Image Entropy for Discrete Dynamical Systems

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Abstract

In the 1950s, Menzel, Stein, and Ulam performed one of the earliest digital simulations of discrete dynamics in two-dimensions. In their work they created a robot mathematician to scan the results of simulations for chaotic attractors. Here I will extend their idea, using the image entropy concept to scan a family of endomorphisms for explosive and catastrophic bifurcations.

1. Introduction

Our joint book (with Laura Gardini and Christian Mira), *Chaos in Discrete Dynamical Systems: A Visual Introduction in 2 Dimensions* of 1997, included detailed studies of two special families of map iterations. The very interesting bifurcation sequences analyzed in the book were originally discovered by very laborious computational work and manual inspection. In this work I will seek an algorithm to automatically draw parts of the bifurcation set of families of maps such as these. I will begin with a short historical review, then report on some preliminary experiments on the image entropy of the two map families. Regarding the recent history of my subject, I would like to thank Clint Sprott, Christian Mira, and Hioshi Kawakami for their generous contributions. Also, I am deeply in debt to Laura Gardini for her substantial support and instruction over the many years.

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PART ONE: HISTORICAL BACKGROUND

2. The Prehistory: 1947-1997

Let us begin with a brief chronology of computational chaos theory, leading up to our joint book, *Chaos in Discrete Dynamical Systems* of 1997.

2.1. Stan Ulam

In *The Chaos Avant-garde* of 2000, Ueda and I defined the computational prehistory of the chaos revolution as the period 1958-1975. Despite the excellent history of discrete dynamical systems included in that book by the chapter by Christian Mira, our focus was primarily on flows, that is, on continuous-time dynamics. So we overlooked the important fact that chaos in discrete dynamical systems predated by more than a decade the famous chaotic flow attractors of Ueda and Lorenz.[1]

I would like now to amend that view by including the earlier works of Ulam and his colleagues, as far back as 1947. Ulam (born in Lvov, Ukraine, in 1909 of a wealthy Jewish family) emigrated to the US in 1938, and worked with John von Neumann (born in 1903 in Budapest in another wealthy Jewish family) and many other important mathematicians in the Manhattan project. He died in 1984. Among his works we find:

U1. 1947. Ulam and Von Neumann.
U2. 1959. Menzel, Stein, and Ulam.
U3. 1962. Ulam.
U4. 1964. Stein and Ulam.
U5. 1987. Stein.

Regarding Item U1.

In earlier times, preceding the World Wide Web, it was common practice in the mathematical research community to publish an abstract of a preliminary result as soon as possible, as publication of a peer-reviewed article took a long time. So it was that an abstract by Ulam and Von Neumann, entitled *On combination of stochastic*

[1] See (Sprott, 2010) for details on these.

and deterministic processes, preliminary report was announced in the 53rd Annual Summer Meeting of the American Mathematical Society in New Haven, Connecticut. This abstract, received by the journal on September 3, 1947, announced the astounding result that iteration of the logistic function, $f(x) = 4x(1-x)$, produced a pseudorandom number sequence. In fact, Ulam and coworkers would use this number generator later in their pioneering Monte Carlo method. This is the first computer study of the logistic iteration that I know, and might be regarded as the alpha point of computational chaos theory.

Regarding Item U2.

Quadratic Transformations: Part I is an 158-page informal report "of an interim nature" from the Los Alamos Scientific Laboratory (LANL) of 1959. (Part II never appeared.) It is the third ever pioneering work of computational dynamics following World War II, after Item U1 and the Fermi-Pasta-Ulam project of 1955¿ These were performed on the MANIAC I at LANL, which came online in 1952. The iterated transformation experiments were done on its successor, the MANIAC II at the LANL, which came online in 1957. These early computers of Von Neumann architecture were programmed in their own machine language, and thus were difficult to program.

I would like to begin with an aside on the authorship of this paper. It is sometimes given as Stein and Ulam, or alternatively, as Mentzel, Stein, and Ulam. Of course Von Neumann is well-known as one of the great mathematicians of all time, and also, as one of the pioneers of computer science. Stein was a physicist and protegee of his. Mentzel, retired in 1991, was a staff programmer at the LANL, and an important member of the team for both the FPU and the iteration projects. She was born Mary Tsingou in 1928 to a Greek family in Milwaukee, and earned an MS in Mathematics at the University of Michigan in 1955, while I was myself an undergraduate there, in engineering mathematics. She went then directly to work at LANL, where she and Mary Hunt were the very first programers of the MANIAC I. She and John Pasta were the first to create computer graphics, using a storage oscilloscope and Polaroid camera for visualization.

Returning to Item U2, this paper refers to Item U1 in footnote 4 on page 11, following the sentence,

> In the limit $n \to \infty$ a variety of behaviors is possible; the vectors may, for example, converge under iteration to a limiting vector $\vec{x}$, they may oscillate between a finite set of limit vectors $\vec{x}_i$, or they may exhibit a more or less chaotic behavior ...

