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The Trouble with Math

RALPH ABRAHAM

Since 1994, the Ross School on Long Island has evolved a curriculum based on a core of world cultural history using an outline I wrote and a long essay written by the historian William Irwin Thompson for the Ross School in 1995. Thompson has devoted considerable time to working with teams of teachers to develop specific course materials for this program, now known as the Ross spiral curriculum. Since each grade at the Ross School is devoted to an epoch of cultural history, and the epochs and grades follow in chronological sequence, it is possible to integrate all of the traditional subjects in the matrix of relationships from which they originally evolved. For example, when the sixth grade is devoted to the axial age, then ancient Greek math, science, art, philosophy, and so on may be taught together, evoking the cultural ambiance of the ancient world. This is the program that has evolved, although the historical sequence and integration of math is not yet complete. This chapter is devoted to describing a proposed program, based on my ideas as well as those of Rupert Sheldrake, William Irwin Thompson, and Courtney Sale Ross, that would integrate math more completely with the core of the spiral curriculum. Since mathematics is the universal language of space-time patterns and the common modeling strategy of all academic disciplines, math skills are highly advantageous for conceptually integrating historical data and understanding the gigantic complex system in which we live. We have therefore proposed that math skills and the concepts of chaos theory be given special emphasis throughout the spiral curriculum.

My ideas for math education have evolved over some fifty years of university teaching at the University of Michigan, University of California Berkeley, Columbia, Princeton, and University of California Santa Cruz. My research in dynamical systems and chaos theory was radically changed by the arrival in Santa Cruz of computer graphics technology in 1974. Early adoption of computer graphics in my research in chaos theory diffused into university teaching, with state-supported grants to introduce more visual representations in the introductory math courses, and the creation of the Visual Math Project at UC

Santa Cruz. A sizable National Science Foundation grant in 1980 was devoted to the validation of the visual approach for math education. We showed that more visuals resulted in less math anxiety. In the 1980s, the Visual Math Project grew into a master's degree program in applied and computational math, and then in the 1990s into a research program, the Visual Math Institute.

In 1987, a brief wave of popularity of chaos theory prompted me to begin writing *Chaos, Gaia, Eros* (1994). In that book I divided world cultural history into three epochs—static, periodic, and chaotic—each of which was tied to a particular development in mathematics. I had become friends with Thompson a few years earlier, and out of respect for his superior knowledge of history I sent him draft chapters for comment. He replied that he had himself devised a similar scheme, most recently published in 1986 in his book *Pacific Shift*. His own scheme, in which world history is outlined as having four major epochs, or *cultural ecologies*, each characterized by a different mathematical style or *mentality*—arithmetic, geometric, dynamic, and chaotic—is credited in *Chaos, Gaia, Eros*. Later I learned that he had been working on these ideas since 1961 in his undergraduate honors thesis influenced by Gregory Bateson, Vico, and Heinz Werner. When Courtney Sale Ross contacted me in 1995, after reading *Chaos, Gaia, Eros*, to request that I help her turn my book into a curriculum for the Ross School, we brought in Thompson as well and together began the ongoing project to create the Ross spiral curriculum, a new school program integrated by world cultural history.

This chapter discusses the problems and strategies of the math component of this history-based program for middle and high schools, as it has evolved in the context of my work with the Ross School over the past fourteen years, and then proposes a new program in outline form. This program has influenced the Ross spiral curriculum but has not yet been fully integrated, partly because of the pernicious influence of standardized tests, as discussed below.

Bifurcations of Cultural History

The word *bifurcation* is taken from the jargon of chaos theory, as explained in great detail in my book *Chaos, Gaia, Eros* (1994). But in our context, it means a major shift in history. My emphasis on historical transformation derives from my conviction that we are experiencing one on a massive scale, as argued by Ervin Laszlo in *The Age of Bifurcation* (1991). Through an understanding of earlier historical transformations, we may improve our chances of surviving this one and creating a viable future for humankind on planet Earth. This level of understanding is an important goal of our program.

The major transformations in world cultural history that Thompson and I, at first independently and then together, have outlined, are expressed in terms of shifts in mathematical thinking. This has enabled us to develop a program that integrates mathematics with cultural history and aims at creating an awareness and understanding of the special role of mathematics in the evolution of consciousness. With immersion in a given cultural ecology, the intertwined paths of math, science/technology, languages/writing/literature, and graphic arts/music become one knowledge. The development of this knowledge through time, with special emphasis on the major points of bifurcation, provides the strongest base for understanding the present and creating the future. Establishing this base is the main goal of our integrated program, of which the math thread is an integral part.

