alpha_f g}, \varphi_{\alpha_f g})$ with $\varphi_{\alpha_f g} : U_{\alpha_f g} \to \Gamma^r(\mathcal{O}_{\alpha_f g}) \subset \Gamma^r(f^*g^*TZ)$ determined by the same spray, $\mathcal{O}_{\alpha_f g} = f^*\mathcal{O}_g$, and $\alpha_f : U_g \to \varphi_{\alpha_f g}$. With respect to these local charts the local representative of α_f is the map

$$A_f : \Gamma^r(\mathcal{O}_g) \to \Gamma^r(\mathcal{O}_{\alpha_f g}) : \gamma \to \gamma \circ f.$$

By the Alpha lemma 10.6, A_f is a continuous linear mapping, so α_f is of class C^s.

Now let X be a compact C^r manifold, Y and Z C^{r+s+2} manifolds admitting partitions of unity, and $g : Y \to Z$ a C^r mapping. Then we have the induced mapping

$$\omega_g : \mathcal{C}^r(X,Y) \to \mathcal{C}^r(X,Z) : f \to g \circ f.$$

11.3 Omega Theorem: If $0 \leq s \leq r$, and g is of class C^{r+s}, then ω_g is of class C^s.

Proof: It is easily verified that the local representative of ω_g with respect to natural charts is of the form

$$\Omega_G : \Gamma^r(\mathcal{O}_f) \to \Gamma^r(\omega_g f^*TZ) \; : \; \gamma \to G \circ \gamma \, ,$$

where $\mathcal{O}_f \subset f^*TY$ and $G : \mathcal{O}_f \to \omega_g f^*TZ$ is of class (C^r, C^{r+s}). Thus ω_g is of class C^s by the Omega lemma, 10.5.

<u>11.4 Theorem</u>: <u>If</u> $g : Y \to Z$ <u>is a closed C^{r+s} embedding</u>, $1 \leq s \leq r$, <u>then</u> $\omega_g : \mathcal{C}^r(X,Y) \to \mathcal{C}^r(X,Z) : f \to g \circ f$ <u>is a closed C^s embedding</u>.

Proof: As Z is of class C^{r+s+2} and admits partitions of unity, and $g(Y) \subset Z$ is a closed submanifold, there exists a spray of the pair $(Z, g(Y))$ by 9.7.

If $f \in \mathcal{C}^r(X,Y)$, then $\omega_g f^*TZ = (g \circ f)^* TZ = f^*g^*TZ$. But under the hypotheses above g^*TZ splits, and we may write $g^*TZ = TY \oplus NY$, where NY is the normal bundle of g. Thus $\omega_g f^*TZ = f^*TY \oplus f^*NY$, and $\Gamma^r(\omega_g f^*TZ) \approx \Gamma^r(f^*TY) \times \Gamma^r(f^*NY)$.

Now let $(U_{\omega_g f}, \varphi_{\omega_g f})$ be a natural chart at $\omega_g f \in \mathcal{C}^r(X,Z)$ determined by the spray $\sphericalangle$ of the pair $(Z, g(Y))$, so

$$\varphi_{\omega_g f} : U_{\omega_g f} \to \Gamma^r(f^*TY) \times \Gamma^r(f^*NY).$$

As $\sphericalangle$ is a spray of the pair, it is immediate that this chart has the submanifold property,

$$\varphi_{\omega_g f} : U_{\omega_g f} \cap \omega_g\left[\mathcal{C}^r(X,Y)\right] \to \Gamma^r(f^*TY) \times 0.$$

Thus ω_g is an embedding. It is of class C^s by the Omega Theorem 11.3, and as $g(Y) \subset Z$ is closed, and X compact, it is evident that $\omega_g[\mathcal{C}^r(X,Y)] \subset \mathcal{C}^r(X,Z)$ is closed.

Many other theorems of this type identifying distinguished submanifolds of $\mathcal{C}^r(X,Z)$, suggest themselves, especially if X has boundary. Several of these have been proved recently by Smale in connection with calculus of variations in the large. Recalling the embedding theorem 7.4, we obtain a useful embedding of $C^r(X,Y)$.

11.5 Corollary: If X is compact C^r and Y is second countable C^{r+s+2} and modelled on a separable Hilbert space, then $\mathcal{C}^r(X,Y)$ can be C^s embedded onto a closed submanifold of a Banach space.

For any C^r manifolds X and Y we have the evaluation mapping

$$ev : \mathcal{C}^r(X,Y) \times X \to Y : (f,x) \to f(x).$$

We now examine the differentiability of this mapping.

11.6 Lemma: If $\pi : E \to X$ is a C^r bundle and X is compact, then the evaluation mapping

$$Ev : \Gamma^r(\pi) \times X \to E : (\gamma,x) \to \gamma(x)$$

of of class C^r.

Proof: We shall show equivalently that Ev is of class (C^r_1, C^r_2) in the sense of the partial derivative rule, 3.6. First, it is evident that Ev is continuous. Now if $x \in X$, we have the partial mapping

$$\mathrm{Ev}_x : \Gamma^r(\pi) \to E : \gamma \to \gamma(x),$$

which is a continuous linear mapping, thus $D_2^p\ \mathrm{Ev} = \mathrm{Ev}$, and Ev is of class C_2^r. On the other hand for each section $\gamma \in \Gamma^r(\pi)$ we have the partial mapping

$$\mathrm{Ev}_\gamma : X \to E : x \to \gamma(x),$$

or $\mathrm{Ev}_\gamma = \gamma$. Thus the partial tangent of Ev is

$$T_1^r\ \mathrm{Ev} : \Gamma^r(\pi) \times T^rX \to T^rE : (\gamma, v_x^r) \to T^r\gamma(v_x^r).$$

Hence if $T^r\pi : T^rE \to T^rX$ denotes the r-th tangent of the bundle π, and

$$T^r : \Gamma^r(\pi) \to \Gamma^o(T^r\pi) : \gamma \to T^r\gamma,$$

we see that T_1^r Ev is the composite of the restriction to $T^r(\Gamma^r(\pi))$ of the mapping

$$\mathrm{Ev}^r : \Gamma^o(T^r\pi) \times T^rX \to T^rE : (\xi, p) \to \xi(p)$$

following

$$T^r \times \mathrm{id} : \Gamma^r(\pi) \times T^rX \to \Gamma^o(T^r\pi) \times T^rX,$$

or $T^r\mathrm{Ev} = \mathrm{Ev}^r \circ (T^r \times \mathrm{id})$. As T^r and Ev^r are continuous, Ev is of class C_1^r, completing the proof.

We suppose now that X is compact C^r and Y is C^{r+s+2} with partitions of unity, so $\mathcal{C}^r(X,Y)$ is a C^s manifold.

11.7 Theorem: If $0 \leq s \leq r$, then the evaluation mapping

$$\mathrm{ev} : \mathcal{C}^r(X,Y) \times X \to Y : (f,x) \to f(x)$$

if of class C^s.

Proof: We shall show equivalently that the enlarged evaluation mapping

$$\mathrm{Ev} : \mathcal{C}^r(X,Y) \times X \to X \times Y : (f,x) \to (x,f(x))$$

is of class C^s. If $f \in \mathcal{C}^r(X,Y)$, and $\mathcal{S}$ is a spray of Y, we have the natural charts (U,φ) at $f \in \mathcal{C}^r(X,Y)$ and (V,ψ) at graph $(f) \subset X \times Y$, where

$$\varphi : U \to \Gamma^r(f^*TY) : g \to \varphi_g,$$

$$\varphi_g(x) = (f^*\mathrm{Exp}^{\mathcal{S}})^{-1}(g(x)), \text{ and}$$

$$\psi : V \to f^*TY : (x,y) \to (f^*\mathrm{Exp}^{\mathcal{S}})^{-1}(y).$$

With respect to these natural charts the enlarged evaluation mapping induces a partially local representative Ev', which is a restriction of the mapping

$$\mathrm{Ev}' : \Gamma^r(f^*TY) \times X \to f^*TY : (\gamma,x) \to \gamma(x)$$

But this is a C^r mapping by the lemma, 11.6.

CHAPTER III

TRANSVERSALITY OF MAPPINGS

In this chapter the notion of transversality or general position of mappings is used to obtain some standard results of differential topology. Throughout this chapter X will be a manifold with boundary, Y a manifold, $W \subset Y$ a submanifold (Y, W without boundary) all of class C^r.

