Chaotic Synchronization in Economic Networks

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and
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Abstract

Synchronization, long an important topic in the theory of nonlinear oscillation, has recently become a research frontier in chaos theory as well. Here we introduce the sync phenomenon for forced and coupled Rössler attractors, and its role in the context of complex economic systems. And we extend the basic framework developed by the late Richard Goodwin in his book, *Chaotic Economic Dynamics*, of 1990 to massively complex dynamical systems of chaotic elements. Recent experimental results and speculative applications to economic networks are presented.[1]

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Dedicated to Richard M. Goodwin, 1913–1996.

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1. Introduction

In the long history of nonlinear dynamics from Lord Rayleigh (1877) to the present, *forced vibrations* has been a major thread.[2] The synchronization, or entrainment, of two oscillators in frequency and in phase, discovered famously by Christiaan Huyghens in 1665, is an aspect of this thread with many important applications in the sciences: physical, biological, and social.[3] With the advent of the modern qualitative theory of dynamical systems in the 1960s, an intuitive geometric theory emerged in which the phase entrainment of an oscillator may be anticipated.[4] As this theory applied to chaotic attractors as well as to periodic ones, the synchronization of chaotic attractors was predicted and verified experimentally in the 1980s.[5] And thus was born the field of *chaos synchronization*. With extraordinary prescience, the late Richard Goodwin applied these ideas to complex economic systems in his classic book of 1990, *Chaotic Economic Dynamics*.

Here we extend his ideas to large scale lattices of Rössler systems, and present preliminary results of computer simulations in which islands (anti-nodes) of synchronization are constrained by nodal lines, as in vibrating membranes. These results, reminiscent of the dynamics of cardiac tissue[6], are highly suggestive for economic systems, and we end with some speculations on these implications. The evolution of lattice dynamics may be summarized in this brief chronology.

- **1732, Physics of the hanging chain, Daniel Bernoulli (1700-1782).**

  He discovered waves in the hanging chain as an early application of the calculus. In the limit of short links relative to the length of the whole chain, this is a hanging rope. At the other extreme, we might consider the case of one link only, a simple pendulum, or two links of equal length, the double pendulum. This is perhaps the earliest forced oscillator.

- **1826, Spatial economics, Johann Heinrich von Thünen (1783-1850).**

  In *The Isolated State* he pioneered the ideas of spatial economics and economic geography. In a mathematical model of agricultural land, he discovered concentric waves of activity.

[2] The highlights are described in (Abraham and Shaw, 1992; ch. 4).
[3] See (Abraham and Shaw, 1992; ch. 5) and (Strogatz, 2003; ch. 4).
[4] See (Abraham, 1984).
[5] Personal communication via Doyne Farmer (1986), reported in (Stone, 1992).
[6] See (Xie, 2007), (Sato, 2009), and references therein.

- **1877, The physics of sound, John William Strutt, Lord Rayleigh (1842-1919).**

  As part of his wave theory of sound, he performed experiments with a mechanically forced pendulum.[7]

- **1917, Biological morphogenesis, D'Arcy William Thompson (1869-1948).**

  In his seminal work, *On Growth and Form*, he drew attention to the emergence of forms in living system, for example, phylotaxis. The development of the reaction-diffusion model for biological morphogenesis followed soon after, in a paper of Ronald Fisher (1890-1962) of 1918. This model predicted waves of gene frequencies in a population.

- **1968, Chemical waves, Boris Belousov (1893-1970) and Anatoly Zhabotinsky (1938-2008).**

  Belousov's discovery of chemical oscillations in the 1950s was announced to the world by Zhabotinsky in 1968. This inspired many people to seek new physico-mechanistic explanations for various physiological functions. Among these, Arthur Winfree (1942-2002) was pre-eminent. He envisioned a lattice of oscillators as a model for the heart muscle.[8]

- **1990, Economic networks, Richard M. Goodwin (1913-1996).**

  Goodwin was among the earliest adopters of nonlinear dynamics and chaos theory in the world of mathematical economics. He regarded the predator-prey model of Lottka and Volterra (1920) as the basis of business cycles. Perhaps through his acquaintance with Otto Rössler, he came to regard the Rössler attractor as a noisy version of the predator-prey model, and therefore of relevance to economic systems. These ideas came together is his book, *Chaotic Economic Systems* of 1990, which concludes with a suggestion to extend the model to economic networks. He died before this plan could be realized, and it this idea of his that we pursue here.