The Changing Nature of Mathematics

Mathematics may even be older than speech: indeed, speech may have evolved from it. Since our ancestors' development of language, math has evolved in giant steps. And since the computer revolution, a new image of the subject is gaining acceptance: the study of space-time patterns. Dynamic math is outpacing the static concepts established by the ancients.

Currently the main branches of mathematics are usually listed as arithmetic, geometry, algebra, and dynamics (a.k.a. analysis). Sometimes logic, topology, chaos theory, and other areas of mathematics are listed as well. We may refer to the branches by the code RGADX (for aRithmetic, Geometry, Algebra, Dynamics, and chaos [Xaos] theory).

Arithmetic is all about numbers, counting, and order. *Geometry* is the study of static spatial patterns such as triangles, circles, cubes, and pyramids. Both ancient branches of math are still evolving. *Algebra* is an extension of arithmetic dealing with the solution of equations, manipulation of polynomials, and so forth. *Geometric algebra* is an intermediate step in which geometrical constructions are used to solve equations (see Katz and Imhausen 2007 for more information). *Dynamics* deals with the analysis of motion in terms of distance, velocity, and acceleration. In its recent development there is an emphasis on so-called qualitative theory and long-term behavior: that is, where will this moving system end up? *Chaos theory* is a further development of dynamics, dealing with behavior that is chaotic—that is, neither fixed nor periodic.

All of the branches of math are useful in daily life as well as in scientific professions. R and G were established in the classical tradition as part of its curriculum, the quadrivium. A arrived in the Middle Ages and became part of the traditional school program in the Renaissance. D is relatively new; it was taught to college seniors in my parents' time and has now descended to high school in Europe and early college in the United States. X has just burst into popular consciousness in the past decade. It has yet to make a dent on most schools, but I am working on changing that. Computers have radically changed the way math is done and taught and the way new knowledge is found, especially in X.

Outline of Our Proposed Program

When Thompson and I began working together, I added an algebraic epoch to the arithmetic, geometric, dynamic, and chaotic epochs that he had proposed in his fourfold scheme of periodization. The sequence of world history that we employ, therefore, is:

1. Prehistoric cultures (arithmetic mentality)
2. Classical civilizations (geometric mentality)
3. Medieval civilizations (algebraic mentality)
4. Modern industrial civilization (Galilean dynamical mentality)
5. Contemporary planetary civilization (complex dynamical mentality)

And the major transformations are:

- R/G Shift, ancient Greece, 500 BCE
- G/A Shift, early Islam, 800 CE
- A/D Shift, Newton and Leibniz, 1660
- D/X Shift, atomic, computer, etc., 1945

To produce the curriculum, we mapped this scheme of history onto the grades. As noted above, each grade at the Ross School is devoted to an epoch of cultural history. The epochs and grades follow in chronological sequence, so that all of the traditional subjects, including mathematics, can be studied together and integrated into the matrix of relationships from which they originally evolved.

Regarding math education in the lower grades one may follow contemporary practice, emphasizing the meaning of numbers before the operations of arithmetic. Sharon Griffin's Number Worlds program is useful in this regard. We therefore begin here by considering the proposed math program for the middle school with the fifth grade.

GRADE 5

Core: The riverine cultural ecologies of ancient Egypt, Mesopotamia, and India, 3500 to 1450 BCE.

Math: In addition to the ongoing program of arithmetic, which includes fractions and decimal notations, begin ancient geometry and archeoastronomy.

GRADE 6

Core: Prophecy and cultural transformation, 1450 to 250 BCE. Moses, Pythagoras, Buddha, Confucius, Lao Tzu. Ancient Greece.

Math: In addition to the ongoing arithmetic program, which includes the elementary number theory of Euclid's *Elements*, begin the two-dimensional constructions of Euclid. There are forty-eight such constructions in books 1 through 6, and instructors should make specific selections. Ultimately, however, the goal is to eventually begin algebra with its roots in geometry. *The placement of* geometry *before algebra is historical.* So the constructions relating to *geometric algebra* should be included. See Katz (1993, 64–70). See also my *Visual Constructions of Euclid*, C#15 (Prop. II-11) and C#16 (Prop. II-14). Here I am beginning to deviate from the spiral curriculum as now taught. This alternative is similar to a proposal of Katz and Imhausen (2007).

GRADE 7

Core: Empires and religions, 350 BCE to 800 CE.

