# BUMPY METRICS

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On a compact Riemannian manifold, $M$, there ought to be infinitely many closed geodesics (a classical conjecture). This is obvious if the isometry group of $M$ has dimension greater than zero, so we should examine the "generic case" of minimal symmetry. For example, suppose $M$ is a 2-sphere embedded in 3-space, with the induced metric. In the case of the standard embedding, every point is in a 1-parameter family of closed geodesics. But if the embedding is perturbed to an ellipsoid with unequal axes, most of these geodesics disappear. Three short ones remain, and there are arbitrarily long spiraling ones as well. Perturbed further by bumps or undulations, this is an example of a *bumpy metric*, with a countable set of closed orbits (finitely many of bounded length), all stable in some sense, and a 0-dimensional isometry group. The definition is stated later, and the main theorem: *on a compact manifold, almost all metrics are bumpy.* We conjecture that every bumpy metric on a compact manifold has infinitely many distinct closed geodesics, and this has been proved for some manifolds by Gromoll and W. Meyer [4].

1. **The definition of bumpy.** We consider closed geodesics from the point of view of Marston Morse. Let $H = H^1(S^1, M)$ be the Hilbert manifold of absolutely continuous maps $c: S^1 \to M$, and $J_g: H \to \boldsymbol{R}$ the energy function for a Riemannian metric $g$ on $M$ (see Palais [6]). Then a critical point $c \in H$ of $J_g$ is a closed geodesic parameterized proportionately to arclength (we identify $S^1 \approx [0,1]/\{0,1\}$). As the group $S^1$ acts continuously on $H$ by $\theta(c)(t) = c(\theta + t)$ for $(\theta, c, t) \in S^1 \times H \times S^1$ and $J_g$ is invariant under this action, the orbit $S^1(c)$ of a critical point $c \in H$ consists entirely of critical points. In fact if $c \in H$ is a $C^\infty$ curve, we may prove that $S^1(c)$ is a $C^\infty$ submanifold of $H$, so this always occurs for critical points (by the usual regularity theorem). Then $S^1(c)$ is a critical manifold of $J_g$ corresponding to a single closed geodesic in $M$.

DEFINITION. A Riemannian metric $g$ on a manifold $M$ is *bumpy* iff for every nonconstant critical point $c \in H$ of $J_g$, $S^1(c)$ is a nondegenerate critical manifold of $J_g$. That is, the index form (or Hessian $d^2J_g(c)$) has a 1-dimensional null space, the tangent space of $S^1(c)$ at $c$.

The constant curves of $H$ are excluded because they comprise a submanifold of $H$ diffeomorphic to $M$, which is a nondegenerate critical manifold of $J_g$ (at least if $M$ is compact) of higher dimension. Because of Bott's formula relating the nullity of an iterated closed geodesic to its Poincaré rotation numbers [3], this definition is equivalent to: every closed geodesic has irrational rotation numbers.

2. **Properties of bumpy metrics.** If $g$ is a $C^{r+4}$ metric on $M$, then $dJ_g$ is a $C^r$ section of the cotangent bundle $T^*H$. Let $H_0 \subset H$ denote the closure of the constant

curves, $X = H/H_0$ the complement, and $\rho_0(g) = dJ_g|X$. If $\mathscr{M}^{r+4}$ is the Banach manifold of $C^{r+4}$ metrics on $M$, and $C^r(T^*X)$ the space of $C^r$ sections of the cotangent bundle of $X$, the map

$$\rho_0 : \mathscr{M}^{r+4} \to \mathscr{C}^r(T^*X)$$

is a $C^r$ representation. That is,

$$ev_{\rho_0} : \mathscr{M}^{r+4} \times X \mapsto T^*X : (g, x) \mapsto \rho_0(g)(x)$$

is $C^r$. It is easy to see that in this context, $g$ is bumpy if $\rho_0(g)$ is as transversal as possible to the zero-section of $T^*X$. As in equivariant Morse Theory, this would be easily made precise if the action $S^1 : X$ were smooth. Unfortunately it is not. Nevertheless, it is possible to construct a vectorbundle $\pi : E \to X$ and a $C^\infty$ representation

$$\rho_1 : \mathscr{M}^{r+4} \to \mathscr{C}^r(\pi)$$

such that $g$ is bumpy iff $\rho_1(g)$ is transversal to the zero-section of $\pi$. Application of an appropriate transversal density theorem then yields the following.