12. Elementary properties

12.1 Definition: A C^r mapping $f : X \to Y$ is transversal to W at a point $x \in X$ iff either $f(x) \notin W$, or $f(x) = w \in W$ and there exists a neighborhood U of $x \in X$ and a local chart (V,ψ) at $w \in Y$ such that

$$\psi : V \to \mathbb{E} \times \mathbb{F} : V \cap W \to \mathbb{E} \times 0,$$

$\pi_1 \circ \psi$ is a diffeomorphism of V onto an open set of $\mathbb{E}$, and $\pi_2 \circ \psi \circ f \mid U$ is a submersion, where $\pi_1 : \mathbb{E} \times \mathbb{F} \to \mathbb{E}$ and $\pi_2 : \mathbb{E} \times \mathbb{F} \to \mathbb{F}$ are the projections. The mapping f is transversal to W on a subset $K \subset X$, $f|K \pitchfork W$, iff f is transversal to W at every point $x \in K$, and f is transversal to W, $f \pitchfork W$, iff $f|X \pitchfork W$.

For proofs of the following see Lang [3, p. 22].

12.2 Theorem: A C^r mapping $f : X \to Y$ is transversal to W at a point $x \in X$ such that $f(x) = w \in W$ iff the composite mapping

$$T_xX \xrightarrow{T_xf} T_wY \to T_wY/T_wW$$

is a splitting surjection.

12.3 Corollary: If $W \subset Y$ has finite codimension, then a C^r mapping $f : X \to Y$ is transversal to W at a point $x \in X$ such that $f(x) = w \in W$ iff

$$T_x f(T_x X) + T_w W = T_w Y.$$

12.4 Theorem: If a C^r mapping $f : X \to Y$ is transversal to W, then the inverse image $f^{-1}(W)$ is a submanifold of X. If also $W \subset Y$ has finite codimension k, then $f^{-1}(W) \subset X$ has finite codimension k.

13. Openness of transversality

In the applications the following question arises. Let $\mathcal{A}$ be a space of mappings from X to Y, $K \subset X$ a subset, and $W \subset Y$ a submanifold. Let $\mathcal{A}_{K,W} = \{f \in \mathcal{A} \mid f|K \pitchfork W\}$. When is $\mathcal{A}_{K,W} \subset \mathcal{A}$ open? In this section sufficient conditions for openness are given.

13.1 Definition: A C^r manifold of mappings from X to Y is a subset $\mathcal{A} \subset \mathcal{C}^r(X,Y)$ which is a C^r manifold such that the evaluation mapping

$$ev : \mathcal{A} \times X \to Y : (f,x) \to f(x)$$

is of class C^r.

Theorems 11.1 and 11.7 imply that the manifolds $\mathcal{C}^r(X,Y)$ (Y of class C^{2r+2}) are C^r manifolds of mappings. Also, any C^r submanifold of $\mathcal{C}^r(X,Y)$ is a C^r manifold of mappings.

We will need a fundamental lemma on linear mappings. Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces, and $L(\mathbb{E}, \mathbb{F})$ the usual Banach space of continuous linear mappings from $\mathbb{E}$ to $\mathbb{F}$. Let $IL(\mathbb{E}, \mathbb{F})$ denote the subset of splitting injections, $SL(\mathbb{E}, \mathbb{F})$ the subset of splitting surjections, and

$BL(\mathbb{E}, \mathbb{F})$ the subset of linear isomorphism.

13.2 Lemma: The subspaces $IL(\mathbb{E}, \mathbb{F})$, $SL(\mathbb{E}, \mathbb{F})$, and $BL(\mathbb{E}, \mathbb{F})$ are open in $L(\mathbb{E}, \mathbb{F})$.

Proof: For the proof that $BL(\mathbb{E}, \mathbb{F})$ is open, see Lang [3, p. 5]. Suppose $T \in IL(\mathbb{E}, \mathbb{F})$, and $\mathbb{M} = Im(T)$. Then if $\mathbb{K}$ is a complement to $\mathbb{M}$ in $\mathbb{F}$, we have the map $T' \in BL(\mathbb{E} + \mathbb{K}, \mathbb{F})$ defined by

$$T' : \mathbb{E} + \mathbb{K} \to \mathbb{M} + \mathbb{K} : (e, k) \to (T(e), k).$$

But $BL(\mathbb{E} + \mathbb{K}, \mathbb{F})$ is open in $L(\mathbb{E} + \mathbb{K}, \mathbb{F})$, and the linear mapping

$$\rho_{\mathbb{E}} : L(\mathbb{E} + \mathbb{K}, \mathbb{F}) \to L(\mathbb{E}, \mathbb{F}) : T' \to T'|\mathbb{E}$$

is open, so there is a neighborhood η of T' in $BL(\mathbb{E} + \mathbb{K}, \mathbb{F})$ such that $\rho_{\mathbb{E}}(\eta) \subset IL(\mathbb{E}, \mathbb{F})$ is a neighborhood of T in $L(\mathbb{E}, \mathbb{F})$. Finally suppose $T \in SL(\mathbb{E}, \mathbb{F})$, $\mathbb{N} = \ker(T)$, and $\mathbb{K}$ is a complement to $\mathbb{N}$ in $\mathbb{E}$. Then we have the map $T' \in BL(\mathbb{E}, \mathbb{N} + \mathbb{F})$ defined by

$$T' : \mathbb{N} + \mathbb{K} \to \mathbb{N} + \mathbb{F} : (n, k) \to (n, T(k)).$$

But again $BL(\mathbb{E}, \mathbb{N} + \mathbb{F})$ is open in $L(\mathbb{E}, \mathbb{N} + \mathbb{F})$, and the linear map

$$\pi_{\mathbb{F}} : L(\mathbb{E}, \mathbb{N} + \mathbb{F}) \to L(\mathbb{E}, \mathbb{F}) : T' \to \pi \circ T',$$

where $\pi : \mathbb{N} + \mathbb{F} \to \mathbb{F}$ is the projection, is open. Thus $SL(\mathbb{E}, \mathbb{F})$ is open in $L(\mathbb{E}, \mathbb{F})$.

Warning: $BL(\mathbb{E},\mathbb{F})$ is not group manifold. X

13.3 Openness lemma: Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces, $U \subset \mathbb{E}$ an open set, $\mathcal{A} \subset \mathcal{C}^r(U, F)$ a C^r manifold of mappings, $K \subset U$ an arbitrary subset, and $\mathcal{A}_{K/S} \subset \mathcal{A}$ the subset of mappings $f : U \to \mathbb{F}$ such that *such that* $f|K$ *is a submersion. Then if* K *is compact,* $\mathcal{A}_{K/S}$ is open.

Proof: Consider the evaluation mapping

$$ev : \mathcal{A} \times U \to \mathbb{F} : (f,x) \to f(x).$$

Then for every point $(f,x) \in \mathcal{A} \times U$ we have the partial mapping $ev_f = ev|\{f\} \times U$, which is identical to f as a mapping of U, and the associated partial derivative

$$D_2 ev : \mathcal{A} \times U \to L(\mathbb{E}, \mathbb{F})$$

which is continuous by hypothesis. Let $SL(\mathbb{E}, \mathbb{F}) \subset L(\mathbb{E}, \mathbb{F})$ denote the subset of linear maps which are splitting surjections. By 13.2 $SL(\mathbb{E}, \mathbb{F})$ is open in $L(\mathbb{E}, \mathbb{F})$. Thus $\mathcal{S} \equiv [D_2 ev]^{-1} [SL(E, F)]$ is open in $\mathcal{A} \times U$. Now suppose $f \in \mathcal{A}_{K/S}$. Then by definition, $\{f\} \times K \subset \mathcal{S}$. Thus if $K \subset U$ is compact, there exists a neighborhood U_f of $f \in \mathcal{A}$ such that $U_f \times K \subset \mathcal{S}$, thus $U_f \subset \mathcal{A}_{K/S}$, and $\mathcal{A}_{K/S}$ is open.

13.4 Openness theorem: *If* $K \subset X$ *is a compact set,* $W \subset Y$ *a closed submanifold, and* $\mathcal{A} \subset \mathcal{C}^r(X,Y)$ *a* C^r *manifold of mappings, then the subset* $\mathcal{A}_{K,W} = \{f \in \mathcal{A} \mid f|K \pitchfork W\}$ *is open in* $\mathcal{A}$.

