The Peregrinations of Poincaré

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Abstract

Dynamical Systems Theory in the spirit of Poincaré (DST) has been in vogue since the controversial award of a prize by King Oscar II of Sweden and Norway, on his 60th birthday, January 21, 1889, to Poincaré. DST diffused Eastward (via Stockholm, Saint Petersburg, and Moscow) to Gorky in Russia, to Japan, and Westward (via Princeton, Mexico City, and Rio), to Berkeley in California. Now in the centennial year of the premature death of Poincaré, it is time for a review of these peregrinations.

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INTRODUCTION

Plotting on a world map the early trajectory of dynamical systems theory from Paris in 1880, into the computational era around 1965, we find a loop, or periodic orbit, as shown in Figure 1. In this chapter we will approach the main sites on this trajectory through biographies of some of the principal agents.[1] Our main focus is on flows, and we have omitted the extensive history of iteration research. I am very grateful to Christophe Letellier for suggesting this writing, and for many suggestions for its scope and presentation, and further to my colleagues Jean-Marc Ginoux, Christian Mira, Miguel A. F. Sanjuan, Charles Tresser, and Yoshisuke Ueda for very abundant historical details and editorial advice. This is my most collaborative writing project ever, and I am both grateful and honored by their participation.

THE ORIGIN

1. Jules Henri Poincaré, 1854-1912

Born in Nancy, France, Poincaré earned a doctorate from the University of Paris in 1879 at age 25. He was professor at the University of Paris until his premature death at age 58. In his short career of 33 years he published 30 books and 500 papers, enriching most branches of mathematics and physics. His popular books introduced the latest ideas of math and science to a wide audience.

Among his accomplishments, the most important for our story is his creation of Dynamical Systems Theory (DST), the geometric (or qualitative) theory of differential equations, in his earliest publications of 1880-1886.[2] The geometric method is introduced in the paper that begins on the very first page of his collected works. Among these methods is the *phase portrait*, a visualization of the flow in the state space, or domain of the system of differential equations, originally the Euclidean plane. He introduced the fundamental idea of a limit cycle, as a closed trajectory in the state space, in 1882.[3]

Later, in 1908, he pointed out that an electronic oscillation is represented by a

[1]Some high points of this journey are recounted in (Abraham, 2000).
[2]For details, see (Barrow-Green, 1997; p. 263).
[3](Poincaré, 1882; p. 261)

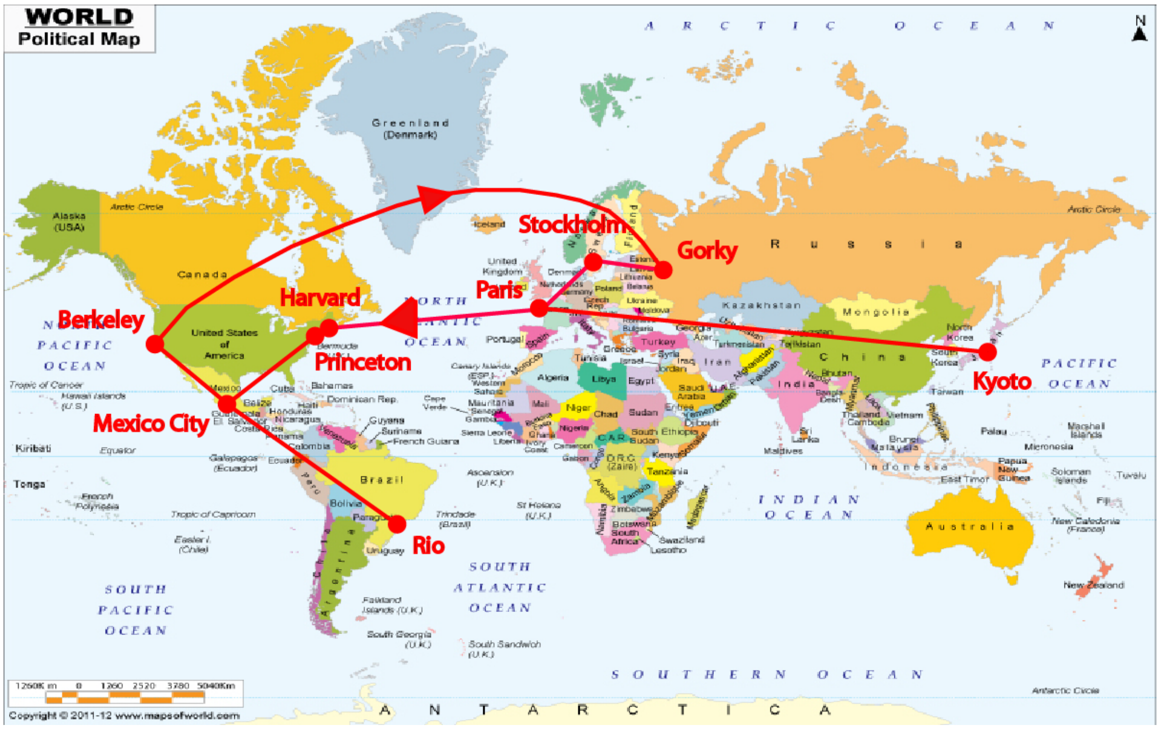

Figure 1: The Trajectory of DST.

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East 1: Paris to Stockholm, Moscow, 1884.

West 1: Paris to Harvard, 1913.

East 2: Moscow to Kiev and Gorky, 1930.

West 2: Kiev to Princeton, 1943.

West 3: Princeton to Mexico City, Rio, and Berkeley, 1958.