This is the first occurrence of *chaotic behavior* in the literature of mathematics, as far as I know. Item U2 deals with a class of homogeneous quadratic maps defined on the unit simplex in Euclidean three-space, $x, y, z \geq 0, x + y + z = 1$. An affine change of coordinates would change the systems into two-dimensional maps on a triangular domain. All 97 iterations of this type were studied numerically on an IBM 704. Of these, only two exhibited "anomalous convergence behavior." One of these is (equation 69, p. 64):

$$\begin{aligned} x' &= 2y + x^2 - 3y^2 \\ y' &= 2y(1-x) \end{aligned} \quad (1)$$

The anomalous convergence behavior consists of a spiral, asymptotic to the boundary of the triangle, as shown in Figure 7 on page 79 of the report.

Regarding Item U3.

This report of a conference presentation refers to Item U2, and concerns cubic maps on the two-dimensional triangle as above. Three photographs of oscilloscope tracings are shown, each exhibiting chaotic behavior. A detailed report, joint with Paul Stein, is promised, and presumably, this is the next item, U4.

Regarding Item U4.

Non-linear transformation studies on electronic computers, written by Mary Mentzel, Paul Stein, and Stan Ulam in January of 1963, is the detailed report promised in Item 3, and may be regarded as a sequel to Item 2. It expands on the cubic maps in two dimensions of Item 3, similar to the treatment of quadratic maps in Item 2, and goes on to study piecewise linear maps as well. Items 1, 2, 3, and 4, taken together, may be regarded as the first great classic of computational chaos theory. This pushes back the origin of computational chaos theory from 1961, as claimed in *The Chaos Avant-garde*, to 1947.

Limit sets are classified in U4 as Class I if there is a single fixed point attractor, Class II if the limit sets are one or more closed curves, Class III or *pseudo-periodic* if a finite set of infinite clusters that permute under the map, and Class IV otherwise.

A total of 9370 cubics were studied experimentally on the MANIAC II computer at Los Alamos. Three quarters of these were found to have limit sets of Class I, 16.5% of simple Class II, 5% showed multiple basins each of Class II, and 3.5% (334 cases) of Class III or IV, that is, chaotic. Appendix II presents photographs of the oscilloscope in several cases. I suspect that an algorithm, a sort of robot mathematician, was created to examine the experimental results and sort the cases into bins for Class I to IV.

Regarding Item U5.

Paul R. Stein met Ulam during WW II as a GI, and returned to the LANL after the war. He was on the Los Alamos staff from 1950 until his recent retirement. He worked with Ulam on several projects until Ulam's death in 1984. These projects are described in this tribute to Ulam, published in 1987. Seven pages are devoted to a highly readable and well illustrated survey of their joint work on iterations. I can think of no better illustration of their work than the frontispiece of Item U5, Paul Stein's reminiscence of 1987 (Figure 1).

2.2. Christian Mira

The pioneering work of Ulam and Stein was accomplished in the decade, 1953-1963. Even before their first publication of 1959, a similar line of work was initiated in Toulouse with the arrival of Igor Gumowski from Quebec in October, 1958. Mira was a student at that time, and following his military service and return to Toulouse in 1963, the two together formed the Toulouse Research Group, which has contributed enormously ever since to iteration theory and related subjects.[2] Gumowski retired in 1987, and Mira followed in 1997. They became interested in iteration theory following contact with Paul Montel, who had written a book on this subject in 1957, and were also influenced by the works of Julia and Fatou. In 1968-1969, they turned their attention to iterations of plane mappings. Here I will outline just a few of their many results:

M1. 1965. Mira.
M2. 1968. Mira.
M3. 1969. Gumowski and Mira, Mira and Roubellat, Roubellat.
M4. 1979. Kawakami and Kobayashi.
M5. 1992. Cathala, Kawakami. and Mira.
M6. 2000. Mira.

Regarding Item M1.

[2]See (Mira, 2000; p. 96) for a detailed account.