Proof: As $W \subset Y$ is closed, there exists an atlas $\{(V_i, \psi_i)\}$ of Y such that if $V_i \cap W \neq \emptyset$, then

$$\psi_i : V_i \to \mathbb{E}_i \times \mathbb{F}_i : V_i \cap W \to \mathbb{E}_i \times 0,$$

the latter a diffeomorphism onto an open set. Suppose $f \in \mathcal{A}_{K,W}$. Then for every $x \in K$ such that $f(x) \in V_i \cap W$, there exists a neighborhood U_x of $x \in X$ such that if $\pi_2^i : \mathbb{E}_i \times \mathbb{F}_i \to \mathbb{F}_i$ is the projection,

$$\pi_2^i \circ \psi_i \circ f \mid U_x : U_x \to \mathbb{F}_i$$

is a submersion. If $f(x) \notin W$, there exists a neighborhood U_x of $x \in X$ such that $f(U_x) \cap W = \emptyset$, for $W \subset Y$ is closed and the evaluation map is continuous. As K is compact there exists a finite covering $\{U_i\}$ of K by open sets of X, such that for each i, $f(U_i)$ is contained in an element of the covering $\{V_j\}$ which we will denote by V_i (by reindexing the cover $\{V_j\}$), and such that whenever $V_i \cap W \neq \emptyset$, then

$$\pi_2^i \circ \psi_i \circ f \mid U_i : U_i \to \mathbb{F}_i$$

is a submersion. Let $K_i = \overline{U_i \cap K}$. Then $\mathcal{a}_{K,W} = \bigcap_i \mathcal{a}_{K_i,W}$, but the $\mathcal{a}_{K_i,W}$ are open by 13.3, so $\mathcal{a}_{K,W}$ is a finite intersection of open sets.

14. Density of transversality

In this section sufficient conditions for the density of transversal maps are given.

14.1 Density lemma: If X has finite dimension n, $W \subset Y$ is a closed submanifold having finite codimension q, $\mathcal{a} \subset \mathcal{C}^r(X,Y)$ is a C^r manifold of mappings with $r > \max(n-q,0)$, and the evaluation mapping is transversal to W at a point $(f,x) \in \mathcal{a} \times X$, then there exists a neighborhood $\mathcal{U}$ of $f \in \mathcal{a}$ and a neighborhood V of $x \in X$ such that $\mathcal{U}_{V,W} \subset \mathcal{U}$ is dense.

Proof: First suppose $f(x) \notin W$. Then, as W is closed and the evaluation map is continuous, there exist neighborhoods $\mathcal{U}$ of $f \in \mathcal{a}$

and V of $x \in X$ such that $ev(\mathcal{U} \times V) \cap W = \emptyset$, so $\mathcal{U}_{V,W} = \mathcal{U}$.

Now suppose $f(x) = w \in W$. In this case the proof depends on the three propositions which follow.

Proposition A: If the evaluation map is transversal to W at (f,x) and $f(x) = w \in W$, there are neighborhoods $\mathcal{U}$ of $f \in \mathcal{A}$ and V of $x \in X$ such that every point $g \in \mathcal{U}$ is contained in a p-dimensional submanifold Σ^p_g, $0 \leq p \leq q$, such that $ev \mid \Sigma^p_g \times V \pitchfork W$.

Proof: By definition 12.1 there exists charts $(\mathcal{U}_o, \eta)$ at $f \in \mathcal{A}$, (V_o, φ) at $x \in X$, and (U_o, ψ) at $w \in Y$ such that

(i) $\eta : \mathcal{U}_o \to \mathbb{E}$, $\eta(\mathcal{U}_o) = \mathcal{U}'_o$,

(ii) $\varphi : V_o \to \mathbb{R}^n$, $\varphi(V_o) = V'_o$,

(iii) $ev : \mathcal{U}_o \times V_o \to U_o$,

(iv) $\psi : U_o \to \mathbb{F} \times \mathbb{R}^q : U_o \times W \to \mathbb{F} \times 0$,

the latter a diffeomorphism onto an open set of $\mathbb{F} \times 0$, and

(v) $\alpha \equiv \pi \circ \psi \circ ev| \mathcal{U}_o \times V_o : \mathcal{U}_o \times V_o \to \mathbb{R}^q$ is a submersion,

where $\pi : \mathbb{F} \times \mathbb{R}^q \to \mathbb{R}^q$ is the projection. Define

$$\beta = \alpha \circ |\eta \times \varphi|^{-1} : \mathcal{U}'_o \times V'_o \to \mathbb{R}^q$$

where $\mathcal{U}'_o \times V'_o \subset \mathbb{E} \times \mathbb{R}^n$. Then by hypothesis $D\beta(0,0)$ is a surjection, so there exists a finite dimensional subspace $\mathbb{R}^p < \mathbb{E}$, $0 \leq p \leq q$, such that

(1) $D\beta(0,0)(\mathbb{R}^p \times \mathbb{R}^n) = \mathbb{R}^q$.

Let $\mathbb{F}$ be a closed complement of $\mathbb{R}^p < \mathbb{E}$, and identify $\mathbb{E}$ and $\mathbb{F} \times \mathbb{R}^p$. Then if $\mathcal{V}'_o$ and $\mathcal{W}'_o$ are neighborhoods of zero in $\mathbb{F}$ and $\mathbb{R}^p$, respectively, with $\mathcal{V}'_o \times \mathcal{W}'_o \subset \mathcal{U}'_o$, we have

$$\beta : \mathcal{V}'_o \times (\mathcal{W}'_o \times V'_o) \to \mathbb{R}^q, \ \mathcal{V}'_o \times (\mathcal{W}'_o \times V'_o) \subset \mathbb{F} \times (\mathbb{R}^p \times \mathbb{R}^n),$$

and the first partial derivative of β with respect to the factor $(\mathcal{W}'_o \times V'_o)$ is surjective at $(0,0)$, by equation (1),

$$(2) \qquad D_2\beta(0,0)(\mathbb{R}^p \times \mathbb{R}^n) = \mathbb{R}^q.$$

But the partial derivative map

$$D_2\beta : \mathcal{V}'_o \times (\mathcal{W}'_o \times V'_o) \to L(\mathbb{R}^{p+n}, \mathbb{R}^q)$$

is continuous, and $SL(\mathbb{R}^{p+n}, \mathbb{R}^q) \subset L(\mathbb{R}^{p+n}, \mathbb{R}^q)$ is open by 13.2, so there exist neighborhoods $\mathcal{V}'_1, \mathcal{W}'_1, V'_1$ of zero in $\mathbb{F}$, $\mathbb{R}^p$, $\mathbb{R}^n$, respectively, with $\mathcal{V}'_1 \times \mathcal{W}'_1 \times V'_1 \subset \mathcal{V}'_o \times \mathcal{W}'_o \times V'_o$, such that

$$(3) \qquad D_2\beta(0,0) : \mathcal{V}'_1 \times (\mathcal{W}'_1 \times V'_1) \to SL(\mathbb{R}^{p+n}, \mathbb{R}^q).$$

Now let $\mathcal{U} = \eta^{-1}(\mathcal{V}'_1 \times \mathcal{W}'_1) \subset \mathcal{U}_o$, and $V = \varphi^{-1}(V'_1) \subset V_o$. For every point $g \in \mathcal{U}$ with $\eta(g) = (v,w) \in \mathbb{F} \times \mathbb{R}^p$, let $\Sigma^p_g = \eta^{-1}(\{v\} \times \mathcal{W}'_1)$. Thus Σ^p_g is a p-dimensional submanifold in $\mathcal{U}$, and equation (3) implies that the restricted evaluation map

$$ev \,\big|\, \Sigma^p_g \times V : \ \Sigma^p_g \times V \to Y$$

is transversal to W.

For the second proposition suppose Σ^p is a p-dimensional submanifold of $\mathcal{A}$, V an open set of X, and $\xi = ev|\Sigma^p \times V$ is transversal to W. Then $W' = \xi^{-1}(W)$ is a submanifold of codimension q of $\Sigma^p \times V$. Let $\sigma : W' \to \Sigma^p$ denote the restriction to W' of the projection $\Sigma^p \times V \to \Sigma^p$.

<u>Proposition B</u>: <u>If</u> σ <u>is transversal to a point</u> $f \in \Sigma^p$, <u>then</u> f <u>is transversal to</u> W <u>on</u> V.