East 3: Berkeley to Gorky, 1963.

East 4: Paris to Kyoto, 1942.

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periodic attractor, or stable limit cycle.[4] This idea is fundamental to the theory of nonlinear oscillations, an important part of the development of DST prior to the advent of chaos theory in the 1960s. Poincaré also pioneered analytical methods based on series expansions.

Even more to the point was his creation of the foundations of chaos theory in his paper on the three-body problem of 1890, which was awarded the prize in the 60th birthday competition of King Oskar II of Norway and Sweden. In a nutshell, the competition was conceived in 1884, announced in 1885, and closed for entries in June of 1888. Poincaré announced his intention to enter the competition to Mittag-Leffler of the University of Stockholm in 1887, explicitly mentioning the first of the four prize problems, on the stability of the three-body problem. At closing time, twelve entries had been submitted. Poincaré's entry, running to 158 pages and very difficult to read, was judged the winner. The result was announced in a local newspaper on King Oskar's 60th birthday, January 20, 1889. The winning paper was prepared for publication in October in the Swedish journal, *Acta Mathematica*, which had been founded by Mittag-Leffler with the support of King Oskar.

Preparation of the paper for publication was entrusted to Lars Edvard Phragmén (1863-1937), one of the editors of the journal, who reported to Mittag-Leffler in July that there were some obscure passages in the paper. Mittag-Leffler wrote to Poincaré for clarifications, and Poincaré replied in December of 1889 that he had found a serious mistake elsewhere in the paper, and he was making major revisions. Prepublication copies of the original prize paper were quickly recalled from circulation and destroyed. Only one copy survived, which was hidden in the Institut Mittag-Leffler until recently.[5] Amazingly, Poincaré was able to submit a revised paper by January of 1890. Fast work indeed, as the original paper purported to prove the stability of the three-body problem, while the revised paper established the opposite! All 270 pages appeared in the *Acta* later in 1890.[6]

And in this hasty work around the New Year of 1890, Poincaré intuitively understood the famous *homoclinic tangle*, basic to chaos theory.[7]

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[4]See (Ginoux and Petitgirard, 2010).

[5](Barrow-Green, 1997; p. 1)

[6]The full story of the prize paper is spelled out in detail in (Barrow-Green, 1997; Ch. 4).

[7]See (Poincaré, 1881-1886), and also the webpage by Jean-Marc Ginoux, "Poincaré's limit cycles" at www.scholarpedia.org.

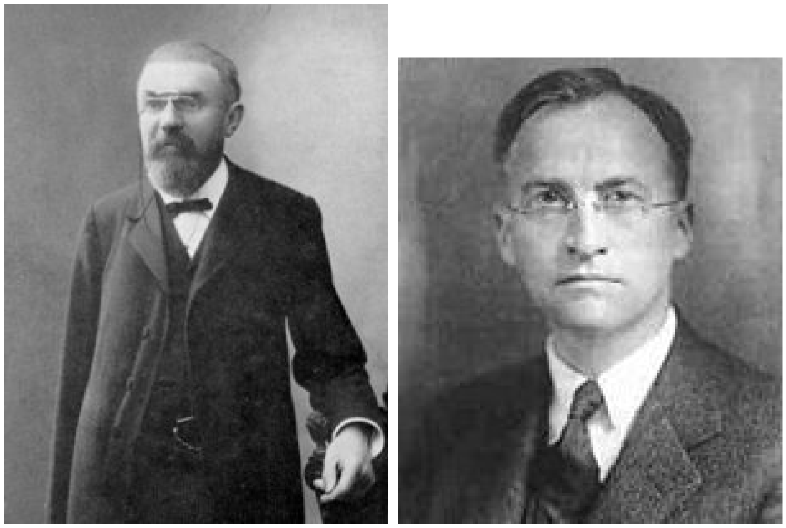

Figure 2: Poincaré and Birkhoff.

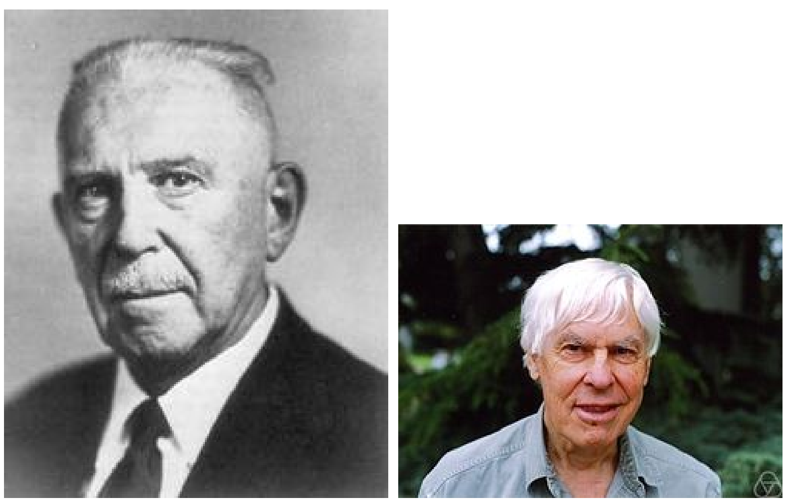

Figure 3: Lefschetz and Smale.