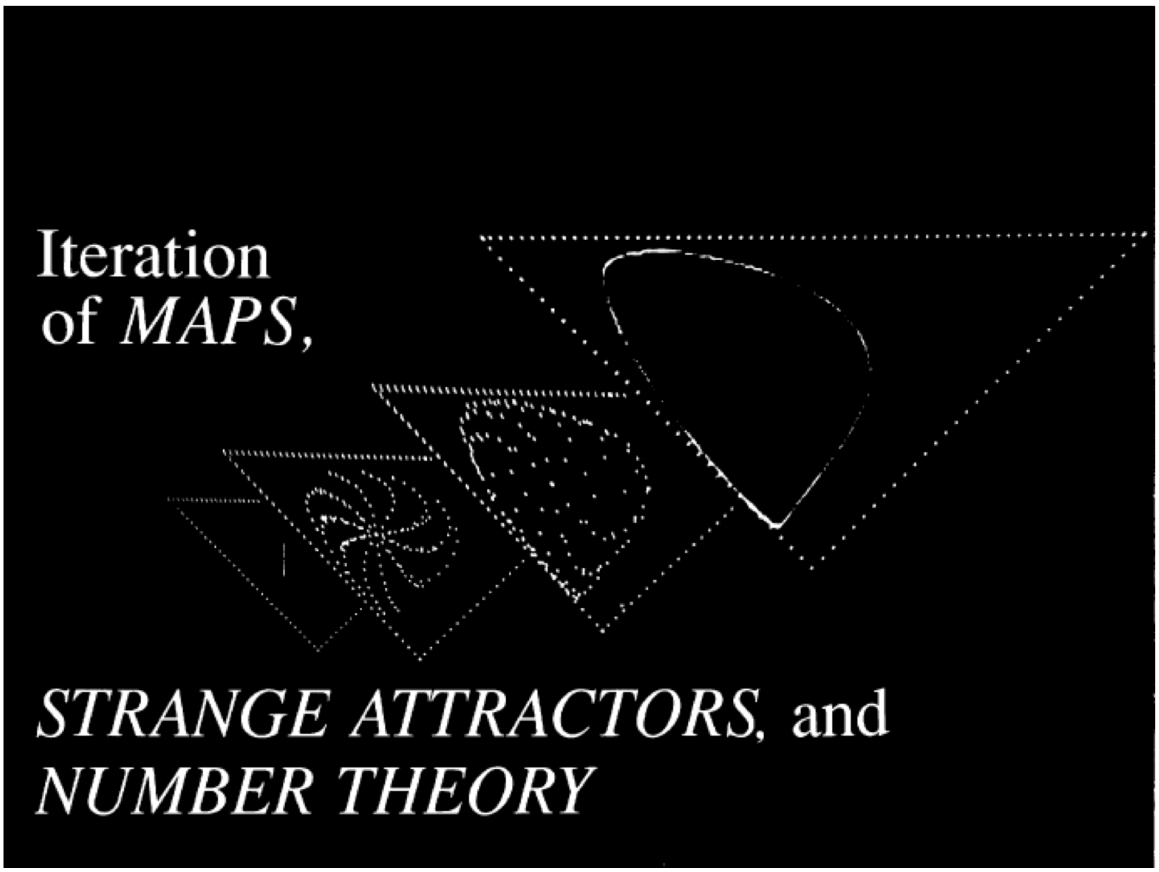

Figure 1: Frontispiece of Stein, 1987.

Gumowski and Mira developed the concept of *critical curve*, which plays a crucial role in our joint book of 1997. It was announced first in Item M1 of 1965.[3]

Regarding Item M2.

This paper presents the first computer graphics from the Toulouse group of chaotic attractors of iterations in the plane.This is therefore a sequel to item U2 of 1959.

Regarding Item M3.

The map family,

$$\begin{aligned} x' &= y \\ y' &= -dx - y - x^2 \end{aligned} \tag{2}$$

is equivalent to the Kawakami-Kobayashi system (1) above, with $a = 1$. These three papers further study this system.

Regarding Item M4.

The Kawakami-Kobayashi system (1) above was introduced in this paper of 1979. Regarding this, Kawakami has said,

> The motivation of this study was the following:
> 1) to obtain the simplest non-invertible map: a quadratic map similar to the Henon map:
>
> $$\begin{aligned} x' &= y + 1 - ax^2 \\ y' &= bx \end{aligned} \tag{3}$$
>
> 2) to obtain a non-invertible two dimensional map which is reduced to the one-dimensional quadratic map defined by $x' = x^2 + b$ when a parameter $a = 0$.[4]

Regarding Item M5.

This paper of 1992 studies the bifurcation curves of the Kawakami-Kobayashi family (1). Some results are also reported in our joint book of 1997.

Regarding Item M6.

[3]See (Mira, 2000; p. 104).
[4]Private communication.

Spectular exhibitions of chaotic attractor images were presented in Toulouse in 1973, and again in 1975. Eight of these color images are included in Item M6, including one on the cover of the book.

2.3. Clinton Sprott

In 1993, Sprott published a highly innovative book/software package, *Stange Attractors: Creating Patterns in Chaos*.

First of all, he was not content to study a simple family of polynomial maps in two or three dimensions. Like Menzel, Stein, and Ulam, he set out, in 1990, to survey them all! Secondly, for some thousands of cases, he not only drew an attractor, but also computed two measures of chaos: the fractal dimension, F, and the first Lyapunov exponent, λ_1. Finally, for 7500 cases of quadratic maps in two dimensions, he obtained a subjective aesthetic value, and plots the resulting data in the (F, λ_1) plane, as shown in Figure 2.[5] Plots of the attractors are shown in the book in 17 cases. I am grateful to him for sending me a nine-page autobiographical email, from which I have taken this account of the genesis of this work.

Recall that Mentzel, Stein and Ulam, in U4 of 1964, studied 9350 cubic maps in the plane, finding only 3.5% having a chaotic attractor. Sprott asked himself how many quadratic maps in the plane exhibit chaos.