To go further we will require a basic vocabulary of complex dynamical systems.

[7] See the illustration in (Abraham and Shaw, 2005; figure 4.1.1).
[8] An excellent account may be found in (Strogatz, 2003; ch. 8).

2. Glossary

Complexity is an ambiguous term referring to some of these ideas: chaos theory, cellular automata, genetic algorithms, networks, system dynamics, and complex systems.[9] A *dynamical system* is either:[10]

- a *flow*, defined by a vectorfield (a system of autonomous ordinary differential equations) defined on a state space, S,
- a *cascade*, defined by a reversible endomorphism of S, or
- an *iteration*, defined by an arbitrary endomorphism of S.

The state space in general is a finite-dimensional differentiable manifold, but here we will consider only the special case, an open set $S \subset R^n$ (cartesian n-space).

A *network* consists of nodes and directed or undirected links. A *complex dynamical system* comprises a directed network together with data: every node contains a *dynamical scheme* (a dynamical system with control parameters), and every directed link connects states at its tail to controls at its head.

3. Rössler Systems

Here we will be concerned with complex dynamical systems with flow schemes at each node. Especially, the case in which the same scheme – for example, the Rössler scheme – is attached to each node:[11]

$$\begin{aligned} u' &= au + v \\ v' &= -u - z \\ z' &= b + (v - c)z \end{aligned} \tag{1}$$

Following Richard Goodwin, we might set $a = b = 0.2$, so we have a three-dimensional flow scheme at each node with a single control parameter, c. The original value for the familiar Rössler attractor is $c = 5.7$. If the control parameter, c, increases from zero to 50 or so, a bifurcation sequence is observed that resembles somewhat the well-known behavior of the logistic family of one-dimensional quadratic maps. Some special cases are shown in Figure 1.

[9] An excellent text is (Mitchell, 2009).
[10] See, for example, (Abraham, 1992, 1997).
[11] Here we have adopted the notation of (Goodwin, 1990; pp. 48, 66), but write z for his k.

Alternatively, we may fix $a = 0.2$, $c = 5.7$, and allow b to be controlled from outside. In a forced system, a directed link will determine the value of b at its head from the state (u, v, z) at its tail. And our simplest networks will consist of only one or two nodes, each containing a Rössler system with identical values of the parameters, a and c, while the value of b is controlled.

In the case of one node, the Rössler scheme will have a single attractor, either fixed, periodic, or chaotic, depending upon the value of the control parameter, c. The bifurcation sequence for this scheme is well known: increasing c from zero, we encounter a single attractor, first a point, then a periodic cycle going once around the central point, the so-called unit attractor. This cycle then undergoes a period-doubling sequence converging to chaos, and exhibiting for some intervals, a periodic window. The bifurcation sequence of this scheme as b is varied also includes period doubling routes to chaos, as well as periodic windows, as shown in Figure 2.[12]

In the case of two nodes and one link, we have a forced Rössler system. These systems may exhibit useful properties of phase entrainment. In the case of two nodes and two links, we have a mutually coupled Rössler system. These systems also exhibit phase entrainment.

Goodwin has employed the Rössler scheme to model economic units such as factories. In this context, the variables u and v may represent wages and profits, respectively, which cycle roughly periodically like predators and prey, while the variable z represents an interactive policy factor.[13] Meanwhile, the control parameter, c, may be manipulated by management or other exogenous factors, changing the shape and character of the unique attractor.