WESTWARD JOURNEY

2. George David Birkhoff, 1884-1944

Christian Mira, who has contributed extensively to dynamical systems theory (DST) and also written much of its history, has said, "Intellectually, Birkhoff was the greatest of Poincaré's disciples."[8]

Birkhoff was born of Dutch ancestry in Overisel, Michigan. He earned the Ph.D. from the University of Chicago in 1907 at the age of twenty-three, and became a professor at Harvard University in 1911, where he stayed throughout his life. He earned many honors, and was regarded the dean of American mathematics. He contributed to many areas of mathematics, especially dynamics and ergodic theory. He became famous through his proof in 1912 of Poincaré's Last Theorem, left unproved by Poincaré at his death in 1912. Thus Birkhoff continued the tradition of Poincaré in dynamics without a gap. His further work in topological dynamics was published in 1927 as *Dynamical Systems*, thus giving a name to the new field we call DST. Of special interest to our story here is his joint paper with Paul Smith of 1928, in which the *homoclinic tangle* discovered by Poincaré in 1881 was drawn and fully understood for the first time.[9] This is shown in our Figure 4.[10] The signature of a homoclinic point was defined as well.[11]

Serendipitously, after I had been working on the Birkhoff and Smith paper in Berkeley for two years, I followed Steve Smale to Columbia University in 1962, and found that Paul Smith was my chairman, having just succeeded Samuel Eilenberg.

Birkhoff's dynamical publications continued for another dozen years, until 1941. Then, for some unknown reason, the Westward trail of Poincaré ended, with an address at the University of Chicago on unsolved problems in dynamics.[12] Perhaps Birkhoff's anti-Semitism, as alleged by Albert Einstein and Norbert Wiener among others, played a role. The entry of the United States into World War II might also have been a factor.

[8] (Mira, 1986; p. 251)

[9] What we now call a heteroclinic point is called a doubly asymptotic, or d.a., point, by Birkhoff and Smith.

[10] (Birkhoff, 1950, vol. 2; p. 365)

[11] This is carefully drawn by Chris Shaw in (Abraham, 1992; Sec. 13.5.)

[12] (Birkhoff, 1950, vol. 2; pp. 710-712)

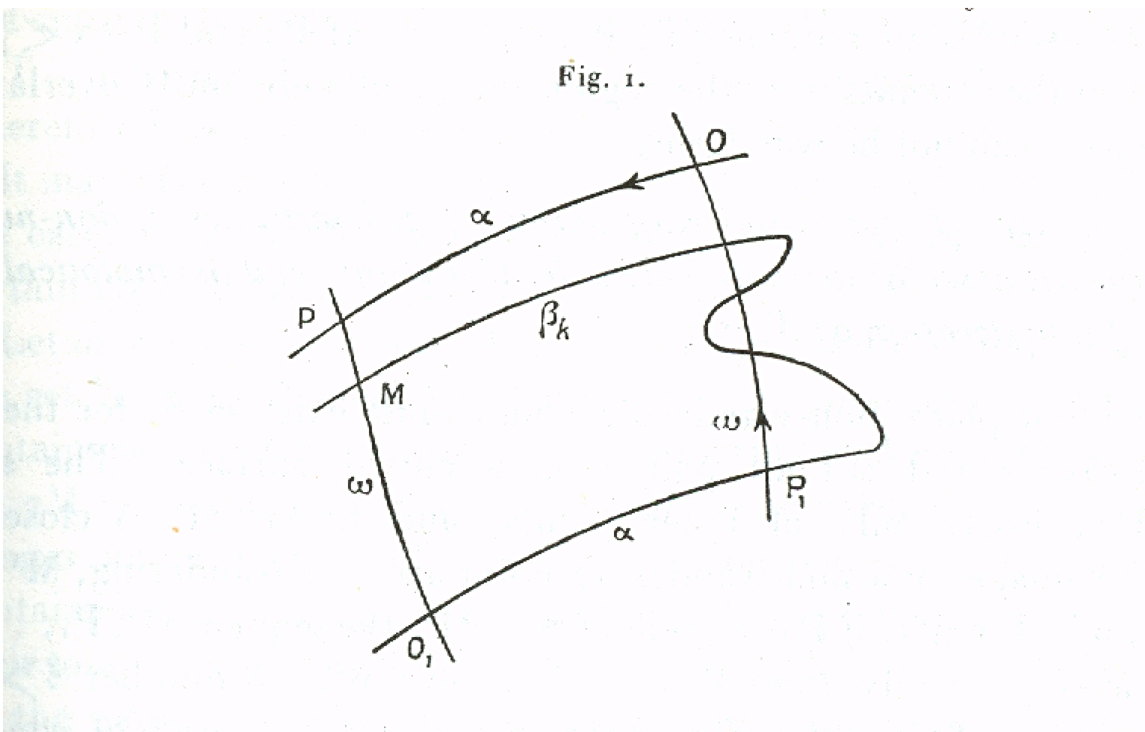

Figure 4: The periodic point O (O_1 is its image) with its inset (stable manifold, two segments labelled ω are shown) and outset (unstable manifold, two segments labelled α are shown) intersect at the homoclinic point, P and its image P_1, and once again at M. This is Fig.1 from Birkhoff and Smith, 1928.

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3. Solomon Lefschetz, 1884-1972

Lefschetz was born in Moscow to a Jewish family and moved to Paris at an early age. French was his first language. He studied engineering in Paris, and moved to the USA after graduation, in 1905. After losing both hands in an accident he migrated into mathematics, earning the Ph.D. in math from Clark University in 1911. He became one of the preeminent pioneers of algebraic topology. He was professor of mathematics at Princeton University from 1924 to 1953, editor of the *Annals of Mathematics* from 1928 to 1958, and president of the American Mathematical Society, 1935-1936. He had 25 excellent Ph.D. students at Princeton, Paul Smith the first of them, in 1926.