Math: More constructions of Euclid, building toward the five Platonic solids. Logic should be introduced from Euclid's theorems, which are intended to prove that the constructions are correct. Begin trigonometry applied to astronomy.

GRADE 8

Core: Climax of the Middle Ages, 800 to 1450.

Math: Introduce algebra as the recapitulation of geometric algebra using the rhetorical method of al-Khwarizmi, and progress to the symbolic method of Descartes.

GRADE 9

Core: Renaissance, Reformation, the New World, 1416 to 1688.

Math: Algebra with symbolic notation, polynomial functions, beginnings of calculus in Kepler and Galileo, anticipating Newton and Leibniz in grade 10.

GRADE 10

Core: Enlightenment and Romanticism, 1688 to 1865.

Math: Advanced algebra, calculus, and matrices.

GRADE 11

Core: Modern era, 1865 to 1948.

Math: Vector spaces, matrices, dynamical systems, modeling and simulation of complex dynamical systems, computational methods, statistics, logic and proofs.

GRADE 12

Core: Contemporary era, 1948 to the present.

Math: Electives, including fractal geometry, chaos theory, and complex systems.

The Sheldrake Principle (1981)

One of our main assumptions in drawing up this curriculum is that it is easiest for individuals to learn a subject in the sequence through which it has developed historically.

I call this the Sheldrake principle because Rupert Sheldrake, a contemporary English biologist and neovitalist, has proposed it as a corollary to his theory that a hypothetical field, the morphogenetic field, is the organizer of forms in nature (Sheldrake 1981, 180–81). Sheldrake and I met in 1982, and this idea emerged in conversation. He explained that I might learn to serve in tennis more easily because so many others had learned it before me. I tried out his idea in my math classes at the University of California Santa Cruz and became convinced it worked.

Such a historical approach to math education has actually been discussed previously in the math education literature, and there are international movements devoted to promoting it (see the International Congress on Mathematics Education, the International Commission on Mathematics Instruction, and the International Study Group on the Relations between History and Pedagogy in Mathematics). This is because the historical sequence of math's development—RGADX: arithmetic, geometry, algebra, dynamics, chaos theory— parallels the sequence of the development of cognitive modes in the individual and evolutionary, providing a natural progression from primitive counting (a skill of newborns; see Dehaene 1997) through visual (preverbal) modes of geometry and geometric algebra, to the rhetorical modes of early algebra and the abstract-symbolic modes of modern mathematics, including dynamics and chaos theory. Thus our curriculum also unfolds in tandem with the main developmental stages of the individual that are outlined by Piaget, in what we have called the rectified Sheldrake principle. Extensive support for this approach may be found in Furinghetti and Radford (2002).

The Importance of Euclid

After the R/G bifurcation around 500 BC, arithmetic, geometry, and geometric algebra were rapidly developed by the Pythagoreans. This rapid development culminated in the Academy of Plato around 350 BC. The results were collected in logical sequence in texts called *stocheia* (elements) by various editors. Euclid was one of these, and his text was so successful that it became the second most published book of all time, after the Bible, and the most influential math text of all time until very recently.

The *Elements of Euclid* develop these topics in thirteen books containing 645 propositions, concerning

- Plane geometry of triangles
- Plane geometry of circles
- Regular polygons in or around a circle
- Proportions (ratios)
- Number theory
- Solid geometry, especially of the regular solids

Of these 645 propositions, 60 are constructions and the others are used to prove that the constructions work reliably. The *kataskeuai* (pronounced *katas-key*, meaning "constructions") may be regarded as the skeleton, the goal, and the most ancient part of the elements. Also known as *sacred geometry* and as *ancient geometry*, they are fundamental to the classical cultural ecology and were embedded in stone by the architects and builders of ancient times. The 48 fundamental constructions of plane geometry constitute the heart of the classical math curriculum.

The early study of Euclid is important in that it can give students a solid historical basis for understanding advanced mathematics. The constructions of Euclid span the history of mathematics from ancient Mesopotamia and Egypt up to the seventeenth century: that is, the time periods of grades 5 through 8 of our program, the epoch of the geometric mentality. They lead naturally into the introduction of the concepts of algebra, which emerged around 800 CE, and its symbolic notations, which came only around 1600, at the end of the geometric period. Given that this stage of the cultural evolution of mathematics was prerequisite to the creation of the dynamics and calculus of Kepler, Newton, and Leibniz, its study is prerequisite for a student learning these concepts, basic to the dynamic mentality. It is also an excellent preparation for understanding chaos theory.