4. Chaos with Periodic Forcing

Consider now an application in which one Rössler scheme, representing a bank let's say, is subject to periodic forcing by an exogenous institution such as a government or central bank. If the forcing is applied subtly, it will approximate the near-periodicity of the chaotic attractor, and seek to stabilize its phase by pushing up in the z-direction as the controlled unit approaches its z-peak, advancing its phase, and pushing down as the controlled unit passes its peak, retarding its phase, The result is better periodic behavior of the forced system. This is the essence of the

[12] From Wikipedia entry for *Rössler attractor*.
[13] (Goodwin, 1990; p. 87)

geometric theory of phase entrainment. Goodwin has considered the case in which a Rössler scheme is subject to periodic exogenous forcing with chaotic results.[14] This master/slave situation may be represented in a conspiracy theory, in which some power group, perhaps imaginary like the Bavarian Illuminati, is forcing up the interest rates or market prices by greedy manipulation. Note that the forcing term effectively alters the control parameter b, thus driving the system over the cliff of the successive bifurcations shown in Figure 2.

5. Chaos with Chaotic Forcing

We now consider a Rössler scheme forced by another Rössler system. We double the equations (1) of Section 3, obtaining for the driving system,

$$\begin{aligned} u_0' &= au_0 + v_0 \\ v_0' &= -u_0 - z_0 \\ z_0' &= b + (v_0 - c)z_0 \end{aligned} \tag{2}$$

and for the driven system,

$$\begin{aligned} u_1' &= au_1 + v_1 \\ v_1' &= -u_1 - z_1 \\ z_1' &= b + f_0 z_0 + (v_1 - c)z_1 \end{aligned} \tag{3}$$

We keep the same constants for each system, $a = b = 0.2$, and $c = 5.7$. Note the addition of a new term, $f_0 z_0$, to the z_1' equation in (3). This is where the spike of the forcing system is pushing the spike of the driven system, with f_0 as a scaling factor (sometimes called the diffusion coefficient).

Recall that the Rössler attractor is roughly a smooth oscillation in the (u, v) plane, accompanied by randomly occurring spikes in the z direction. This behavior is similar to that of a Hodgkin-Huxley neuron model, with periodic equal spiking replaced by a rough equivalent, which is chaotic both in the timing and the strength of the spikes. With no connection between two Rössler systems, $f_0 = 0$, the two (u, v) nearly periodic oscillators will be uncorrelated, and the spikes as well. But with weak coupling from the master z-spike to the b-parameter of the slave system, partial synchronization occurs.

[14] (Goodwin, 1990; final chapter)

In Figure 3 we show the parallel trajectories of the master system (orange) and the slave (yellow), drawn in 3D by a NetLogo simulation.[15]

In Figure 4 we can see the time variation of the z coordinate of the master (green) along with that of the the slave (blue). Note that stronger spikes are frequently in sync, while weaker ones are not.

We may obtain a rough measure of the synchronization of the spikes in this figure as follows. Let us set a threshold value, 1.0, to define a spike. That is, when the w-coordinate of a system exceeds 1.0, we say it is spiking. Now during a simulation run, add together all the time intervals of the master system in which it is spiking, s_0, and all the time intervals, s_b, that both master and slave are spiking, and express s_b/s_0 as a percentage. This approximates the correlation of the spikes of the two systems. It takes account of the slave spikes that are entrained by the master, while ignoring slave spikes that are not so entrained. Thus it is more appropriate for a forced system than for a mutually coupled system.

In the run of Figure 4, the spike correlation we found was 57%, while the same run with no forcing showed spike correlation only 7%.

6. Mutually coupled chaos

Our model in Section 5 with two Rössler attractors is a complex dynamical system with two nodes and one link. We now add a second link to symmetrize the model. So here to our equations (3) we add a second forcing term, obtaining,

$$\begin{aligned} u_0' &= au_0 + v_0 \\ v_0' &= -u_0 - z_0 \\ z_0' &= b + f_1 z_1 + (v_0 - c)z_0 \end{aligned} \tag{4}$$

and for the driven system,

$$\begin{aligned} u_1' &= au_1 + v_1 \\ v_1' &= -u_1 - z_1 \\ z_1' &= b + f_0 z_0 + (v_1 - c)z_1 \end{aligned} \tag{5}$$

[15]NetLogo is powerful and rapid development system for agent based modeling, see ccl.northwestern.edu/netlogo.