In 1943 he met Nicolai Minorsky (1885-1970) who had studied in Saint Petersburg and received the Ph.D. in 1914. Minorsky emigrated to the US in 1918, eventually becoming a professor at Stanford University. His knowledge of the Russian school of dynamical systems theory, presented in his book of 1947 on nonlinear oscillations, gave great impetus to the resumption of mathematical dynamics in the United States. Through Minorsky, Lefschetz discovered some of the Russian works on dynamical systems.[13]

During World War II he continued his studies on dynamics, reviving the Poincaré tradition, and founding the new field of global analysis. His translation of the early works of Andronov and his students into English, published in 1943, 1946, and, 1949, gave a strong boost to the reestablishment of the Poincaré tradition following its recent demise in the Birkhoff line after 1941.

He was an extraordinarily altruistic person, and decided in 1944 to further the development of mathematics in Latin America by teaching graduate courses at the Universidad Nacional Autónoma de México, UNAM, in Mexico City. It was there that I met him in 1959, while working on my Ph.D. thesis on general relativity under Nathaniel Coburn at the University of Michigan.

He attracted graduate students from all over the Americas, and in this group came

[13]Especially those of Andronov, Khaikin, Vitt, Krylov, and Bogolyubov. Minorsky's books cover an enormous range of topics, from European as well as Russian authors (especially, Poincaré and Van der Pol).

the revival of the concept of structural stability of Andronov and Pontriagin, and the related idea of generic properties.[14] Peixoto improved on earlier work of H. F. de Baggis, another student of Lefschetz.

After retirement from Princeton in 1953, he moved to the Research Institute for Advanced Studies, RIAS, in Baltimore, which flourished from 1958 to 1964, and then created the Lefschetz Center for Dynamical Systems at Brown University.

A boom in DST and global analysis was triggered by Mauricio Peixoto (b. 1921) of Brazil, who had studied with Lefschetz in Princeton in 1957. Lefschetz encouraged him to work on dynamical systems, and Peixoto proved an important result on structural stability in two dimensions in 1958.

4. Stephen Smale, b. 1930

Smale was born in Flint, Michigan, and received the Ph.D. in mathematics from the University of Michigan in 1956. That summer, at a conference in Mexico, he met René Thom, Moe Hirsch, and Elon Lima from Brazil. In the fall, all four were at the University of Chicago, along with Dick Palais and Ed Spanier, and Thom was lecturing on transversality theory. Later that year, through Lima, Smale met Peixoto in Princeton, learned of his result, and proposed an extension to higher dimensions, using Thom's ideas on transversality in combination with his own extensive experience and ingenuity in higher dimensional differential topology.

In Mexico City in the summer of 1959, where I had met Lefschetz while working on my thesis. I had noticed that a conference was in progress. At this conference, Smale met Lefschetz, and delivered a talk on Morse-Smale dynamical systems. Early in 1960 Smale visited Brazil, where he studied homoclinic points on the beaches of Rio. After arriving in Berkeley in the summer of 1960, he perfected his horseshoe example, which was published in 1963. In his autobiographical writings, Smale gives credit to the Andronov school of Gorky.[15]

All these mathematicians converged in Berkeley in the fall of 1960, just as I arrived for my first academic job following my own Ph.D. at the University of Michigan. At the daily tea in the math department in Campbell Hall, I met them all. I began

[14] The concept of structural stability is due to Andronov and Pontryagin in 1937.

[15] (Smale, 2000; p. 21)

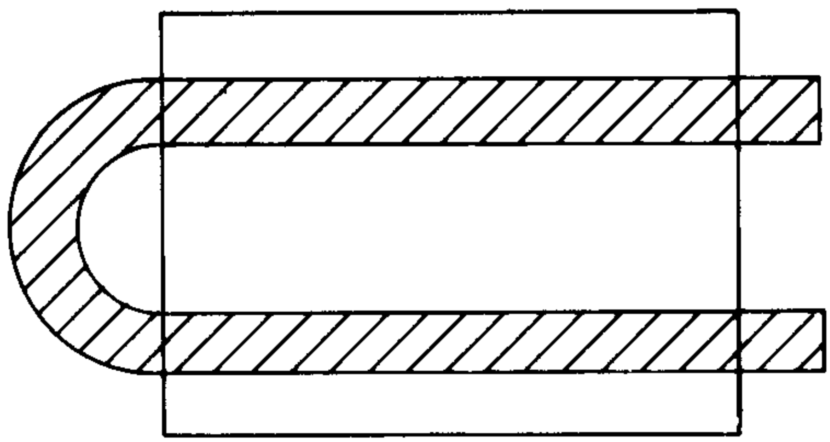

Figure 5: The horseshoe mapping of Smale.
The shaded horseshoe is the image of the plain rectangle under one iteration of the map. From Smale, 1980, p. 150.

working with Thom on transversality theory, and with Smale on homoclinic points and other problems of global analysis.

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EASTWARD JOURNEY

5. Sophie Kovalevsky, 1850-1891

The number of women in mathematics is lately on the rise, overturning a gender bias on a planetary scale of long standing. As recently as 1974, a sympathetic survey of the history of mathematics from ancient times turned up only eight significant names,[16] Kovalevsky was seventh on this list. Among her many mathematical ac-

[16] See (Osen, 1974).

complishments, her contribution to dynamical systems theory (DST) may be the least known, and here we will tell her story.