> I did not set out to produce art; rather it was an unintended but delightful byproduct of my chaos research. My initial foray into the field of chaos was to quantify the likelihood that a simple equation chosen at random would have a chaotic solution. In 1990, personal computers were just becoming powerful enough to address this question through a brute-force search. After running millions of cases over several months of continuous computing, I was able to find that a few percent are chaotic. The computer could be programmed to test for chaos automatically and to record the conditions under which the chaos occurred in a compact form, initially only eight bytes for each case found.

[5] (Sprott, 1993; p. 316)

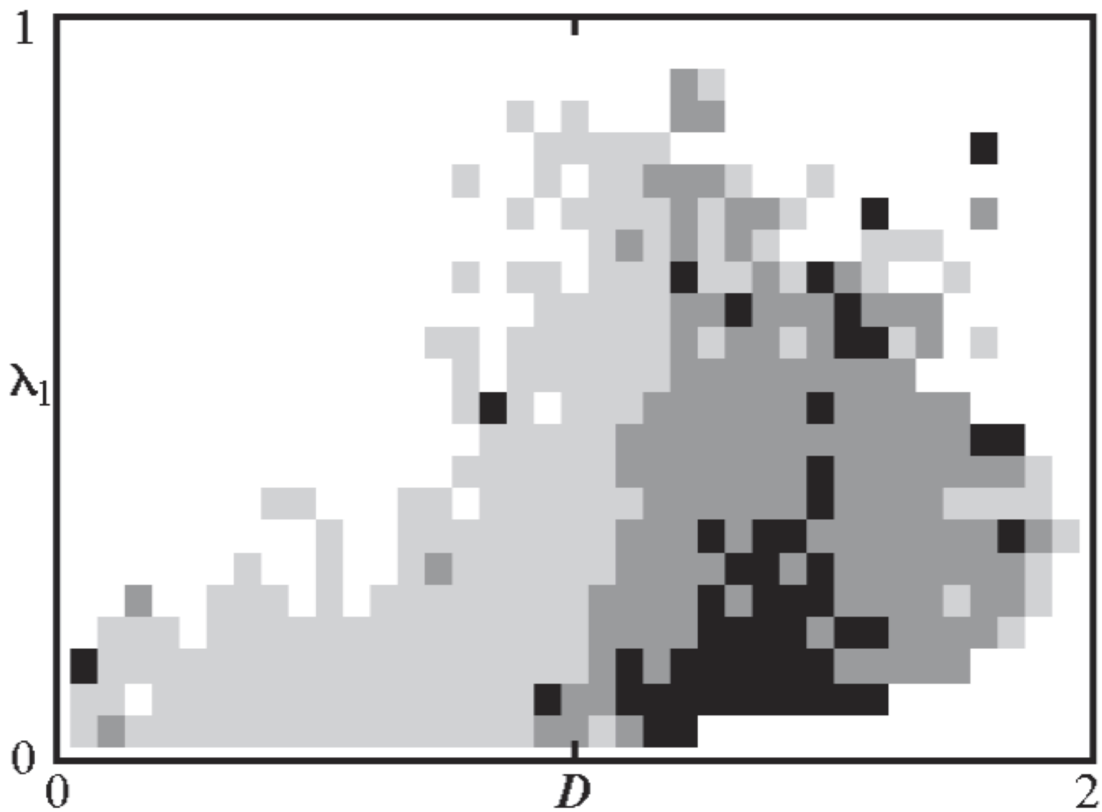

Figure 2: Figure 8-4 of Sprott, 1993. Darker indicates more aesthetic.

3. My History: 1974-1997

Following my work in pure mathematics, 1960-1974, I had a conversion to computational work, which has occupied me ever since. Some highlights:

A1. 1974. Received preprint of Mira's graphics, Tektronix graphics device, ORBIT program of Richard Palais
A2. 1991. Siena Workshop on Mathematical Economics
A3. 1992. Fellowship at NASA Goddard Supercomputer Center
A4. 1993. Urbino (Gardini), Toulouse (Mira)
A5. 1997. Joint book

Regarding Item A1.

There were three triggers for my conversion from pure to computational mathematics.. First was the arrival in my university of one of the first commercially available computer graphic devices, a Tektronix 4006 storage scope, in 1974. Second was the arrival of a 1973 preprint from Mira, showing beautiful images of chaotic attractors in the plane.[6] And third was a BASIC program written for me by my colleague Richard Palais for drawing chaotic attractors on the Tektronix scope, connected to our mainframe computer. All this came together in a new direction for my teaching and research.

Regarding Item A2.

In 1972 I had given a two-week short course on nonlinear dynamics at the University of Florence, at the invitation of Professor Francesco Gherardelli. I had made the acquaintance of several Italian mathematicians, including Franco Gori, who had been a PhD student of Gherardelli, and Gerald Goodman, an American expatriate. Most of the pure mathematicians in Italy are teaching officially in economics departments, and thus it happened that I was invited to a conference on mathematical economics in Siena in the Spring of 1991, organised by Franco Gori, Lucio Geronazzo, and Marcello Galeotti. I was greatly impressed by the computer graphics and technical expertise of two mathematicians there, Laura Gardini and Christian Mira. I already knew of Mira's work, but Gardini was new to me.