Again, we keep the same constants for each system, $a = b = 0.2$, and $c = 5.7$. Note the addition of a new term, $f_1 z_1$ to the z_0' equation in (4). So now we have two mutual forcing terms, and we are going to consider the symmetry case, $f_0 = f_1 = f$.

Again running some simulations with NetLogo, we observed the spike correlations as the force coefficient, f, increases from zero to around 2. The correlation increases approximately monotonically to nearly 100%, very strong correlation. Moreover, the amplitudes of the spikes increase due to the positive feedback from the strong correlation.

With our strongest forcing, $f = 1.8$, we found a surprising bifurcation to periodic behavior of the mutually coupled system, one period including around 12 spikes. The trajectory of the mutually coupled system, shown in Figure 5 as two curves (green for system 0 and blue for system 1), shows the behavior from time 1500 to 2000, after the chaotic transient has converged to the attractor. This is clearly seen to be periodic. We found the spike correlation to be 99 percent. For comparison, with $f = 1.4$, the behavior is still chaotic after time 2000, with spike correlation only 61 percent.

Let us consider the mutually coupled system with $f_0 = f_1 = f$, a complex system with two nodes, as a single six-dimensional flow scheme with one control parameter, f. Then we may understand this emergent periodic behavior as evidence of a periodic window in the response diagram for this large scheme.

We now proceed to consider coupling in massively complex systems, such as the world economy.

7. Chaos Synchronization in Networks

Goodwin also considered the global economy as a massively complex dynamical network of chaotic nodes. In this context, the emergence of a large clique of nodes in spontaneous synchrony, like a school of fish or flock of birds, may function as a conspiracy, without any explicit manipulation. This possibility, a sort-of pseudo-conspiracy, actually occurs in computer simulations of complex dynamical networks of identical nodes, such as a reaction-diffusion lattice of Rössler schemes, as we show below.

In general, the lattice is an arrangement of the nodes of the network underlying the complex dynamical system in a regular grid in a Euclidean n-dimensional space, called the *physical substrate* of the system. In our case, $n = 2$, a 2D lattice. The links connect each node with its four nearest neighbors.

8. Simulation Results

Thanks to the spectacular decline in the cost of massively parallel supercomputers, we are able to present here the result of long simulations of two-dimensional lattices of Rössler schemes. These have square arrays of 400 by 400 nodes, each coupled by diffusion of the z-variable from its four nearest neighbors. This means, approximately, that the two-dimensional laplacean of z at a given node modulates its control parameter b in equations (1) above. The response diagram of the Rössler scheme, with b in the range $[0, 2]$, shown in Figure 2, shows that the chaotic attractors with b around 0.2 are bracketed by three-periodic windows.

The nodes begin with random initial conditions, chosen in the domain, $u \in [-10, 10], v \in [-10, 10], z \in [0, 20]$, and all have the same value of the control parameters, $a = b = 0.2$ and $c = 5.7$, in the chaotic regime. Thus, there is no master bank manipulating the controls, and sync develops solely from mathematical cooperation of the policy decisions of the nodes. The diffusion rate in the z direction is chosen in the range, $D_z \in [0, 5]$.

This corresponds to coupling many copies of the scheme:

$$\begin{aligned} u' &= au + v \\ v' &= -u - z \\ z' &= b + (v - c)z + d \end{aligned}$$

where d is now the control parameter, linked to the z values of the four nearest neighbors by diffusion. In other words, the b parameter is modulated around a base value of 0.2 by the diffusion factor, d. The boundary conditions in this simulation are periodic, that is, the lattice resides on a 2D torus.

Our main visualization strategy is to show the instantaneous states of the z value at each node as a color: from blue (at $z = 0$) to red (at $z = 25$). Where we see a blue zone, a ring or disc for example, we are observing phase synchrony in the z variable, the one with the spike behavior. The color code is shown in Figure 6.

We may understand this phase coherence, and the variation of synchronous phases across the two-dimensional lattice of nodes, by means of Zeeman's visualization strategy.[16] We imagine the lattice as a square paper napkin. At a given instant in the simulation, the scheme at each node has instantaneous values of (u, v, z) and also d. Thus we may map the paper napkin into the standard cell, the four-space of (u, v, z, d).