Proof: If σ is transversal to $\{f\}$, then for every point $(f,x) \in \Sigma^p \times V$ such that $f(x) = w \in W$ we have by 12.3,

$$T_{(f,x)}\sigma(T_{(f,x)}W') = T_f \Sigma^p.$$

Thus in $\Sigma^p \times V$, the tangent space at (f,x) is the sum

$$(4) \qquad T_{(f,x)}(\Sigma^p \times V) = T_{(f,x)}W' + T_x V.$$

But by hypothesis the restricted evaluation map ξ is transversal to W, so by 12.3 again

$$(5) \qquad T_{(f,x)}\xi\left[T_{(f,x)}(\Sigma^p \times V)\right] + T_w W = T_w Y.$$

Substituting (4) in (5) we have

$$(6) \qquad T_x f(T_x X) + T_w W = T_w Y,$$

as $T_{(f,x)}\xi(T_{(f,x)}W') < T_w W$, and $T_{(f,x)}\xi(T_x V) = T_x f(T_x X)$. Thus for every $x \in V$ such that $f(x) = w \in W$, equation (6) holds, so $f|V \pitchfork W$.

The final proposition is a well known theorem of Sard [5]. If $f : X \to Y$ is any C^1 mapping, a point $y \in Y$ is a <u>critical value</u> of f iff it is false that $f \pitchfork \{y\}$. Let $\mathcal{X}_f \subset Y$ be the set of all critical values of f.

<u>Proposition C (Sard)</u>: <u>If</u> $f : \mathbb{R}^s \to \mathbb{R}^t$ <u>is of class</u> C^r <u>with</u> $r > \max(s-t,0)$, then $\mathcal{X}_f \subset \mathbb{R}^t$ <u>has outer measure zero</u>.

Now to prove the density lemma, see $\dim(\Sigma^p \times V) = p + n$, codim $(W') = q$, so $s = \dim(W') = \max(p+n-q,-1)$, and locally $\sigma: \mathbb{R}^s \to \mathbb{R}^p$, so $t = p$ and $\max(s-t, 0) = \max(n-q, 0)$. Thus the lemma follows at once from the three propositions.

Recall that a <u>residual set</u> in a topological space is a countable intersection of open dense sets, a <u>Baire space</u> is one in which every residual set is dense, and by the Baire category theory every Banach manifold is a Baire space.

<u>14.2 Density theorem</u>: <u>Let</u> X <u>be an n-manifold with boundary</u>, $K \subset X$ <u>any subset</u>, Y <u>a Banach manifold (without boundary), and</u> $W \subset Y$ <u>a closed submanifold (without boundary) of finite codimension</u> q, <u>all of class</u> C^r. <u>Let</u> $\mathcal{A} \subset \mathcal{C}^r(X,Y)$ <u>be a</u> C^r <u>manifold of mappings and</u> $\mathcal{A}_{K,W} = \{f \in \mathcal{A} \mid f|K \pitchfork W\}$. <u>If the evaluation map of</u> $\mathcal{A}$

$$ev : \mathcal{A} \times X \to Y : (f,x) \to f(x)$$

<u>is transversal to</u> W <u>on</u> K <u>and</u> $r > \max(n-q, 0)$, <u>then</u> $\mathcal{A}_{K,W} \subset \mathcal{A}$ <u>is residual</u>.

Proof: First, suppose $K \subset X$ is compact and $f \in \mathcal{A}$. Then by the density lemma 14.1 there exists a finite set of pairs $\{(\mathcal{U}^i, V^i)\}_{i=1}^N$ of neighborhoods $\mathcal{U}^i$ of $f \in \mathcal{A}$ and open sets V^i of X such that $\{V^i\}_{i=1}^N$ covers K and for each i, $\mathcal{U}^i_{V^i,W} \subset \mathcal{U}^i$ is dense. Let $\mathcal{U}^f = \bigcap_{i=1}^N \mathcal{U}^i$ and $V = \bigcup_{i=1}^N V^i$. Then $\mathcal{U}^f_{V,W} = \bigcap_{i=1}^N \mathcal{U}^i_{V^i,W}$ is dense in $\mathcal{U}^f$ and $\mathcal{U}_{V,W} \subset \mathcal{U}_{K,W}$. As every $f \in \mathcal{A}$ has such a neighborhood $\mathcal{U}^f$, $\mathcal{A}_{K,W} \subset \mathcal{A}$ is dense. By the openness theorem 13.4, $\mathcal{A}_{K,W}$ is also open.

Now let $K \subset X$ be arbitrary. Then there is a countable covering $\{K^i\}$ of K by compact sets, and from the above, $\mathcal{A}_{K^i,W} \subset \mathcal{A}$ is open

and dense for each i, so $\bigcap_i \mathcal{A}_{K^i,W}$ is residual. Note $\bigcap \mathcal{A}_{K^i,W} \subset \mathcal{A}_{K,W}$.
Next let $\{\mathcal{U}_\alpha\}$ be a countable set of coverings $\mathcal{U}_\alpha = \{K^i_\alpha\}$ such that $\mathcal{K}_\alpha = \bigcup_i K^i_\alpha$ and $\bigcap_\alpha \mathcal{K}_\alpha = K$. Then for each α $\mathcal{A}_{\mathcal{K}_\alpha,W} = \bigcap_i \mathcal{A}_{K^i_\alpha,W}$ is residual, so $\mathcal{A}_{K,W} = \bigcap_\alpha \mathcal{A}_{\mathcal{K}_\alpha,W}$ is residual.

In the density theorem the condition that $W \subset Y$ be closed is undesirable and unneccary. In fact the embedding $e : W \hookrightarrow Y$ can be replaced by an arbitrary mapping $f : W \to Y$ as follows.

Let X be a manifold with boundary, Y and Z manifolds (without boundary), $f : X \to Y$ and $g : Z \to Y$ differentiable maps.
Let $\Delta \subset Y \times Y$ denote the diagonal,

$$\Delta = \{(y_1,y_2) \in Y \times Y \mid y_1 = y_2\}.$$

Clearly $\Delta \subset Y \times Y$ is a closed **submanifold.**

14.3 Definition: The mappings $f : X \to Y$ and $g : Z \to Y$ are transversal at points $x \in X$ and $z \in Z$ iff the product $f \times g : X \times Z \to Y \times Y$ is transversal to Δ at (x,z) in the sense of definition 12.1. The mappings are transversal on sets $K \subset X$ and $M \subset Z$, $f|K \pitchfork g|M$, iff $f \times g|K \times M \pitchfork \Delta$, and they are transversal, $f \pitchfork g$, iff $f \times g \pitchfork \Delta$.

This definition displays an inherent symmetry of the notion of transversality. The definition can obviously be extended to n-tuples of maps having a common target. Further, to make the symmetry complete one should let all sources have boundaries, in which case their product would be a manifold with corners. All the considerations of this section may be extended in both of these directions without substantial modification, but we omit these sophistications as they will not be required in our applications.

For a final version of the density theorem, consider C^r manifolds of mappings $\mathcal{A} \subset \mathcal{C}^r(X,Y)$ and $\mathcal{B} \subset \mathcal{C}^r(Z,Y)$, X with boundary, Y and Z without, X, Y and Z finite dimensional. Let $K \subset X$ and $M \subset Z$ be arbitrary subsets, and

$$\mathcal{A} \underset{K \times M}{\times} \mathcal{B} = \{(f,g) \in \mathcal{A} \times \mathcal{B} \mid f|K \pitchfork g|M\}.$$

Let $ev_{\mathcal{A}}$ and $ev_{\mathcal{B}}$ denote the evaluation maps of $\mathcal{A}$ and $\mathcal{B}$, respectively.

<u>14.4 Corollary</u>: <u>If</u> $ev_{\mathcal{A}} \mid \mathcal{A} \times K \pitchfork ev_{\mathcal{B}} \mid \mathcal{B} \times M$ <u>and</u> $r > \max(\dim X + \dim Z - \dim Y, 0)$, <u>then</u> $\mathcal{A} \times \mathcal{B}_{K \times M} \subset \mathcal{A} \times \mathcal{B}$ <u>is residual</u>.

