NONGENERICITY OF Ω-STABILITY

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We prove here that in general, Ω-stable diffeomorphisms are not dense in Diff(M), the space of C^r diffeomorphisms on a C^∞ manifold M, with the uniform C^r topology, $1 \leq r \leq \infty$. Recall from [1] that if $f \in$ Diff(M), then $x \in M$ is a *non-wandering point* of f if and only if for every neighborhood U of $x \in M$ there is a nonzero integer $m \in Z$ such that $f^m(U) \cap U \neq \emptyset$. The set $\Omega = \Omega(f)$ of all non-wandering points of f is a closed invariant set. If $\Lambda \subset M$ is a closed invariant set, Λ has a *hyperbolic structure* if and only if the tangent bundle of M restricted to Λ, $T(M)$, splits into a sum of C^0 subbundles E^s and E^u, invariant under the tangent of f, $Tf: T_\Lambda(M) \to T_\Lambda(M)$ such that Tf is expanding on E^u and contracting on E^s (see [1] for complete definitions). Then f satisfies *Axiom* A if and only if:

(Aa) $\Omega(f)$ has a hyperbolic structure, and

(Ab) The periodic points of f are dense in $\Omega(f)$.

If $f, g \in$ Diff(M), they are *Ω-conjugate* if and only if there exists a homeomorphism $h: \Omega(f) \to \Omega(g)$ such that $gh = hf$, and f is *Ω-stable* if and only if there is a neighborhood $N(f)$ of $f \in$ Diff(M) such that every $g \in N(f)$ is Ω-conjugate to f.

In this paper we construct an open set $N \subset$ Diff($T^2 \times S^2$) such that every $g \in N$ violates both (Aa) and Ω-stability. The basic idea is to construct $f \in$ Diff(M) with disjoint closed invariant sets Λ_1 and Λ_2, having hyperbolic structures of different dimensions, such that an orbit goes from Λ_1 to Λ_2, and another goes from Λ_2 to Λ_1. This implies that Λ_1, Λ_2, and the two orbits are contained in $\Omega(f)$, which therefore cannot have a hyperbolic structure. Further, this "pathology" is stable under perturbations of f in the C^1 topology.

In §1 we establish a criterion for the behavior described above, and in §2 we construct a diffeomorphism satisfying the criterion. §3 establishes Ω-instability for this example.

1. We begin by recalling some aspects of the Stable Manifold Theorem [1, §7.3], or [2], or [3].

If Λ is a compact invariant set of $f \in$ Diff(M) with hyperbolic structure, $T(M) = E^s + E^u$, then there is defined for each $x \in \Lambda$, a stable manifold $W^s(x)$ which is a one-to-one immersed cell in M, and consists of points $y \in M$ with the property that $d(f^m(x), f^m(y)) \to 0$ as $m \to \infty$. Then $W^u(x)$ is defined as the stable manifold at $x \in \Lambda$ for f^{-1}. Then define $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$. Finally, $W^s(x)$ varies smoothly on compact sets as x varies in Λ.

The *type* of Λ is the pair (a, b) where a = fiber dim E^s and b = fiber dim E^u.

DEFINITION. *A subbasic set* for $f \in$ Diff(M) is a compact invariant set $\Lambda \subset M$ with hyperbolic structure such that f/Λ is topologically transitive and the periodic points are dense in Λ.

If Λ_1 and Λ_2 are disjoint subbasic sets of f, we write $\Lambda_1 < \Lambda_2$ when $W^s(\Lambda_1) \cap W^u(\Lambda_2) \neq \emptyset$, and $\Lambda_1 \ll \Lambda_2$ when there are points $p_i \in \Lambda_i$ such that $W^s(p_1)$ and $W^u(p_2)$ have a point of transversal intersection.