Regarding Item A3.

[6] Later published as (Bernussou Liu Hsu and Mira, 1976). Some were also published in M6.

One of the projects I wished to pursue involved a two-dimensional lattice of two-dimensional oscillators. This project was delayed for years as the super-computational demands were too great. But after some years, serendipity struck. My friend Jack Corliss obtained a fellowship at the NASA Goddard Space Flight Center in 1991-1992. Through him I also was awarded a grant for extended visits to Goddard, to enable us to work together on the oscillator lattice simulation. Goddard maintained a huge supercomputer center, which was intended to provide researchers with the latest and greatest machines, which they called "testbeds". At that time, the leading supercomputer was the Massively Parallel Processor, or MPP. This giant machine spoke only its own language, which was known only to one person, its designer, John Dorband.

Fortunately, Dorband was employed at Goddard to assist MPP users. So Jack Corliss and I got spend many hours with John Dorband in the control room of the MPP, known as "the blue room". We learned about MPP programming from John, and he learned about chaos theory from us. Shortly after my return to Santa Cruz from Goddard, I received a small spool of 16mm film from John. It contained a computer graphic masterpiece of chaos research, a three-dimensional tour through the portrait of a one-parameter family of two-dimensional chaotic attractors that I came to call "the Dorband family". This is shown in equations (2) above. The movie exhibited very unusual bifurcation features (transformations from one type of attractor into another) that fairly demanded further research.

Regarding Item A4.

The graphics of Dorband's movie were very similar to those I had seen of Gardini and MIra that I had seen in Siena. In an effort to understand the phenomena exhibited by the Dorband family, I went to see Laura Gardini in Urbino, not once but several times, 1993-1996 In this way I met Mira, visiting him once in Toulouse, and learned some of the theory of critical curves.

Regarding Item A5.

The outcome of this program was our joint book of 1997, which includes very detailed analyses of the Kawakami-Kobayashi and Dorband families of maps. The companion CD of the book, showing our own view of the Dorband family, was made primarily by Ronald Record, then a PhD student of mine in Santa Cruz.

PART TWO: IMAGE ENTROPY EXPERIMENTS

4. Image Entropy Defined

In the literature of chaos theory, many measures of chaos have been developed. The fractal dimension and Lyapunov exponents have been mentioned above, as attributes of a chaotic attractor. They are closely related.[7] The notion of entropy as a measure of disorder, familiar from thermodynamics, as been adapted to chaotic dynamics by Kolmogorov and Sinai.[8] Like fractal dimension, the K-S entropy is also related to the Lyapunov exponents. All these measures of a chaotic attractor are computed from the time series of a trajectory asymptotically approaching the attractor, and are difficult to compute. In place of these highly precise measures, we shall employ a simpler measure, the *image entropy*.

4.1. Definition

We define a grid in the domain of the iterated map, chose an initial point, and iterate the map some number of times. The resulting trajectory, a finite set of points in the domain, will populate some of the cells (we call them *patches*) of the grid repeatedly. We count the number of points of the trajectory in each patch. This count is called the *hits* of the patch. The data may be visualized as a gray-scale image in case the domain is two or three-dimensional: let black indicate zero hits, white indicate the maximum of the hits values, *maxhits*, and shades of gray in between.

The image entropy is easily computed from the list of hits values. It will depend to some extend on the length of the trajectory and the refinement of the grid, but it should converge if the trajectory is bounded. Here is the algorithm. We construct a list of indices,

$$indexlist = [0, \ldots, maxhits] \tag{4}$$

For each index, count up all the patches which are populated by that number of hits, $patchcount(hits)$, and list them;

$$freqlist = [patchcount(0), \ldots, patchcount(maxhits)] \tag{5}$$

[7]See (Sprott, 2003; pp. 121-122, 311) for an excellent explanation.
[8]See (Shaw, 1984).

This is the histogram of the shades of gray in the image of the trajectory, as described above. Now if *numbpatchs* is the total number of patches in the grid, we may calculate the probability, p, of each hits value by,

$$p(hits) = patchcount(hits)/numbpatches \tag{6}$$

and now list these results,

$$problist = [p(0), \ldots, p(maxhits)] \tag{7}$$

In other words, the *problist* is the $freqlist$ divided by $numbpatches$.
Note. The sum of all the terms of *problist* is 1.

Now let the function $e(p)$ be defined by

$$e(p) = p * log_2(p) \tag{8}$$

Applying this function to each term of the *problist*, we obtain,

$$entlist = [e(p(0)), \ldots, e(p(maxhits))] \tag{9}$$

Finally, the image entropy, E, of the trajectory is obtained by summing up all the terms of *entlist*, and dropping the minus sign.