This follows at once from 14.2 and 14.3. For a special case generalizing 14.2, take $\mathcal{B}$ to be a single map $\mathcal{B} = \{g\}$ in 14.4. Then if $\mathcal{A}_{K,g} = \{f \in \mathcal{A} \mid f|K \pitchfork g|Z\}$, we have:

<u>14.5 Corollary</u>: <u>If</u> $ev_{\mathcal{A}} \mid \mathcal{A} \times K \pitchfork g$ <u>and</u> $r > \max(\dim X + \dim Z - \dim Y, 0)$, <u>then</u> $\mathcal{A}_{K,g} \subset \mathcal{A}$ <u>is residual</u>.

If in 14.5 g is an embedding, and W its image, we obtain 14.2 with the condition "W closed" removed.

<u>15. Jets</u>

We shall give several applications of the density theorem, but the first of these requires a digression on jets. Let X and Y be C^r manifolds (without boundary) and $T^k X = T(T^{k-1} X)$ the k-th iterated tangent bundle space of X. If $f : X \to Y$ is a C^k mapping, $0 \leq k \leq r$, let $T^k f = T(T^{k-1}f) : T^k X \to T^k Y$ denote the iterated tangent mapping.

15.1 Definition: Two mappings $f, g \in \mathcal{C}^k(X,Y)$ are k-equivalent at a point $x \in X$, $f \underset{x}{\overset{k}{\sim}} g$, iff $T^k_x f = T^k_x g$.

It is evident that k-equivalence implies p-equivalence for $0 \leq p \leq k$. Thus we have the following local characterization of k-equivalence. Let (U,φ) and (V,ψ) be local charts at $x \in X$ and $y \in Y$, $f, g \in \mathcal{C}^k(X,Y)$ such that $f(x) = g(x) = y$, $f, g : U \to V$, and let $f', g' : U' \to V'$ denote the local representatives of f and g, where $U' = \varphi U$, $V' = \psi V$.

15.2 Lemma: The mappings f and g are k-equivalent at $x \in X$ iff $D^p f'(x') = D^p g'(x')$, $p = 0,\ldots,k$.

15.3 Definition: If $f \in \mathcal{C}^k(X,Y)$ and $f(x) = y$, let $[f]^k_{(x,y)}$ denote the $\underset{x}{\overset{k}{\sim}}$-class of f. The set $J^k(X,Y)$ of all equivalence classes $[f]^k_{(x,y)}$ with k fixed and $f \in \mathcal{C}^k(X,Y)$ is the k-jet bundle of X and Y. The natural map

$$\pi^k : J^k(X,Y) \to X \times Y : [f]^k_{(x,y)} \to (x,y)$$

is the k-jet projection. If $f \in \mathcal{C}^k(X,Y)$, the induced mapping

$$j^k f : X \to J^k(X,Y) : x \to [f]^k_{(x,f(x))}$$

is the k-jet extension of f.

(Beware: The jet bundle is not a Banach bundle.)

We shall describe some natural local charts for the k-jet bundle. Let (U,φ) and (V,ψ) be local charts on X and Y, respectively, $\varphi : U \to \mathbb{E}$,

$\psi : V \to \mathbb{F}$. If $f \in \mathcal{C}^k(X,Y)$, let $f' \in \mathcal{C}^k(U',V')$ denote the induced local representative of f. Let $W = (\pi^k)^{-1}(U \times V)$, and

$$\eta : W \to U' \times V' \times L(\mathbb{E}, \mathbb{F}) \times \ldots \times L^k_s(\mathbb{E}, \mathbb{F}) :$$
$$: [f]^k_{(x,y)} \to (x', y', Df'(x'), \ldots, D^k f'(x'))$$

where $(x,y) \in U \times V$, $y = f(x)$, $x' = \varphi(x)$, and $y' = \psi(y)$. By lemma 15.2 it is clear that η is a bijection, and $\eta(W)$ is an open set in the Banach space $\mathbb{E} \times \mathbb{F} \times L(\mathbb{E}, \mathbb{F}) \times \ldots \times L^k_s(\mathbb{E}, \mathbb{F})$. The pair (W,η) is a natural local chart for the set $J^k(X,Y)$. If (U^i,φ^i) and (V^α,ψ^α) are C^r atlases for X and Y, respectively, then the associated natural local charts $(W^{i\alpha}, \eta^{i\alpha})$, where $W^{i\alpha} = (\pi^k)^{-1}(U^i \times V^\alpha)$, comprise a natural atlas for $J^k(X,Y)$.

15.4 Theorem: If X and Y are C^r and $k \leq r$, then every natural atlas of the k-jet bundle $J^k(X,Y)$ is of class C^{r-k}.

Proof: Let (U,φ_1) and (U,φ_2) be local charts X with $\varphi_i : U \to \mathbb{E}$, $i = 1,2$, and (V,ψ_1) and (V,ψ_2) local charts on Y with $\psi_i : V \to \mathbb{F}$, $i = 1,2$. Let $W = (\pi^k)^{-1}(U \times V)$ and

$$\eta_i : W \to U' \times V' \times L(\mathbb{E}, \mathbb{F}) \times \ldots \times L^k_s(\mathbb{E}, \mathbb{F}),\ i = 1,2,$$

the local chart mappings defined by $\{\varphi_1,\psi_1\}$ and $\{\varphi_2,\psi_2\}$, respectively. It is sufficient to show that the map $\xi = \eta_2 \circ \eta_1^{-1}$ is of class C^{r-k} in the two cases: (i) $\varphi_1 = \varphi_2$ and (ii) $\psi_1 = \psi_2$.

(i) Let $f \in \mathcal{C}^r(X,Y)$, $\varphi_1(U) = U'$, $\psi_1(V) = V'$, $\psi_2(V) = V''$, and $f' : U' \to V'$ and $f'' : U' \to V''$ the induced local representatives of f, so $f'' = \beta \circ f'$, where $\beta = \psi_2 \circ \psi_1^{-1}$. Then if for

$$\xi : U' \times V' \times L(\mathbb{E}, \mathbb{F}) \times \ldots \times L^k_s(\mathbb{E},\mathbb{F}) \to U' \times V'' \times L(\mathbb{E}, \mathbb{F}) \times \ldots \times L^k_s(\mathbb{E}, \mathbb{F})$$

we write $\xi : (u', v', \ell_1, \ldots, \ell_k) \to (u', \xi_o, \ldots, \xi_k)$, and if $f'(u) = v'$,

$D^p f'(u') = \ell_p$, we have $\xi_p = D^p f''(u')$, or from the composition mapping formula 3.3,

$$\xi_p(u',v',\ell_1,\ldots,\ell_k) = \sum_{j=1}^{p} D^j \beta(v') \circ \sum \sigma_j^p(i_1,\ldots,i_k)\ell_{i_1} \oplus \cdots \oplus \ell_{i_k}$$

so ξ_p is of class C^{r-p}, and ξ is of class C^{r-k}.

(ii) With f as above and $\varphi_1(U) = U'$, $\varphi_2(U) = U''$, we have $f' : U' \to V'$, $f'' : U'' \to V'$, and f'' and $f' \circ \alpha^{-1}$, where $\alpha = \varphi_2 \circ \varphi_1{}^{-1}$. Assuming $D^p f'(u') = \ell_p$ we have $\xi_p = D^p f''(u'')$, or from 3.3,

$$\xi_p(u',v',\ell_1,\ldots,\ell_k) = \sum_{j=1}^{p} \ell_j \circ P_j^p\, \alpha^{-1}(\alpha(u'))$$

and thus ξ_p is of class C^{r-p} and ξ of class C^{r-k}.

Consider next the k-jet extenstion. If $f \in \mathcal{C}^r(X,Y)$ and $k \leq r$, $j^k f : X \to J^k(X,Y)$ has local representatives in natural charts of the form $(u',f'(u'),Df'(u'),\ldots,D^k f'(u'))$, so clearly $j^k f \in \mathcal{C}^{r-1}(X,J^k(X,Y))$, and we may regard the k-jet extension as a mapping

$$j^k : \mathcal{C}^r(X,Y) \to \mathcal{C}^{r-k}(X,J^k(X,Y)).$$

Let X and Y be C^{2r+2} manifolds admitting partitions of unity, X compact, so by 11.1, $\mathcal{C}^r(X,Y)$ and $\mathcal{C}^{r-k}(X,J^k(X,Y))$ are C^r and C^{r-k} manifolds, respectively.