It should be noted that Euclid himself, in editing the *Elements*, was meticulously faithful to the Sheldrake principle, the preservation of historical order. This is the reason that constructions of the golden section appear twice: in book 4, according to the Pythagoreans, and in book 6, after the theory of proportions, as Heath notes in his commentary. On a larger scale, the presentation first of geometry, then of geometric algebra, and finally of algebra is in accordance with the historical development of mathematics, unlike the standard curriculum today, in which the historic order of geometry and algebra is reversed.

How did this change occur? After the Enlightenment, circa 1800, the logic of Aristotle was blown up into a new paradigm for mathematics, called *formalism*. The months were renamed (e.g., *July* became *Brumaire* in French) to avoid the taint of history, and math was reorganized according to its logical as opposed to its historical order. The *Elements* of Euclid followed the traditional names of the months into the dustbin of history, and there began a disease of our current cultural ecology, of which widespread math anxiety, or damaged ability to do mathematics, was just one of the symptoms. As the *Elements* were abandoned and replaced by inferior works, the math crisis and math anxiety rapidly grew.

Math Anxiety

Math anxiety is a pathology produced in many individuals during their school years. It is a cognitive disability in which the natural human capacity for doing math is damaged. Even some people who succeed in technical and mathematical professions may be functioning at a fraction of their full potential because of the debilitating effects of this acquired disability. Its primary symptom is a flush of anxiety (math anxiety) or the urge to flee (math avoidance syndrome) when math comes up.

In her important book *Overcoming Math Anxiety* (1978/1993), Sheila Tobias points to word-solving problems as the crux of the problem of math anxiety. She describes the main flaws of the current school math program as:

- Domain disintegration (assuming knowledge from another discipline that has not actually been developed)
- Time disintegration, or anachronism (requiring concepts that are going to be, or should be, introduced later)
- Cultural disintegration (using metaphors foreign to the cultural context of the skills required)
- Mode disintegration (making use of too few, or inappropriate, cognitive modes)

Tobias's list highlights the very deficiencies that our own program counters through coordination of all disciplines, presentation of topics in their historical order, integration of mathematics with cultural history, and the use of multimedia technology. While Tobias's book is aimed at curing math anxiety, our program aims at preventing it.

The traditional math curriculum fosters math anxiety. My own theory is that math anxiety arises when a math topic is presented to students prematurely, at a time in their schooling (frequently around grade 7 or 8) before the prerequisite cognitive skills are well developed, so that they are unable to fully understand it, do not master it, and are left with the erroneous impression that they are at fault. More specifically, the concept of an unknown number (basic to algebra) is introduced too early, and geometric constructions, which build visual intelligence, are omitted or introduced very late when in fact they should be included from the early grades onward.

A curriculum that presented mathematics in historical rather than logical sequence, in particular restoring Euclid as a basic math text for the middle and high schools, would thus be a giant step toward the elimination of math anxiety. Further, it has long been known that math requires the coordinated modes of verbal, visual, and symbolic representation: the dynapic technique, for example, the method that mathematicians use to communicate among themselves, combines verbal description multiplexed with symbolic statements and a line-by-line drawing. But traditional math curricula underutilize some of these modes or do not use them at all; visual representations are particularly likely to be missing. Multimedia supplemental materials and the dynapic technique may be used to augment the traditional textbook-based program beginning no later than middle school. Howard Gardner's work on multiple intelligences has supported this view. We believe that correct math training encourages the coordination of the various cognitive modes and their balanced development in childhood. On a larger scale, such a revised program would require the creation of adequate new texts, the retraining of teachers to use graphics and to follow the sequence geometry, geometric algebra, algebra, and the abandonment of standardized tests that misrepresent math.

The tried and proven traditional program for school math is a tried and proven cause of math anxiety. It is unfortunate that so often the response to widespread math failure is merely to intensify the same efforts. For instance, a *San Jose Mercury News* article from July 27, 1999, entitled "Math: A Great Divide: Dismal Scores Reveal How Far California Schools Lag in Achieving Standards Called among the Toughest in U.S.," offers three suggestions for improving academic performance in math: train teachers in advanced math concepts; try out and buy new textbooks; and focus on preparing students to take algebra in eighth grade. School administrators in the San Jose Unified School District plan to begin upreparing students for algebra as early as fourth grade. According to the theories presented in this chapter, nothing could be worse than these suggestions, especially considering the fact that geometry is not presented until the ninth grade, if at all.