Born Sofia Vasilyevna Korvin-Krukovskaya in Moscow in a long line of mathematicians, she grew up far away from the cultural centers of Russia, near the border with Lithuania. At seventeen, she went to Saint Petersburg to study mathematics. Her determination to continue her studies in university was frustrated as European universities were open only to men. She entered a convenience marriage in 1868 with a friend, Vladimir Kovalevsky, to obtain permission to leave Russia, and together they went to Heidelberg to continue their studies. Outside Russia, she began to use the name Sophie Kovalevsky. Although the university in Heidelburg was also closed to women, she was able to attend lectures there by Leo Königsberger (1837-1921), who had been a student of Karl Weierstrass (1815-1897) in Berlin. Weierstrass was then the most famous mathematics professor in Europe, and regularly lectured to audiences of 300 or more.

In 1870 she moved to Berlin, with a strong recommendation from Königsberger, to study with Weierstrass, but found his university also closed to women. He accepted her as a private student, and after four years as his star pupil, with already three publications, she was awarded a Ph.D. from Göttingen. She returned to Russia in 1874 with Vladimir. Here for some time, unable to find a job, she occupied herself writing theater reviews.[17] After the birth of her daughter Fufa in 1878, she returned to Berlin to continue mathematical work. In 1882 she moved on to Paris, on the advice of Weierstrass, to interact with the mathematical community around Hermite.[18] That year she met Poincaré, who was a prominent member of this group.[19]

Among her fellow students in Berlin were many who eventually achieved note in the history books, among them, Gösta Mittag-Leffler (1846-1927). After his Ph.D. in 1872 and a career of several years as professor of mathematics in various Scandinavian universities, he became the first professor of mathematics at Stockholm University in 1881. Kovalevsky and Mittag-Leffler became friends in Saint Petersburg in 1876. Mittag-Leffler was an activist for women's rights, and he campaigned for a position in his university for Kovalevksy. In 1883, she moved to Stockholm and began giving lectures there in German. In 1884 she was appointed Professor, the first woman in this post in any European university. In the academic year 1887-1888, she gave a

[17](Kennedy, 1983; p. 177)
[18](Kennedy, 1983; p. 206)
[19]See her autobiography in (Kovaleskaya, 1978; pp. 213-229), written in 1890.

Figure 6: Kovalevsky and Andronov.

course in Poincaré's work on DST.[20] In 1888, she was awarded the Prix Bordin by the French Academy for her paper on what is now known as Kovalevsky's top.

During these years, Kovalevsky was close friends with Mittag-Leffler's sister, Anna Charlotte Leffler (1849-1892), a writer. They maintained a weekly salon, and King Oscar sometimes attended. Also, during these years, Kovalevsky made frequent trips to Saint Petersburg, Moscow, and Paris. In this way, the King became interested in the problem of the stability of the solar system, and offered the prize competition for his 60th birthday in 1889. The inspiration of this prize in 1884 was due to King Oskar himself, or Mittag-Leffler, or Kovalevsky, or all three – we do not know.[21]

The prize was offered for four problems, one of which was the stability of the three-body problem. Dirichlet (1805-1859) had announced a proof in a letter to Weierstrass in 1858, but died before publishing it. Weierstrass suggested this as a prize problem in a letter to Kovalevsky.[22]

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[20] (Kennedy, 1983; p. 323)
[21] (Barrow-Green, 1997; p. 51)
[22] (Moser, 1973; p. 9)

Figure 7: Shilnikov and Hayashi.

Figure 8: Ueda at the grave of Poincaré in Paris. Courtesy of Professor Ueda.

6. Aleksandr Andronov, 1901-1952

The trail of Poincaré in Russia begins with Alexandr Mijailovich Lyapunov (1857-1918), from whom many branches diverge. He was the first and most famous of Poincaré's followers in Russia. Inspired by the works of Poincaré and Kovalevsky, he wrote his memorial paper "Probléme Générale de la Stabilité du Mouvement" in 1892. Here he proved a useful condition for the stability of a rest point of a dynamical system.

In the 1930s, a derivative school was established in Kiev (southwest of Moscow), known especially through the works of Krylov and his student Bogolyubov. Their book of 1935 was translated by Lefschetz from French to English in 1943. Another such school was established in Gorky (also Gorkii, and now called Nizhnii Novgorod, northeast of Moscow) by Andronov in 1932. Andronov was a student of Leonid Isaakovich Mandel'shtam (or Mandelstam, 1874-1944), a Jewish mathematical physicist in Moscow. Mandelstam had a good knowledge of the works of Poincaré as he had been a student in Strasbourg, 1899-1914.[23]

In his thesis of 1930, Andronov was the first to connect the oscillations of the Van der Pol system with the limit cycle concept of Poincaré.[24] His group included his wife, E. A. Leontovich.[25] The group produced important books, including *The Theory of Oscillations* written in 1937, jointly authored with S. E. Khaikin and A. A. Vitt, and translated from Russian to English by Lefschetz in 1949. The head of the Gorky group, for many years until his recent death, was L. P. Shilnikov.[26]

7. Leonid Pavlovich Shilnikov, 1934-2011

Shilnikov was a student of Andronov and received the Ph.D. in mathematical physics at Gorky State University. In his thesis work in the late 1950s he was already interested in bifurcations of flows involving homoclinic trajectories. His first major discovery on spiral chaos in three dimensions was published in 1962 in Russian, and in English in 1970. In this work he made use of Smale's paper on the horseshoe

[23] (Bissell, 2001; p. 2)
[24] See (Mira, 1997a; p. 174) and (Bissell, 2001; p. 2).
[25] (Smale, 1980; p. 147)
[26] (Afraimovich, 2010; p. 105)

map, published in 1963, thus completing the closed orbit shown on the world map in Figure 1.