15.5 Theorem: The k-jet extension is an embedding of $\mathcal{C}^r(X,Y)$ into $\mathcal{C}^{r-k}(X,J^k(X,Y))$.

Proof: It is sufficient to consider the case in which Y is a Banach space. Then $J^k(X,Y)$, $\mathcal{C}^r(X,Y)$ and $\mathcal{C}^{r-k}(X,J^k(X,Y))$ are Banach spaces and obviously j^k is a continuous linear map. We have the projections $\pi^k : J^k(X,Y) \to X \times Y$ and $\pi_Y : X \times Y \to Y$, and if $\pi_Y^k = \pi_Y \circ \pi^k$, see that $\pi_Y^k \circ j^k(f) = f$, so j^k is injective. Finally, the mapping

$$\omega_{\pi_Y^k} : \mathcal{C}^{r-k}(X,J^k(X,Y)) \to \mathcal{C}^{r-k}(X,Y) : F \to \pi_Y^k \circ F$$

is clearly a continuous linear surjection, so if $F = j^k f$, $f \in \mathcal{C}^r(X,Y)$, $\omega^{-1}_{\pi_Y^k}(F)$ is a closed complement to $j^k[\mathcal{C}^r(X,Y)]$ at F. Thus j^k is an embedding.

It is easy to generalize all of the above to the case in which X and/or Y have boundaries. In what follows we take this for granted.

16. Applications

We turn now to the first application of the density theorem: the Thom transversality theorem. Let X be a C^r manifold with boundary and Y a C^r manifold (without boundary). The set $\mathcal{C}^r(X,Y)$ may be given a topology as follows.

16.1 Definition: The C^r topology of compact convergence on $\mathcal{C}^r(X,Y)$ is the topology induced by j^r from the compact-open topology on $\mathcal{C}^0(X,J^r(X,Y))$.

Hereafter $\mathcal{C}^r(X,Y)$ denotes the space of C^r maps with the C^r topology of compact convergence. If X is compact and Y C^{2r+2}, the C^r manifold topology of $\mathcal{C}^r(X,Y)$ is the same as this new topology. It is well known that $\mathcal{C}^r(X,Y)$ is a Baire space.

Suppose now that X and Y are C^{2r+2} manifolds, and W a C^{r-k} manifold, all finite dimensional, X with boundary, Y and W without. If

$F \in \mathcal{C}^{r-k}(W, J^k(X,Y))$, let $\mathcal{C}^r_F(X,Y)$ denote the subspace of mappings $f \in \mathcal{C}^r(X,Y)$ such that $j^k f \pitchfork F$.

16.2 Thom Transversality Theorem: *If*

$$r > \max\{\dim X + \dim W - \dim J^k(X,Y), 0\},$$

then the subspace $\mathcal{C}^r_F(X,Y) \subset \mathcal{C}^r(X,Y)$ *is residual for every* C^{r-k} *mapping* $F : W \to J^k(X,Y)$.

Proof: First we suppose X is compact so that $\mathcal{C}^r(X,Y)$ is a manifold. Let $\mathcal{A} = j^k[\mathcal{C}^r(X,Y)]$. From 15.5 it is clear that $\mathcal{A}$ is a C^{r-k} manifold of mappings in the sense of 13.1. Furthermore, a standard computation with a local chart and a C^r characteristic function shows that for all $(j^k f, x) \in \mathcal{A} \times X$, $T(ev)(j^k f, x)$ is surjective, so the evaluation map is transversal to any mapping $F : W \to J^k(X,Y)$. Thus if $r > \max\{\dim X + \dim W - \dim J^k(X,Y), 0\}$, the openness theorem 13.4 and the density theorem in the form 14.5 imply that $\mathcal{A}_{X,F} \subset \mathcal{A}$ is open and dense.

Now let X be arbitrary, $\{X^i\}$ a countable covering of X by compact manifolds with boundary, and

$$\mathcal{A}^i = j^k[\mathcal{C}^r(X^i, J^k(X,Y))] .$$

Then we have the restriction map

$$\rho_i : \mathcal{A} \to \mathcal{A}^i : j^k f \to j^k f|X^i.$$

Clearly ρ_i is continuous, and hence $\mathcal{A}_{X^i,F} = \rho_i^{-1}(\mathcal{A}^i_{X^i,F})$ is open and dense in $\mathcal{A}$. But $\mathcal{A}_{X,F} = \bigcap_i \mathcal{A}_{X^i,F}$, so $\mathcal{A}_{X,F}$ is residual in $\mathcal{A}$.

As an application of the Thom theorem we shall show that for manifolds of appropriate dimensions any mapping may be approximated by an immersion (see 4.5 for definition). Let X be a C^{2r+2} manifold with boundary, of finite dimension s, Y a C^{2r+2} manifold (without boundary) of finite dimension t, and $\mathrm{Im}^r(X,Y) \subset \mathcal{C}^r(X,Y)$ the subspace of immersions (C^r topology of compact convergence).

<u>16.3 Whitney Immersion Theorem</u>: <u>If</u> $r \geq 2$ <u>and</u> $t \geq 2s$, <u>then</u> $\mathrm{Im}^r(X,Y)$ <u>is residual in</u> $\mathcal{C}^r(X,Y)$.

Proof: Let $W^k \subset J^1(X,Y)$ be the set of jets $j^1f(x)$ such that in a natural chart $Df(x)$ has rank, $k = 0,\ldots,s$. This condition is independent of the chart used, and W^k is seen to be a submanifold of codimension $q_k = st - k(s + t - k)$. Let $W = W^0 \cup W^1 \cup \ldots \cup W^{s-1}$. Then $f \in \mathcal{C}^r(X,Y)$ is an immersion iff $j^1f(x) \cap W = \emptyset$. The smallest of the codimensions $\{q_0,\ldots,q_{s-1}\}$ is clearly $q_{s-1} = t - s + 1$. As $r \geq 2$ and $t \geq 2s$, we have $s - q_k \leq s - q_{s-1} = 2s - t - 1 \leq -1$, so (1) $r > s - q_k$ for all $k = 0,\ldots,s - 1$, and (2) $j^1f \pitchfork W^k$ implies $j^1f(X) \cap W^k = \emptyset$ for all $k = 0,\ldots,s - 1$. From (1), the Thom transversality theorem 16.2 implies that the set $\mathcal{C}^r_W(X,Y)$ of mappings $f \in \mathcal{C}^r(X,Y)$ such that $j^1f \pitchfork W^k$ for all $k = 0,\ldots,s - 1$ is residual. From (2), $\mathcal{C}^r_W(X,Y) = \mathrm{Im}^r(X,Y)$, as $j^1f \pitchfork W^k$ implies $j^1f(X) \cap W^k = \emptyset$ for $k = 0,\ldots, s - 1$.

Next we give a direct application of the density theorem. Let X be a compact C^r manifold with boundary, of finite dimension s, Y a C^{2r+2} manifold (without boundary) of finite dimension t, so that $\mathcal{C}^r(X,Y)$ is a C^r manifold of mappings. Let $\mathrm{Inj}^r(X,Y) \subset \mathcal{C}^r(X,Y)$ be the subspace of injective mappings.

16.4 Theorem: *If X is compact and* $t \geq 2s + 1$, *then* $\mathrm{Inj}^r(X,Y)$ *is an open and dense submanifold of* $\mathcal{C}^r(X,Y)$.

Proof: Let $\mathcal{A}$ be the diagonal of $\mathcal{C}^r(X,Y) \times \mathcal{C}^r(X,Y)$, Δ_X the diagonal of $X \times X$, $K = X \times X \setminus \Delta_X$, and Δ_Y the diagonal of $Y \times Y$. Then the bijection

$$\delta : \mathcal{C}^r(X,Y) \to \mathcal{A} : f \to (f,f)$$

clearly induces a structure of C^r manifold of mappings on $\mathcal{A}$. As $t \geq 2s + 1$, we have $\dim(X \times X) - \operatorname{codim}(\Delta_Y) = 2s - t \leqslant -1$, so (1) $r > \dim(X \times X) - \operatorname{codim}(\Delta_Y)$ for all r, and (2) $\mathcal{A}_{K,\Delta_Y} = \delta[\mathrm{Inj}^r(X,Y)]$, as $(f,f)|K \pitchfork \Delta_Y$ implies $(f,f)(K) \cap \Delta_Y = \emptyset$. From (1), the density theorem 14.2 and the openness theorem imply that $\mathcal{A}_{K,\Delta_Y} \subset \mathcal{A}$ is an open and dense submanifold. As $\delta : \mathcal{C}^r(X,Y) \to \mathcal{A}$ is a diffeomorphism by definition, the theorem follows from (2).