At the start, the paper is very crumpled up, and positioned randomly in the cell. As the simulation advances step-by-step, the napkin is moved about in the cell and all image points are attracted to the locus of the Rössler attractor in the cell. The color code lives in the cell, as layers of different color, parameterized by z. As the folded napkin image moves about in the cell, the colors it moves through are pulled-back to the original lattice, and these we see in the simulation as a movie.

A single frame of this movie is shown in Figure 7. But different blue zones, for example, may be in phase sync with phase difference 360 degrees. Thus image points in between are pulled around the attractor ring, and are stuck in phase sync of relative phases between 0 and 360 degrees. Our simulations are recorded as movies, each frame representing an instantaneous state of the entire lattice, one node per pixel. Thus a blue node, in a single frame, is frozen in an instant of an actual (u, v) nearly periodic motion in the (u, v) plane, mostly with low (blue) z. The z value has a spike-like behavior, like a neuron but not as sharp. So the blue zones are analogous to the Chladni nodal curves (quiet zones, low z) of a vibrating membrane, while the red zones are roughly synchronous spikes, analogous to the anti-nodes of a vibrating membrane.

In the movie there is an apparent near-periodicity of the full images. To reveal this we computed the image entropy of each frame. This is a measure of image complexity, similar to the fractal dimension.[17] A plot of the image entropy versus time is shown in Figure 8, where time varies from 500 to 1000.

As a secondary visualization strategy, we may show the instantaneous phase of the rotation around the origin in the (u, v) plane, ϕ, at each node, as a color: from blue (at $\phi = 0$) to red (at $\phi = 360$). Where we see a red zone, for example, we are observing phase synchrony in the (u, v) variables. A single frame from a movie of the lattice seen this way is shown in Figure 9.

[16](Zeeman, 1977; p. 117)
[17]See (Abraham, 2012) for the definition.

9. Speculations on the Global Economy

Our experiments on a 2D lattice of Rössler attractors are inspired partly by pure mathematics, and partly by recent research on the nonlinear dynamics of the human heart, using 2D coupled lattices of oscillators.[18] Goodwin's suggestion of 1990 to regard the global economy as a complex dynamical system probably requires a neural network model with long links, rather than a biological organ model like the heart, which has only nearest-neighbor connections. Nevertheless, we may regard our Rössler lattice as a simple first step in the direction of understanding the behavior of the global economy as a complex dynamical network of chaotic elements. Along this line, then, we may give an economic interpretation to our simulation results.

The chief feature of dynamic behavior of our lattice is that of the propagating red islands of anti-nodal (tight spike) synchronization, in a blue sea of nodal (loose oscillatory or predator-prey) synchronization. This feature, which may look like a golf-course conspiracy, is actually of purely mathematical origin. It is an emergent feature of a complex dynamical system.

In the human heart, red islands in a blue sea might correspond to arrhythmias such as atrial or ventricular fibrillation. Recent studies suggest that red islands must satisfy certain size restrictions in order to propagate, and may then trigger a fatal fibrillation.[19]

It is not impossible that models such as this one might inform efforts to stabilize, or even to destabilize, the global economic network. The present troubles of the global economy, especially in the United States and Europe, may actually be evidence of a propagating red node, that is, an economic fibrillation.

10. Conclusion

Richard Goodwin foresaw the advent of network economics in 1990:

> The modern economy consists of a great variety of separate activities intimately linked, directly or indirectly through markets, with all or most of the other sectors.[20]

[18] For example, see (Sato, 2009).
[19] See (Xie, 2007) and (Sato, 2009; p. 2985).
[20] Final page, (Goodwin, 1990; p. 130).

Our simulations of massively complex dynamical systems composed of the models suggested by Richard Goodwin for chaotic economic systems suggest that pseudo-conspiracies may develop from mathematical reasons alone. Further, instabilities of our global economic system may be better understood from the further study of these Goodwin networks through extensive simulations. We plan to pursue this direction in future work.