The cylinder C shown in Figure 9 is the Poincaré cross-section, the domain of the first-return map, in which the horseshoe appears, as shown in that figure. This early work was greatly extended in the sequel works of Shilnikov and his students in Gorky, and those of Charles Tresser and his coworkers, Pierre Coullet and Alain Arnéodo, in Nice in the early 1980s.[27]

We do not cover here the history of cascades (iterations of noninvertible maps), although that is also part of the tradition of Poincaré. However, it is relevant that there has been a cross-over from cascades to flows. In the context of cascades, the period doubling route to chaos was discovered by Myrberg in 1963 for the quadratic map, and its topologically universal character by Metropolis, Stein and Stein in 1973 for interval maps. The metrically universal aspect was then discovered by Feigenbaum in 1975 and independently by Coullet and Tresser. Coullet and Tresser discovered and explained its abundance in more general maps, and also in flows. Shilnikov also studied period doubling for flows.[28]

Subsequently, Shilnikov ran the laboratory of the Department of Differential Equations headed by E. Leontovich-Andronova, and became head of that department in 1984. He wrote over 200 articles and co-authored several books. In a memorial for his 75th birthday, his students wrote:

> His works greatly influenced the overall development of the mathematical theory of dynamical systems, as well as nonlinear dynamics in general. Shilnikov's findings have became classics, and have been included in the most text and reference books which are used worldwide by mathematics students and nonlinear dynamicists to study the qualitative theory of dynamical systems and chaos. The elegance and completeness of his results let them reach to the heart of the matter, and provide applied researchers with in-depth mathematical understanding of the outcomes of natural experiments. No doubt that this popularity is due the status of a living classic that Professor Shilnikov has attained over several decades for his continuous hard work on the bifurcation theory of multi-dimensional dynamical systems, mathematical chaos theory, and

[27]See (Tresser, 1983, 1984) and references therein. This figure, and the simple geometric description of it given in (Ruelle, 1989), derives from joint work of Arnéodo, Coullet, and Tresser.

[28]Many thanks to a private communication from Charles Tresser for this bit of history.

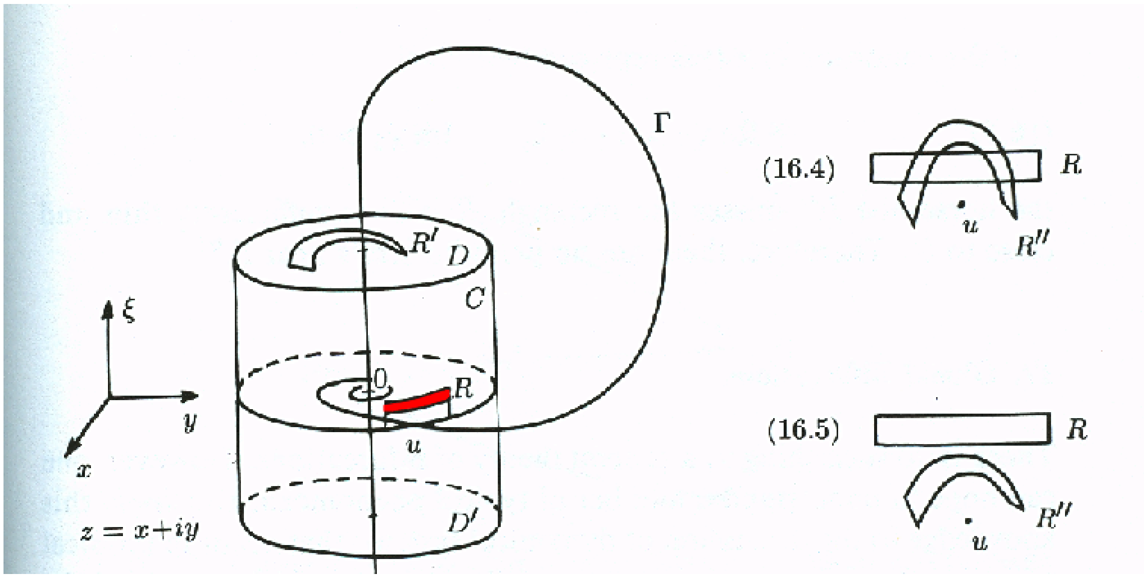

Figure 9: Shilnikov's spiral chaos.
The image under the Poincaré cross-section map of the shaded rectangle R is the horseshoe R'' shown in the upper right. From Ruelle, 1989, p. 117.

the theory of strange attractors.[29]

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8. Chihiro Hayashi, 1911-1987

Hayashi began a program of experimental work as professor in the Kyoto University Department of Electrical Engineering in 1942. He published a book on this work in English in 1953, and visited MIT in 1955-56. He had an excellent command of English, and had most likely learned the new results in dynamics from the books of Minorsky and Lefschetz.[30]

[29](Afraimovich, 2010; p. 101)

[30](Minorsky, Andronov and Chaikin, Stoker (1950), and McLachlan (1947) are explicitly acknowledged in (Hayashi, 1953; p. vii). And in the revised and expanded 1964 edition of this work, he further credits Professors Rudenberg of Harvard and Den Hartog and Paynter of MIT.

9. Yoshisuke Ueda, b. 1936

Ueda was a student of Hayashi, earning his Ph.D. at Kyoto University in 1965. He observed chaotic behavior in computer simulation of a three-dimensional flow in 1961, and was among the first to do so. Due to Hayashi's devotion to a pre-chaotic worldview, Ueda's discovery was not published until 1970.[31] His chaotic attractor is organized around an attractive homoclinic tangle, as shown in Figures 10 and 11.