THEOREM 1. *If $M$ is compact and $r \geq 1$, the bumpy metrics in $\mathscr{M}^{r+4}$ comprise a residual subset.*

Some of the essential lemmas in the proof yield these properties.

THEOREM 2. *If $g$ is a bumpy $C^5$ metric on $M$, then*

(a) *the Poincaré rotation numbers of all closed nonconstant geodesics of $g$ are irrational,*

(b) *every closed nonconstant geodesic $c$ of $g$ is geometrically isolated, that is, for all $\varepsilon > J_g(c)$, there exists a neighborhood $S$ of $c \in H$ (or $\Delta$ of $c(S^1) \subset M$) such that if $c'$ is a closed nonconstant geodesic of $g$ with $c' \in \delta$ (or $c'(S^1) \subset \Delta$), then $J_g(c') > \varepsilon$,*

(c) *if $B > 0$, and $g$ is a bumpy $C^5$ metric, then $g$ has at most a finite number of closed nonconstant geodesics of energy (or length) less than $B$, and*

(d) *the set $\overline{\Gamma}(g) =$ closure $\{c(S^1)|c$ nonconstant geodesic$\}$ is lower semicontinuous in $g$, that is, the function $\overline{\Gamma} : \mathscr{M}^5 \to 2^M_0$ is lower semicontinuous at $g$, where $2^M_0$ is the set of closed subsets of $M$ with the Hausdorf topology.*

Of these, only the last is nontrivial, and requires condition (C) of Palais-Smale. The appropriate transversality theorem is a mildly strengthened form of the usual one [2].

THEOREM 3. *Let $\mathscr{M}$, $X$, and $E$ be $C^r$ Banach manifolds, $W \subset Y$ a $C^r$ submanifold, and $\rho : \mathscr{M} \to C^r(X, E)$ a $C^r$ representation. Then $\{m \in M|\rho(m) \pitchfork W\} \subset \mathscr{M}$ is residual if*

(i) *for all $m \in M$, $\dim(\rho(m) \cap W) \leq k_1$,*

(ii) *for all $m \in M$ and $x \in X$ such that $\rho(m)(x) \in W$, $T_x\rho(m)$ has closed split range and finite-dimensional kernel, with $\dim \ker T_x\rho(m) \leq k_2$,*

(iii) *$r > \max\{0, k_2 - k_1\}$, and*

(iv) *$ev_\rho \pitchfork W$.*

The proof of Theorem 1 is analogous to the proof of the Kupka-Smale Theorem in [2]. Because condition (iv) of Theorem 3 fails at iterated closed geodesics in the representation $\rho_1$, a device similar to the Peixoto induction argument in [2] is used. For the induction step, a local reduction in $\mathscr{M}$ is made, using properties (c) and (d) of Theorem 2. Then Theorem 3 is applied to the reduced representation to complete the induction. In this application, only hypothesis (iv) is difficult to establish, as in the Kupka-Smale case. It follows from direct construction of a perturbing metric, using Fermi coordinates at a fixed closed geodesic of the original metric.

3. **Remarks.** These generic properties of Riemannian metrics are interesting not only because of geometric questions about closed geodesics [**4**], [**8**], but also as candidates for generic qualitative features of Hamiltonian systems. Analogous properties have recently been proved to be generic in the Hamiltonian case by K. Meyer [**5**] and Robinson [**7**].

If there were a closing lemma for geodesic flows, then Theorem 2(d) would imply that for almost every metric, closed geodesics are dense in the manifold.

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