Acknowledgements

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Figures

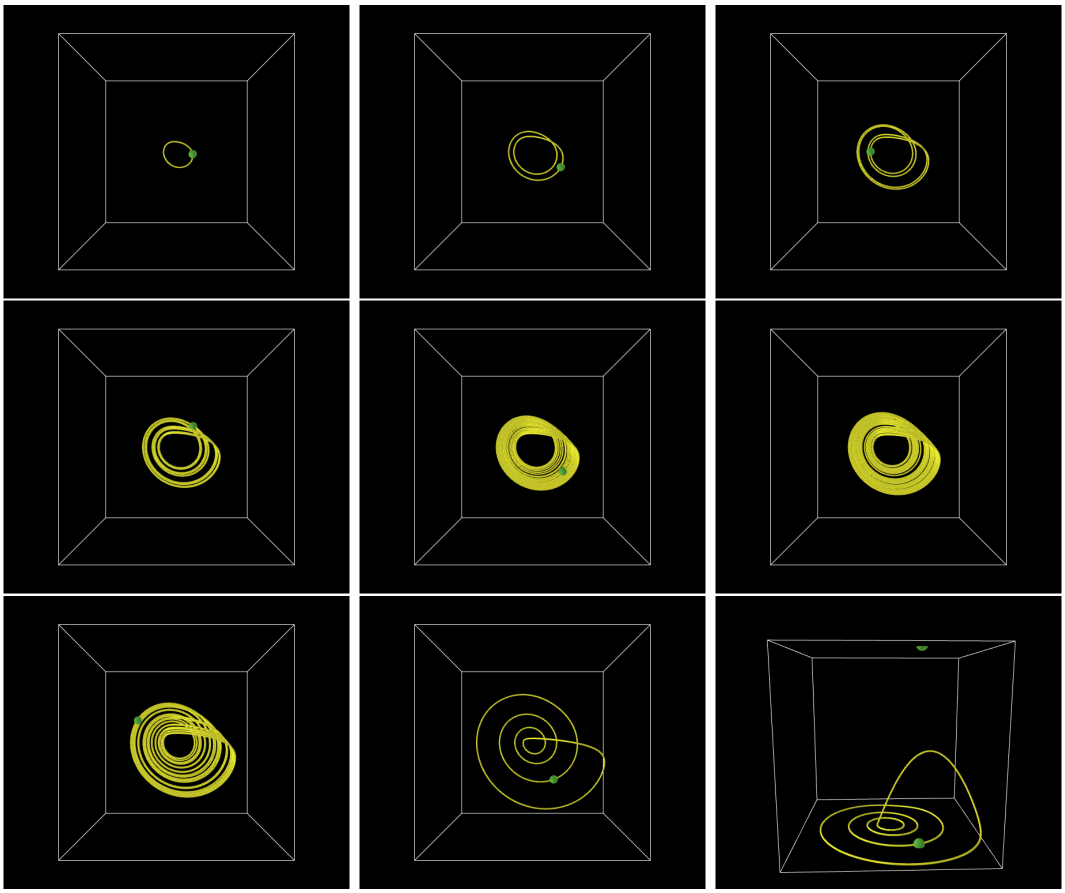

Figure 1: Dependence of Rössler attractor upon control c. The successive values of c are $1.6, 3.0, 3.9, 4.2, 4.5, 5.0, 5.7$ (the standard value), and 7.0, This latter case belongs to a periodic window. The views of these eight cases are looking down from the positive z axis. The ninth image is again $c = 7.0$, but seen from the positive u axis.