Let X and Y be as above and $\mathrm{Em}^r(X,Y) \subset \mathcal{C}^r(X,Y)$ the subspace of embeddings (see 4.5 for definition). As an injective immersion of a compact manifold is an embedding we obtain the following immediately by intersecting 16.3 and 16.4.

16.5 Corollary: *If* $r \geq 2$, $t \geq 2s + 1$, *and* X *compact, then* $\mathrm{Em}^r(X,Y)$ *is an open and dense submanifold of* $\mathcal{C}^r(X,Y)$.

This result may be generalized for noncompact source X by means of this easy consequence of Sard's theorem.

16.6 Lemma: If X is a paracompact finite dimensional C^r manifold (without boundary) and $r > \dim(X)$, then there exists a countable set $\{X^i\}$ of compact finite dimensional C^r manifolds with boundary such that $\dim(X^i) = \dim(X)$, $X^i \subset \operatorname{int}(X^{i+1})$ and $\bigcup X^i = X$.

Proof: First, note that on any C^r manifold (without boundary) X, non-empty, with countable base and admitting C^r partitions of unity, there exists a proper positive C^r function. For if $\{U_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ are countable coverings of X with U_n open, K_n compact, $K_n \subset K_{n+1}$ and $K_n \subset U_n$ for all n, and $\{g_n\}$ is an associated C^r partition of unity ($g_n | K_n = 1$, $g_n | X \setminus U_n = 0$, $\Sigma\, g_n = 1$), then $f = \Sigma\, n g_n$ is positive, C^r and proper. Next, as $r > \dim X - 1$, the critical values of f are nowhere dense in $\mathbb{R}$ by Sard's theorem (4.1C) so for each positive integer i there is a point $y_i \in (i-1, i]$ such that $f \pitchfork \{y_i\}$, so $f^{-1}(y_i)$ is a submanifold of codimension one in X. Let $Y^i = [0, y_i]$ and $X^i = f^{-1}(Y^i)$. As f is proper X^i is compact, so X^i is a compact manifold with boundary $\partial X^i = f^{-1}(y^i)$, and $\dim(X^i) = \dim(X)$.

Combining 16.5 and 16.6 we obtain this classical result.

16.7 Whitney Embedding Theorem: If X is a paracompact C^r manifold (without boundary) of finite dimension s, Y a C^{2r+2} manifold (without boundary) of finite dimension t, $r \geq \max(\dim X, 2)$ and $t \geq 2s + 1$, then $\mathrm{Em}^r(X,Y) \subset \mathcal{C}^r(X,Y)$ is residual.

17. Nondegenerate functions

As a final application of the Thom theorem we shall show that any differentiable function on a manifold can be approximated by a nondegenerate function. To define nondegenerate functions we will need two new notions: the linear connector and the covariant differential. Let X be a C^{r+2} manifold and $\sigma : E \to X$ a C^{r+1} bundle. Then we have the obvious C^{r+1} bundles $\pi_{TX} : TX \oplus E \to TX$, $\pi_E : TX \oplus E \to E$, and $T\sigma : TE \to TX$.

<u>17.1 Definition</u>: A <u>C^r connector</u> of σ is a C^r mapping $\Gamma : TX \oplus E \to TE$ which induces

(a) an exact sequence $0 \to \pi_E \to \tau_E$ of bundle maps, and

(b) a bundle map $\pi_{TX} \to T\sigma$.

A <u>C^r linear connector on</u> X is a C^r connector of the tangent bundle of X.

Differential geometers will see that every connection is the image of a unique connector. Also, the local representative of a connector is the analog of the classical "components" of the corresponding connection. In fact if $U \subset \mathbb{F}$ is an open set of a Banach space, $\mathbb{G}$ a Banach space, $E = U \times \mathbb{G}$, and $\sigma : E \to U$ the obvious local bundle, then a connector $\Gamma : TX \oplus E \to TE$ has the form

$$\Gamma : U \times (\mathbb{F} \times \mathbb{G}) \to (U \times \mathbb{G}) \times (\mathbb{F} \times \mathbb{G}) :$$
$$: (e;f,g) \to (e,g;f,\Gamma'(e;f,g))$$

where $\Gamma'(e)$ is bilinear. Thus it is easy to construct connectors using partitions of unity, proving the following.

<u>17.2 Lemma</u>: <u>If</u> X <u>is a</u> C^{r+2} <u>manifold admitting partitions of unity and</u> $\sigma : E \to X$ <u>is a</u> C^{r+1} <u>bundle, then there exists a</u> C^r <u>connector of</u> σ.

For the second notion, let X be a C^r manifold and $\mathbb{F}$ a Banach space. If $f \in \mathcal{C}^1(X, \mathbb{F})$ then $Tf : TX \to T\mathbb{F}$. But $T\mathbb{F} = \mathbb{F} \times \mathbb{F}$, where $\{v\} \times \mathbb{F} = T_v \mathbb{F}$. Let $\pi_V : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ denote the projection onto the second factor.

<u>17.3 Definition</u>: If $f \in \mathcal{C}^r(X, \mathbb{F})$, the mapping $\nabla f = \pi_V \circ Tf : TX \to \mathbb{F}$ is the <u>differential</u> of f. Similarly, if $2 \leq k \leq r$,

$$\nabla^k f = \nabla(\nabla^{k-1} f) : T^k X \to \mathbb{F}$$

is the <u>k-th differential</u> of f. If $\Gamma : TX \oplus TX \to T^2X$ is a C^{r-2} linear connector on X $(r > 2)$ then the map

$$^{\Gamma}\nabla^2 f = \nabla^2 f \circ \Gamma : TX \oplus TX \to \mathbb{F}$$

is the <u>second covariant differential</u> (determined by Γ) of f.

We now turn to real-valued functions.

<u>17.4 Definition</u>: If $f \in \mathcal{C}^1(X,\mathbb{R})$, then $x \in X$ is a <u>critical point</u> of f iff $\nabla f | T_x X$ is zero.

Suppose X is C^{r+2} with C^r linear connector Γ, and $f \in \mathcal{C}^{r+2}(X, \mathbb{R})$. If $x \in X$, let $H_x^{\Gamma} f = {}^{\Gamma}\nabla^2 f | T_x X \oplus T_x X$.

<u>17.5 Lemma</u>: <u>If x is a critical point of f, then $H_x^{\Gamma} f$ is a symmetric bilinear form on $T_x X$ independent of Γ. If $X \subset \mathbb{E}$ is an open set of a Banach space, then $H_x^{\Gamma} f = D^2 f(x)$.</u>

Proof: It sufficed to prove the final assertion. Here $\nabla f : X \times \mathbb{E} \to \mathbb{R} : (x,e) \to Df(x).e$, so by the partial derivative rule 3.6,

$$\nabla^2 f : (X \times \mathbb{E}) \times (\mathbb{E} \times \mathbb{E}) \to \mathbb{R}$$
$$: (x, e_1; e_2, e_3) \to Df(x) \cdot e_3 + D^2 f(x) \cdot (e_1, e_2).$$

We have already seen that

$$\Gamma : X \times (\mathbb{E} \times \mathbb{E}) \to (X \times \mathbb{E}) \times (\mathbb{E} \times \mathbb{E})$$
$$: (x; e_1, e_2) \to (x, e_2; e_1, \Gamma_3(x; e_1, e_2))$$

so that for every $x \in X$,

$$H_x^\Gamma f(x; e_1, e_2) = Df(x) \cdot \Gamma_3 + D^2 f(x) \cdot (e_2, e_1)$$

from which the proof is immediate.

<u>17.6 Definition</u>: If $f \in \mathcal{C}^{r+2}(X, \mathbb{R})$ and $x \in X$ is a critical point of f, <u>the Hessian of f at x</u> is the form

$$H_x f = H_x^\Gamma f : T_x X \times T_x X \to \mathbb{R}$$

defined by any C^r linear connection Γ. A critical point $x \in X$ of f is <u>nondegenerate</u> iff $H_x f$ is nondegenerate (induces an isomorphism $H_x f_L : T_x X \to (T_x X)^*$), and f is a <u>nondegenerate function</u> iff every critical point of f is nondegenerate.