THEOREM. *If Λ_1 and Λ_2 are disjoint subbasic sets of $f \in \mathrm{Diff}(M)$ and $\Lambda_2 \ll \Lambda_1 < \Lambda_2$, then*

$$W^s(\Lambda_1) \cap W^u(\Lambda_2) \subset \Omega = \Omega(f).$$

COROLLARY. *If Λ_1 and Λ_2 are disjoint subbasic sets of $f \in \mathrm{Diff}(M)$, $\Lambda_2 \ll \Lambda_1 < \Lambda_2$, and type $(\Lambda_1) \neq$ type (Λ_2), then f does not satisfy Axiom A.*

PROOF. Suppose f satisfies Axiom A. If $\Lambda_2 \ll \Lambda_1 < \Lambda_2$ and $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2)$, then $x \in \Omega(f)$ by the theorem above, while $f^m(x) \to \Lambda_1$ as $m \to \infty$, and $f^m(x) \to \Lambda_2$ as $m \to -\infty$. As the orbit closure $\overline{O(x)} \subset \Omega(f)$, Λ_1 and Λ_2 must be in the same basic set Ω_i of f (that is, the same indecomposable piece of $\Omega(f)$, see [1, 6.2]). As Ω_i has a hyperbolic structure and is indecomposable, $\dim E_x^s$ is constant for all $x \in \Omega_i$, a contradiction.

The proof of the theorem requires the following

LEMMA. *Let $f: \Lambda \to \Lambda$ be a topologically transitive homeomorphism of a compact metric space with periodic points dense in Λ. Then given nonempty open sets V_1, V_2 in Λ, there is a periodic point $p \in V_1$ such that $f^m(p) \in V_2$ for some m.*

PROOF. From the topological transitivity $f^m(V_1) \cap V_2 \neq \emptyset$ for some m. Let q be a periodic point in this intersection and $p = f^{-m}(q)$.

PROOF OF THE THEOREM. Let $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2)$ and let U be a neighborhood of x.

Now by the hypothesis $\Lambda_2 \ll \Lambda_1$, $W^u(p_1)$ and $W^s(p_2)$ have a point of transversal intersection for $p_1 \in \Lambda_1$, $p_2 \in \Lambda_2$. Let V_1 be a neighborhood of p_1 in Λ_1, U_1 a neighborhood of p_2 in Λ_2 such that for every $p \in V_1$, $q \in U_1$, $W^u(p)$ and $W^s(q)$ have a point of transversal intersection. Choose an open set V_2 in Λ_1, U_2 in Λ_2 such that if $q' \in V_2$, $p' \in U_2$, then $W^s(q')$ and $W^u(p')$ intersect U.

Now apply the previous lemma to obtain periodic orbits γ_1 meeting V_1, V_2 and γ_2 meeting U_1, U_2. Thus $W^u(\gamma_1)$ and $W^s(\gamma_2)$ have a point of transversal intersection while $W^s(\gamma_1)$ and $W^u(\gamma_2)$ both meet U. Now apply the argument of [1, 7.2] to obtain that $f^m(U) \cap U \neq \emptyset$ for some m. This proves the theorem.

2. We now construct a diffeomorphism $f \in \mathrm{Diff}(T^2 \times S^2)$ having a subbasic set configuration $\Lambda_1 \ll \Lambda_2 < \Lambda_1$. Let $g \in \mathrm{Diff}(T^2)$ be the Thom diffeomorphism, defined by the linear isomorphism of R^2 having matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

(see [I, §1–3]), and $p \in T^2$ the fixed point corresponding to the origin, which is of type (1, 1), that is, $\dim E_p^s = 1$, $\dim E_p^u = 1$. Let $h \in \mathrm{Diff}(S^2)$ be the horseshoe diffeomorphism [I, §1–5], which has two fixed points, q_1 and q_2, of type (1, 1), at each of which the map h is linear in a C^∞ coordinate chart. In addition, recall that $\{q_1\} \ll \{q_2\} \ll \{q_1\}$ with respect to h. Let $f_0 = g \times h \in \text{Diff}(T^2 \times S^2)$. Then $\Lambda_1 = T^2 \times \{q_1\}$ is a 2-dimensional subbasic set of type (2, 2) and $\Lambda_2 = \{(p, q_2)\}$ is a one point subbasic set of type (2, 2). Using local coordinate charts on T^2 and S^2, at p and q_2 respectively, with respect to which g and h have linear local representatives, we modify f_0 through a curve of linear maps so that the fixed point Λ_2 becomes fixed of type (1, 3). We may do this in such a way that the new local diffeomorphism at Λ_2 can be extended so as to agree with f_0 outside a small neighborhood of Λ_2, and so that the new diffeomorphism f satisfies:

$$W^u_{f_0}(\Lambda_2) \subset W^u_f(\Lambda_2)$$

and $W^s_f(\Lambda_2)$ contains a connected part of the 1-dimensional stable manifold $W^s(q_2)$ in S^2 which contains q_2 and a point $y \in S^2$ of transversal intersection, $y \in W^s(q_2) \cap W^u(q_1)$.