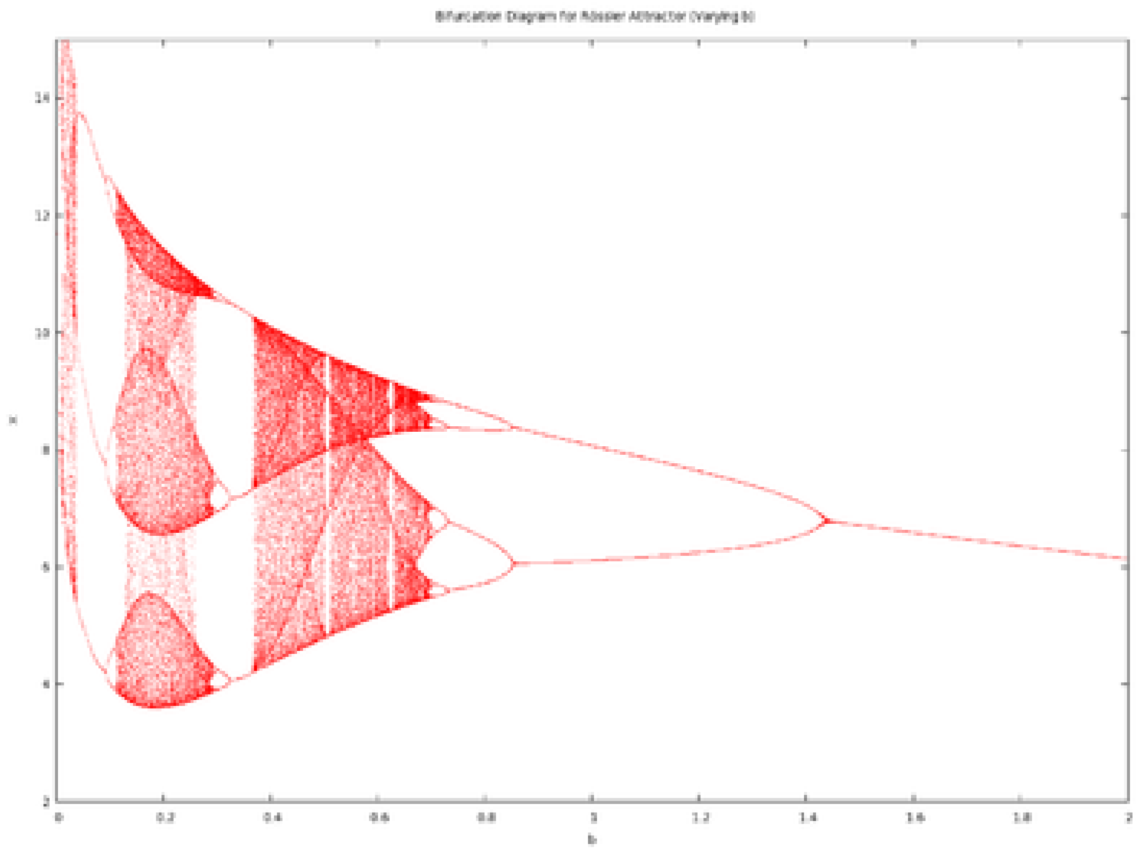

Figure 2: Dependence of Rössler attractor upon control b.
21

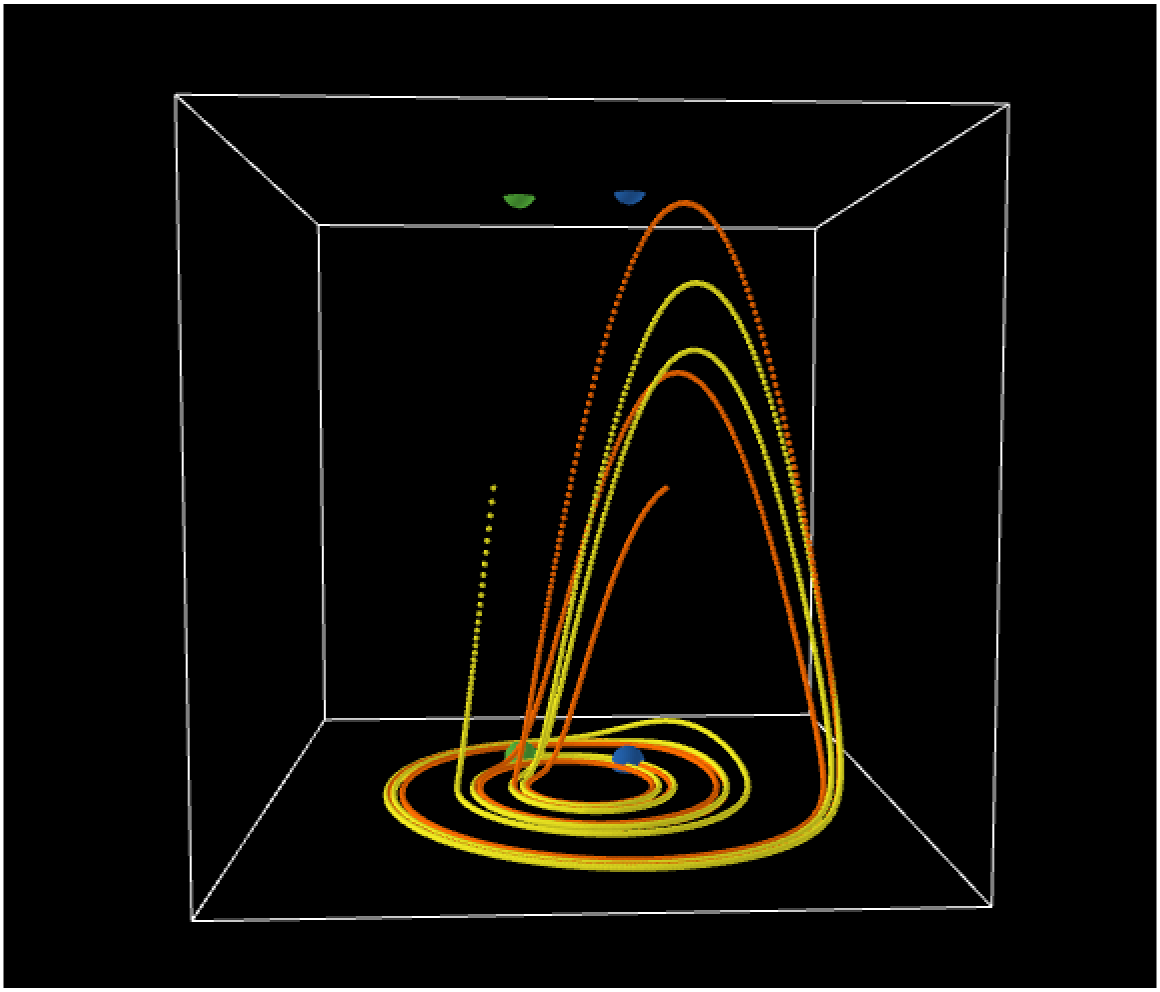

Figure 3: Master (yellow) and slave (orange).