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CONCLUSION

Following the famous song by Tom Lehrer: From Paris to Stockholm to Saint Petersburg to Moscow to Gorky we shall run. And then: to Princeton to Mexico City to Rio to Berkeley and back to Gorky we shall run. And lastly, back to Paris. Here are the approximate schedules for these voyages.

Westward Journey, 1880-1960

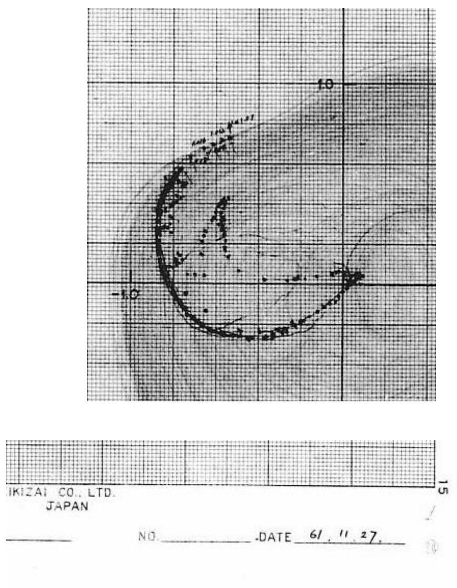
- 1880, origin at Paris (Poincaré)
- 1912, flight from Paris to Harvard (Birkhoff)
- 1928, stop at Harvard (Birkhoff and Smith)
- 1941, end of this voyage at Harvard
- 1943, arrival of new flight from Kiev to Princeton (Lefschetz)
- 1958, stop at Princeton (Peixoto)
- 1959, arrival at Mexico City (Lefschetz, Smale)
- 1960, arrival at Rio, on to Berkeley (Smale)

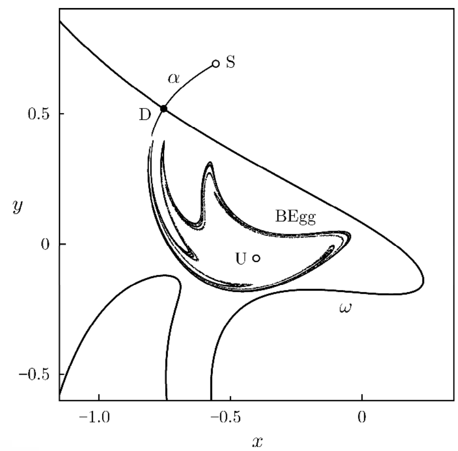
Eastward Journey, 1880-1935

- 1880, origin at Paris (Poincaré)
- 1887, stop at Stockholm (Kovalevsky)

[31] For the full story, see (Ueda, 1993).

Part of Experimental Data and Phase Portrait





ODE for the subject:

$$\left.\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= \mu(1-\gamma x^2)y - x^3 + B\cos\nu t\end{aligned}\right\}$$

Figure 10: Ueda's original plot of the Broken Egg (chaotic) attractor of 1961, left, and computer graphic showing the attractor as the outset of a homoclinic tangle, on the right. The figures belong to a Poincaré cross-section of a forced oscillator (three-dimensional) flow. This figure is from a lecture of Ueda, November, 2011, in Kyoto, on the 50th anniversary of his discovery of the Broken Egg. Courtesy of Professor Ueda.