The Tyranny of Testing

As the *San Jose Mercury News* article suggests, low test scores in mathematics fuel renewed efforts to teach to the test, but such efforts only produce more math failure because standardized tests uphold traditional methods of teaching mathematics that are producing the low test scores in the first place. Standardized tests dominate the programs in schools, public or private, and cripple the evolutionary tendency of teachers, pupils, and parents to adapt the curriculum to individual skills and needs. This worldwide tendency is antievolutionary in that the standardized tests are very expensive and therefore slow to change. Perhaps more in math than in other subjects, the cognitive modes used in the tests are poorly adapted to the subject, and training for the test deviates from a balanced and coordinated development of modes. In particular, the visual mode is underutilized or uncoordinated with verbal and symbolic modes.

Math programs in schools worldwide—especially in the United States—are stuck in a loop. A faulty program presents the wrong material, out of sequence, without adequate cognitive modes, and set students up for failure. Young people of all levels of natural ability are convinced that they cannot learn math. These people become parents with math anxiety, and they pass this on to their children; it becomes a family disease. Then this anxiety manifests in the creation of standards and standardized tests. Standardized tests such as the SAT discourage curricular reform in that schools are under strong pressures to focus on aspects of a discipline that the test emphasizes and to give scant or no attention to aspects that the test omits. Regarding the math section of the SAT in particular, two problems might be instanced. First, under "Arithmetic" are included "word problems involving such concepts as: rate/time/distance, percents, averages." Yet dynamics is not an arithmetic concept. The difficulties students have with word problems may stem from this in part, while word problems on weights and measures (under "Geometry") may be easier. Problems on dynamics probably should be moved to the physics section of the test. Their mistaken inclusion under "Arithmetic" inclines school math programs to undertake these concepts too early. Second, "formal geometric proofs" are excluded from "Geometry." Of course these may be difficult to test with multiple-choice questions. But the explicit exclusion of this material from the SAT makes likely its exclusion from the entire high school program, which has unfortunately led school geometry programs to focus exclusively on the content, as opposed to the method, of geometry.

The Importance of Teaching Chaos Theory

Chaos theory was not known by this name until 1975 or so. For almost a century it was called dynamical systems theory, a topic of pure mathematics little known to the scientific community. After the computer revolution, chaotic motions became visible on computer graphic screens, and an awareness of their significance began to spread among scientists. In 1971 this awareness materialized as a technical report published in a journal of theoretical physics, and the chaos revolution was on. It took an additional fifteen years to sweep throughout the sciences and reach public awareness. All of this amounts to a major paradigm shift as chaotic behavior moved from mystery to familiarity (Abraham and Ueda 2000).

The most frequent application of dynamical systems theory is to the technology of modeling complex natural systems. The importance of chaos theory has been in this context. Because of the new wisdom of chaotic motions, many more complex systems now have useful models: the biosphere, the global economy, climate change, the human immune system, and so on. Different models for subsystems, created by scientists of disjoint specialties, may now be combined into a single complex supermodel, thanks to chaos theory. It provides a new technique for the unification of the sciences. Offhand it is not obvious that the chaotic motions of chaos theory have any direct bearing on the chaotic experiences of everyday life. However, as the applications of chaos theory to the social sciences evolve, more and more everyday chaos, including world cultural history, is brought into the embrace of chaos theory.

Since chaos theory is a new branch of math, it is relatively independent of the main topics of the traditional program. Therefore, it is quite accessible to people without an extensive math background. Calculus, for example, is not required. The plane geometry of Euclid is an excellent preparation. The applicability of chaos theory to the complex space-time patterns observed in nature and in human society make it an important subject for everyone to learn. And the paucity of background knowledge required makes it accessible to all. Math anxiety is usually triggered by high school algebra and its arcane symbolic notations, while most students feel comfortable with geometry. Since chaos theory builds upon geometry without requiring algebra, it provides a fresh start for those afflicted with math anxiety and may actually restore math confidence.

Our proposed program, then, relies on the synergistic operation of several features—integration of mathematics with other disciplines; adherence to a curricular sequence based on the historical order of the field's evolution and the developmental order of individual capacities; and recruitment of multiple modes of representation, particularly the visual. It prevents math anxiety; illuminates the cultural meanings of mathematics, thereby deepening students' knowledge and interest; and builds a strong foundation for understanding not only mathematics as it has evolved up to this point but mathematics as it is currently unfolding in chaos theory.

NOTE

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