Thus with respect to f, $\Lambda_1 \ll \Lambda_2 < \Lambda_1$, completing the construction.

Finally we claim there exists a neighborhood N of $f \in \text{Diff}(T^2 \times S^2)$ such that for all $g \in N$, there are subbasic sets $\Lambda_i(g)$, homeomorphic to Λ_i, $i = 1, 2$, such that

$$\Lambda_1(g) \ll \Lambda_2(g) < \Lambda_1(g).$$

First, we observe that the local Ω-stability Theorem [3, Proposition 3.1] applies to yield the following: There is a neighborhood N_0 of $f \in \text{Diff}(T^2 \times S^2)$ and for every $g \in N_0$, subbasic sets $\Lambda_i(g)$ and a conjugating homeomorphism $h(g)$: $\Lambda_1(g) \cup \Lambda_2(g) \to \Lambda_1 \cup \Lambda_2$, $fh(g) = h(g)g$. Furthermore, the stable manifolds $W^s(\Lambda_1(g))$ depend continuously (C^r topology on compact subsets in the fibers, C^0 topology on $\Lambda_1(g)$) on g (C^r topology, $r \geq 1$). This follows from the continuity conclusion of the Stable Manifold Theorem [2]. Thus $\Lambda_1 \ll \Lambda_2$ is always an open condition. The relation $\Lambda_2 < \Lambda_1$ can in general be destroyed by arbitrarily small C^1 perturbations, but for this particular diffeomorphism $f \in \text{Diff}(T^2 \times S^2)$ both Λ_1 and Λ_2 are submanifolds, so $W^u(\Lambda_1)$ and $W^s(\Lambda_2)$ are smoothly immersed, with $W^u(\Lambda_1)$ transversal to $W^s(\Lambda_2)$. But nonempty transversal intersections are preserved even by C^0 perturbations of the submanifolds (that is, intersection but not transversality is preserved), so in this case $\Lambda_2 < \Lambda_1$ is an open condition also. Thus there is a neighborhood N of $f \in \text{Diff}(T^2 \times S^2)$ contained in N_0 such that every $g \in N$ has a configuration of subbasic sets of the form $\Lambda_1(g) \ll \Lambda_2(g) < \Lambda_1(g)$, and $\Lambda_1(g)$ has type (2, 2), while $\Lambda_2(g)$ has type (1, 3). Thus every $g \in N$ violates condition (Aa) by the corollary of §1, and thus violates Axiom A.

3. In this section we show that the neighborhood N of f in $\text{Diff}(T^2 \times S^2)$ has the property that every diffeomorphism $g \in N$ is not Ω-stable.

We argue that if $g \in N = N(f)$, then either

(a) $W^s(\Lambda_2) \cap W^u(\Lambda_1)$ contains at least one point in some $W^s(\Lambda_2) \cap W^u(z)$ where $z \in \Lambda_1$ is periodic or

(b) not (a).

Nongenericity of Ω-stability now follows from the following facts:

(A) A diffeomorphism $g \in N$ satisfying (a) may be approximated by one satisfying (b).

(B) A diffeomorphism $g \in N$ satisfying (b) may be approximated by one satisfying (a).

(C) If $g, g' \in N$ satisfy (a), (b) respectively, then g, g' are not Ω-conjugate.

Now (A) is a consequence of a general approximation theorem (see for example [4, p. 100]). Fact (B) is proved easily since the periodic points are dense in Λ_1 and the $W^u(z)$ for z periodic are dense in $W^u(\Lambda_1)$.

Finally we see the truth of (C) by following the orbit of $W^s(\Lambda_2) \cap W^u(z)$ and its image under a possible conjugacy.

References

1. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817.
2. M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, these Proceedings, vol. 14.
3. S. Smale, *The Ω-stability theorem*, these Proceedings, vol. 14.
4. R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York, 1967.

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