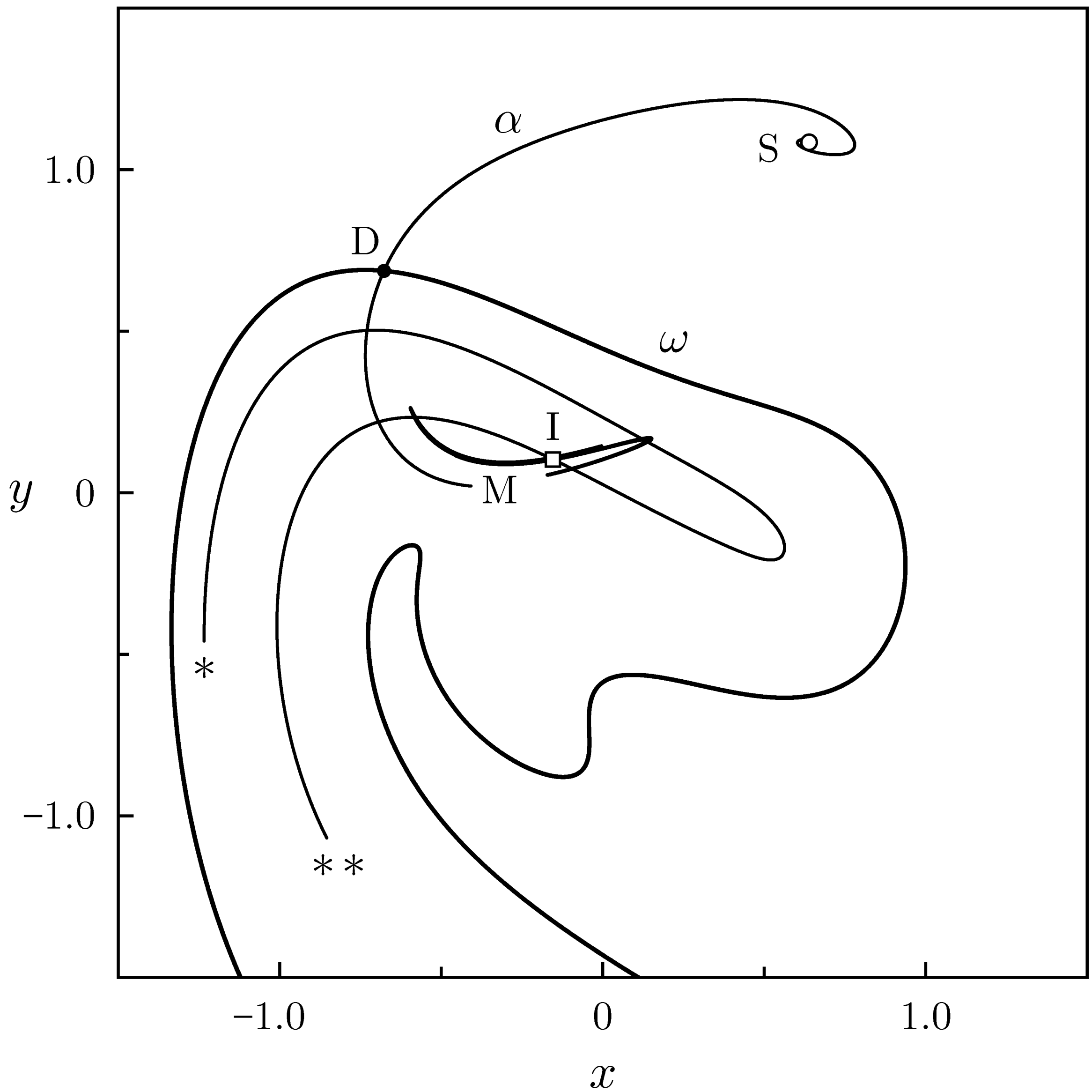


Figure 11: Another computer graphic of the Broken Egg tangle, showing the homoclinic intersections. From (Ueda, 1992; Fig. 2, p. 102), originally published in 1973 in a paper by Ueda, Akamatsu, and Hayashi. Improved version courtesy of Professor Ueda. A similar figure (Ueda, 1982; Fig. 4, p. 86) created in 1969, is among the first such drawings to be published.

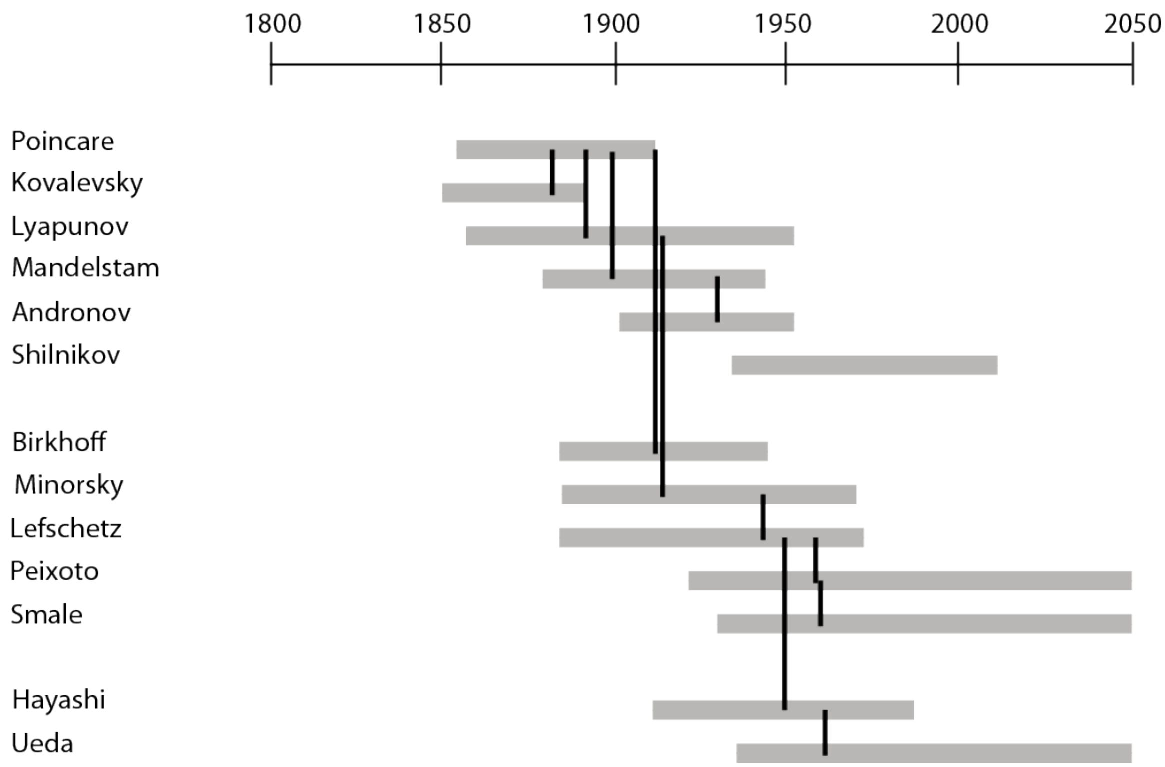


Figure 12: A flow chart for the tradition of Poincaré.

- 1892, stop at Moscow (Lyapunov)
- 1932, arrival at Gorky (Andronov)
- 1935, arrival at Kiev (Krylov)
- 1963, flight arrives from Berkeley (Smale's horseshoe)

Reunion, 1963-1970

- 1963, The arrival of Smale's horseshoe in Gorky connects the two journeys in a round trip
- 1970, The work of Charles Tresser and coworkers on Shilnikov's bifurcation brought the tradition back home to Paris

Following this time, an explosion of mathematical work extended the Poincaré tradition worldwide.

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