4.2. The Unit Simplex and the Function E

Let $N = maxhits$, Then we may consider *problist* as a point, P, in euclidean $(N+1)$-dimensional space, the *probability vector*. The *unit simplex* in this space is the set of points,

$$(x_1, \ldots, x_{N+1}), x_i \geq 0, \Sigma x_i = 1 \tag{10}$$

which belongs to a hyperplane of dimension N. In case $N = 2$, the unit simplex is a triangle in 3-space with vertices on the unit points of each of the three axes. In any case, the probability vector, P, lies in the unit simplex, as its coordinates sum to one as noted above.

As the image entropy of a trajectory, E, depends only on the entries in its *problist*, we may think of its computation in two steps: first the calculation of it probability vector, P, in the unit simplex, and second, the evaluation of the real-valued function, $E(P)$, defined on the unit simplex by adding up all the $e(p_i)$ values of the coordinates p_i of P.

Proposition. Let C denote the center of the unit simplex is the point $C = (c, \ldots, c)$ where $c = 1/(N+1)$. Then the function $E(P)$ for P in the init simplex achieves its maximum value at the point $P = C$.

This proposition is easily proved using vector calculus.

4.3. Examples

As the function E involves the logarithm base 2, we may simplify the arithmetic by considering the case of a trajectory having $maxhits = 2^q - 1$ for some small positive integer q. Then the maximum value of $E(C)$ at the center of the unit simplex is simply q.

Example 1.

Suppose we have a planar dynamical system with a point attractor. After discarding a transient, we obtain an image which has one white point in a black background. Suppose we determine a square region in the domain of the map, and choose a grid of 100 by 100 patches. Then the number of patches is 10,000, and we have run a trajectory of 100 points. Then we have 100 hits in one patch, and 0 in the other 9999. Thus, $maxhits = 100$, and we have the two lists of length 101,

$$\begin{aligned} indexlist &= [0.1...., 99, 100] \\ freqlist &= [9999, 0, ..., 0, 1] \end{aligned}$$

To find the probabilities we divide the frequencies by the number of patches, 10,000, getting,

$$problist = [0.9999, 0, ..., 0, 0.0001]$$

Note that the sum of the entries is 1. And now for the *entlist* we use the approximate values,

$$\begin{aligned} log_2(0.9999) &\simeq 0 \\ log_2(0.0001) &\simeq -13.3 \end{aligned}$$

so,

$$loglist = [0, ..., 0, -13.3]$$

and multiplying *loglist* entry by entry by *problist*, we get,

$$entlist \ = [0, ..., 0, -0.0133]$$

and adding entries and dropping the minus sign, we find the approximate entropy for this point attractor,

$$E \ = 0.0013$$

Example 2.

Now consider a wandering trajectory of 100 points, each in a different patch (no matter which ones). Thus,

$$\begin{aligned} maxhits &= 1 \\ indexlist &= [0,1] \\ freqlist &= [9900,100] \\ problist &= [0.99,0.01] \\ log_2((0.99) &\simeq -0.014 \\ log_2(0.01) &\simeq -6.64] \\ entlist &\simeq [-0.014,-0.0664 \\ E &\simeq 0.08 \end{aligned}$$

Example 3.

Consider a 2D image, like a flag, with four horizontal stripes of equal height. The lowest stripe is black, that is, all its patches have zero hits. The stripe above is dark gray, all patches have one hit. The next, light gray, two hits, and the top stripe, white, or 3 hits. Then we find,

$$\begin{aligned} maxhits &= 3 \\ indexlist &= [0,1,2,3] \\ freqlist &= [2500,2500,2500,2500] \\ problist &= [0.25,0.25,0.25,0.25] \\ log_2(0.25) &= -2 \\ entlist &= [-0.5,-0.5,-0.5,-0.5] \\ E &= 2 \end{aligned}$$

This is consistent with the proposition above, and also puts the preceding two examples in context. They are, respectively, tiny, and very small. We now go on to some more realistic examples.

5. Entropy Results

In joint work with Gardini and Mira, we analyzed bifurcations in two families of plane endomorphisms. The first family,

$$\begin{aligned} u &= ax + y \\ v &= b + x^2 \end{aligned} \tag{11}$$

with a and b each in the interval $[-2, 2]$, was introduced by Kawakami and Kobayashi in 1979, and subsequently studied by Mira and coworkers. In our book, we studied the bifurcations along three curves in the plane of the parameters (a, b), defined by $a = 0.7, 1.0$, and -1.5, and b in the intervals, $[-1.0, -0.4]$, $[0.593, 0.600]$, and $[-2.115, -1.5]$, respectively.

The second family,

$$\begin{aligned} u &= (1-c)x + 4cy(1-y) \\ v &= (1-c)y + 4cx(1-x)) \end{aligned} \tag{12}$$

with c in the interval [0.1], was introduced by John Dorband in 1991, and studied by Gardini and coworkers in the 1990s. In our book we studied a sequence of bifurcations as c increased from 0.6 to 0.75. Let's begin with this first family.