Note that nondegenerate critical points are defined here only for C^3 manifolds modelled on self-dual Banach spaces and admitting C^r partitions of unity. However nondegeneracy is a local property, and the Hessian can be defined more generally using a neighborhood of the critical point rather than the entire manifold.

Now let X be a finite-dimensional C^3 manifold. Then our final application of the Thom theorem is the following (originally proved by Thom).

17.7 First Theorem of Morse Theory: *If* $\dim X \geq 2$ *and* $r \geq 3$, *then the subspace* $\mathcal{C}^r_{ND}(X, \mathbb{R}) \subset \mathcal{C}^r(X, \mathbb{R})$ *of nondegenerate functions is residual in the* C^r *topology of compact convergence.*

Proof: Let $W \subset J^2(X, \mathbb{R})$ be the set of 2-jets $j^2f(x)$ having local representatives $(x', Df'(x'), D^2f'(x'))$ such that $Df'(x') = 0$ and $D^2f'(x')$ has rank less than $n = \dim X$. This condition is clearly independent of the local chart used, and W is a finite union of submanifolds W^i of codimension $3(n - 1) + i = q_i$, $i = 1,2,\ldots$. As $\max (n - q_i, 0) = 0$ for all i if $n \geq 2$, $r - 2 > \max (n - q_i, 0)$ if $r \geq 3$. Thus by the Thom theorem 16.2, the subspace $\mathcal{C}^r_W(X, \mathbb{R}) \subset \mathcal{C}^r(X, \mathbb{R})$ of mappings f such that $j^2f \pitchfork W^i$, $i = 1,2,\ldots,$ is residual. But if $n \geq 2$, $\operatorname{codim} W^i = 3(n - 1) + i > n$ for all i, so $j^2f \pitchfork W^i$ all i implies $j^2f \cap W = \emptyset$, and f is a nondegenerate function.

While on the subject of nondegenerate functions we may give a characterization of the behavior of a function in the neighborhood of a nondegenerate critical point. As the characterization is local it suffices to consider functions on a Banach space, and the form we give is based on a letter from R. Palais [4]. Let $\mathbb{E}$ be a self-dual Banach space and U a neighborhood of the origin.

17.8 Lemma: *If* $f \in \mathcal{C}^3(U, \mathbb{R})$ *has the origin as a nondegenerate critical point with Hessian* H_of, *there exist neighborhoods* $V, W \subset U$ *of the origin and a* C^1 *diffeomorphism* $\varphi : V \to W$, $\varphi(0) = 0$, *such that if* $x \in V$, *then* $f(\varphi(x)) = H_of(x,x)$.

Proof: Applying Taylor's formula [2, p. 186] we may write $f(x) = B_x(x,x)$ where for each $x \in U$, B_x is the symmetric bilinear form

$$B_x = \int_0^1 \int_0^1 t\, D^2 f(st\, x)\, ds\, dt.$$

Let $\beta_x : \mathbb{E} \to \mathbb{E}^*$ denote the linear map induced by B_x. As $B_0 = H_0 f$ there is a neighborhood $U_1 \subset U$ of the origin such that if $x \in U_1$, β_x is an isomorphism. Let $\gamma_x = \beta_x^{-1} \circ \beta_0$ for each $x \in U_1$. As γ_0 is the identity there is a neighborhood $U_2 \subset U_1$ of the origin such that $\alpha_x = \gamma_x^{1/2}$ is defined for each $x \in U_2$ as a convergent power series in $I - \gamma_x$. Let $\varphi : U_2 \to \mathbb{E}$ be defined by $\varphi(x) = \alpha_x^{-1}(x)$. Then as $\beta : U_2 \to L(\mathbb{E}, \mathbb{E}^*) : x \to \beta_x$ is of class C^1, so is $\gamma : U_2 \to \mathrm{Laut}(\mathbb{E}) : x \to \gamma_x$ and $\alpha : U_2 \to \mathrm{Laut}(\mathbb{E}) : x \to \alpha_x$. Thus φ is of class C^1, and in fact its derivative is easily seen to be

$$D\varphi(x) = \alpha_x^{-1} + D\alpha^{-1}(x)\cdot x\ .$$

Hence $D\varphi(x) \in \mathrm{Laut}(\mathbb{E})$, and as $\mathrm{Laut}(\mathbb{E}) \subset L(\mathbb{E}, \mathbb{E})$ is open and $D\varphi$ is continuous, there exists a neighborhood $U_3 \subset U_2$ of the origin such that $\varphi : U_3 \to \mathbb{E}$ is a local chart. Let $W = \varphi(U_3) \cap U$ and $V = \varphi^{-1}(W)$. Then $\varphi : V \to W$ is a C^1 diffeomorphism, and we shall show that $f(\varphi(x)) = B_0(x,x)$.

First, note that as B_x is symmetric, β_x is self-adjoint (we identify $\mathbb{E}$ and $\mathbb{E}^{**}$ by the canonical isomorphism). Also $\beta_x \circ \gamma_x = \beta_0$ clearly, so $\beta_x \circ \gamma_x$ is self-adjoint, or $\beta_x \circ \gamma_x = \gamma_x^* \circ \beta_x$. Furthermore, as $\alpha_x = \gamma_x^{1/2}$ is a convergent power series in $I - \gamma_x$, it follows that $\beta_x \circ \alpha_x = \alpha_x^* \circ \beta_x$. Thus $\gamma_x^* \circ \beta_x = \beta_0$ or $\alpha_x^* \circ (\alpha_x^* \circ \beta_x) = \beta_0$ or

$\alpha_x^* \circ \beta_x \circ \alpha_x = \beta_o$, so the diagram

$$\begin{tikzcd} \mathbb{E}^* & \mathbb{E}^* \arrow[l, "\alpha_x^*"'] \\ \mathbb{E} \arrow[u, "\beta_o"] \arrow[r, "\alpha_x"] & \mathbb{E} \arrow[u, "\beta_x"'] \end{tikzcd}$$

commutes. Finally, we see that

$$\begin{aligned} f(\varphi x) &= \beta_{\varphi x}(\varphi x)\cdot\varphi x = \beta_o(\alpha_x \circ \varphi x) \cdot (\alpha_x \circ \varphi x) \\ &= \beta_o(x)\cdot x. \end{aligned}$$

Thus $$f(\varphi x) = H_o f(x,x).$$

In the case $\mathbb{E}$ is a Hilbert space a further reduction can be achieved by a linear change of local chart, and the following corollary obtained.

<u>17.9 Second Theorem of Morse Theory</u>: <u>If</u> $f \varepsilon \mathcal{C}^3(U, \mathbb{R})$ <u>has the origin as a nondegenerate critical point, there exist neighborhoods</u> $V,W \subset U$ <u>of the origin, a</u> C^1 diffeomorphism $\varphi : V \to W$, $\varphi(0) = 0$, <u>and a direct sum decomposition</u> $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$, <u>such that if</u> $x_1 + x_2 \varepsilon V$,

$$f(\varphi(x_1 + x_2)) = \langle x_1, x_1\rangle - \langle x_2, x_2\rangle .$$

Proof: The proof, an exercise in spectral theory, is taken from Palais [4]. The spectrum of A splits into two portions, to the left and to the right of 0 on the real line. Taking the characteristic functions of these portions and applying them to A gives us two projection operators P_1 and P_2 which commute with A and such that $P_1 + P_2 = I$,

$P_1P_2 = P_2P_1 = 0$. Also, $P_1^2 = P_1$, and $P_2^2 = P_2$, and P_1, P_2 are symmetric. Thus

$$\begin{aligned}\langle Ax,x\rangle &= \langle P_1\, Ax,x\rangle + \langle P_2\, Ax,x\rangle \\ &= \langle A\, P_1x,P_1x\rangle + \langle A\, P_2x,P_2x\rangle .\end{aligned}$$

In this way we have split our Hilbert space into two complementary subspaces such that f is represented by the positive definite operator $P_1A = AP_1$ on the first, and the negative definite operator $P_2A = AP_2$ on the second. Finally, we put f in normal form. Say $f(x) = \langle Ax,x\rangle$ where A is positive definite. We make the change of chart $\psi(x) = \sqrt{A^{-1}}\, x$ which transforms f into the square of the Hilbert norm. If A is negative definite, we apply the same argument to -f.

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