5.1. Bifurcations in the Kawakami-Kobayashi Family

Our book begins the exploration of this family in Chapter 4 with the case $a = 0.7$, and decreasing b in the interval, $[-1.0, -0.4]$. The first event noted is a Neimark-Hopf bifurcation between $b = -0.4$ and -0.5.

The next event noted in Chapter 4 is an explosive bifurcation between $b = -0.78$ and -0.80. For these values of the parameters, we find $E = 0.024$ and $E = 0.360$ respectively. The relevant figures in the text are Fig. 4-15 and 4-16.

The next value of a considered is $a = 1.0$ in Chapter 5. The first event noted is a contact bifurcation between $b = -0.593$ and -0.59500. For these values of the parameters, we find $E = 0.321$ and $E = 0.361$ respectively. The relevant figures in the text are Fig. 5-1 and 5-5.

Finally, the case $a = -1.5$ is treated in Chapter 6. An explosive bifurcation is noted between $b = -1.98$ and -2.1. For these values of the parameters, we find $E = 0.045$

and $E = 0.173$ respectively. The relevant figures in the text are Fig. 6-23 and 6-28. A graph of E versus $-b$ is shown in Figure 3. Note that the entropy plot clearly locates the explosive bifurcation between $-b = 2.08$ and 2.09.

5.2. Bifurcations in the Dorband Family

Our book dissects in detail a bifurcation between $c = 0.64218$ and 0.64219. This is a "type II CCB of the second kind." In it, a 14-cyclic attractor becomes a 7-cyclic attractor by merging pieces in pairs. Computing the image entropy for these two values of the control parameter, c, yields the approximate values $E = 0.084$ and 0.075, respectively.

Nearby is a type II bifurcation of the first kind, between $c = 0.64392$ and 0.64400, in which the 7-cyclic chaotic attractor explodes into a larger 1-piece chaotic attractor. Computing the image entropy for these two values of the control parameter, c, yields the approximate values $E = 0.134$ and 0.184, respectively.

In each of these cases, a bifurcation is revealed by a jump in the entropy. By the way, the NetLogo programming language is extremely convenient for this work. Writing the program is very rapid, but the computations are rather slow. On a Mac Pro with eight cores running at 2.8 GHz, these runs each took about an hour. With $c = 0.740$, $E = 0.84$ was obtained after about five hours. A screen shot is shown in Figure 4.

6. Conclusion

Image entropy as been tested as a robot mathematician in the spirit if Stein and Ulam, using the families of plane endomorphisms studied in our joint book, *Chaos in Discrete Dynamical Systems.* Important bifurcations were successfully discovered, revealed by jump discontinuities in the image entropy as a function of control parameters. The investigation was greatly facilitated by use of the NetLogo programing language, in which the entropy calculation requires only a few lines of code. A goal for the future would be to produce automatically a bifurcation plot such as Figure 7 in (Kawakami and Kobayashi, 1979).

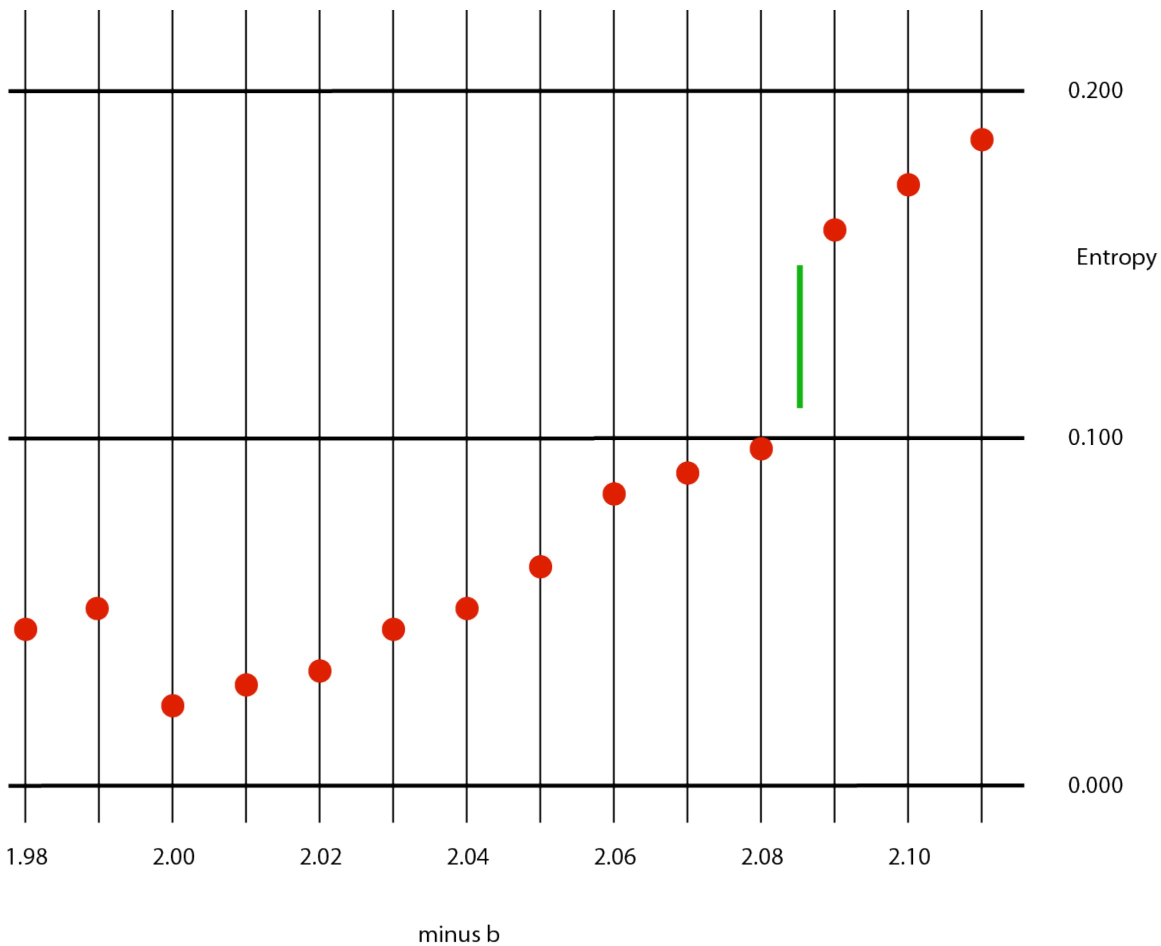

Figure 3: E vs $-b$ for the Kawakami-Kobayashi family, $a = -1.5$.

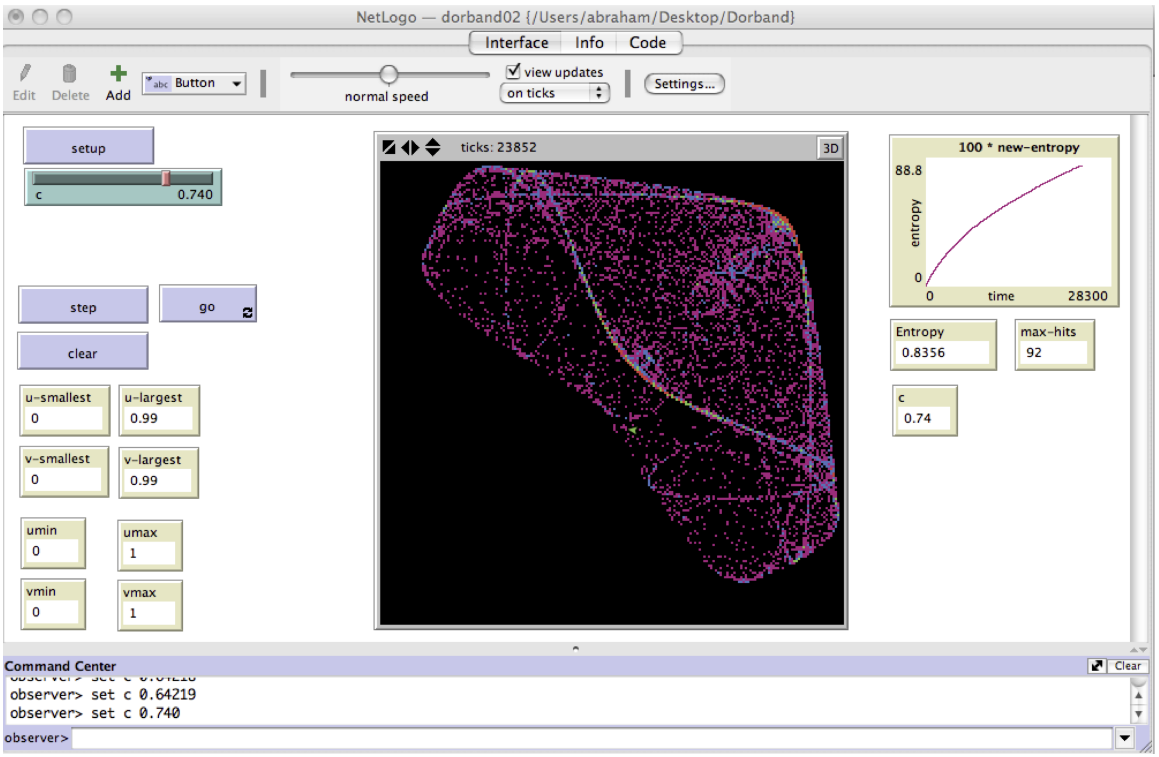

Figure 4: Interface of the NetLogo model for the Dorband map, $c = 0.740$.

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Regarding Mary Mentzel, see: http://philosophyofscienceportal.blogspot.com/2008/04/wrong-righted.html

Regarding the MANIAC computers, see: http://en.wikipedia.org/wiki/MANIAC_I