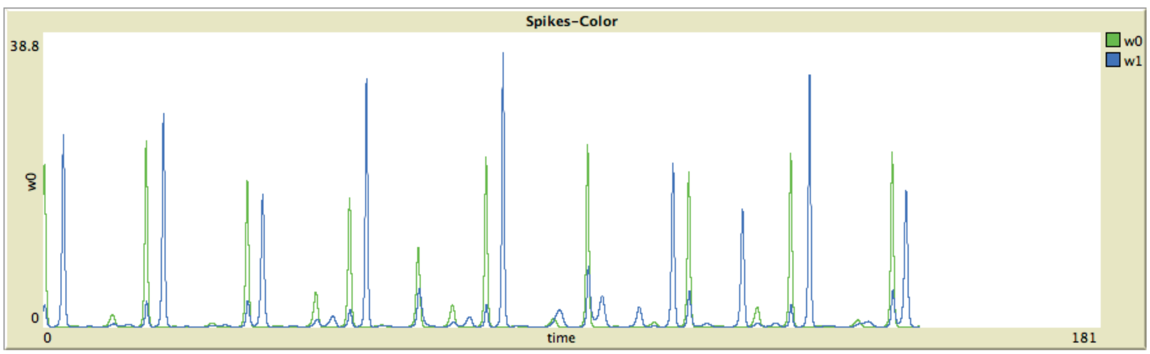

Figure 4: Rössler forced Rössler, Master (green) and Slave (blue), forcing constant 2.0, showing strong synchronization. The simulation time shown is from 1000 to 1150.

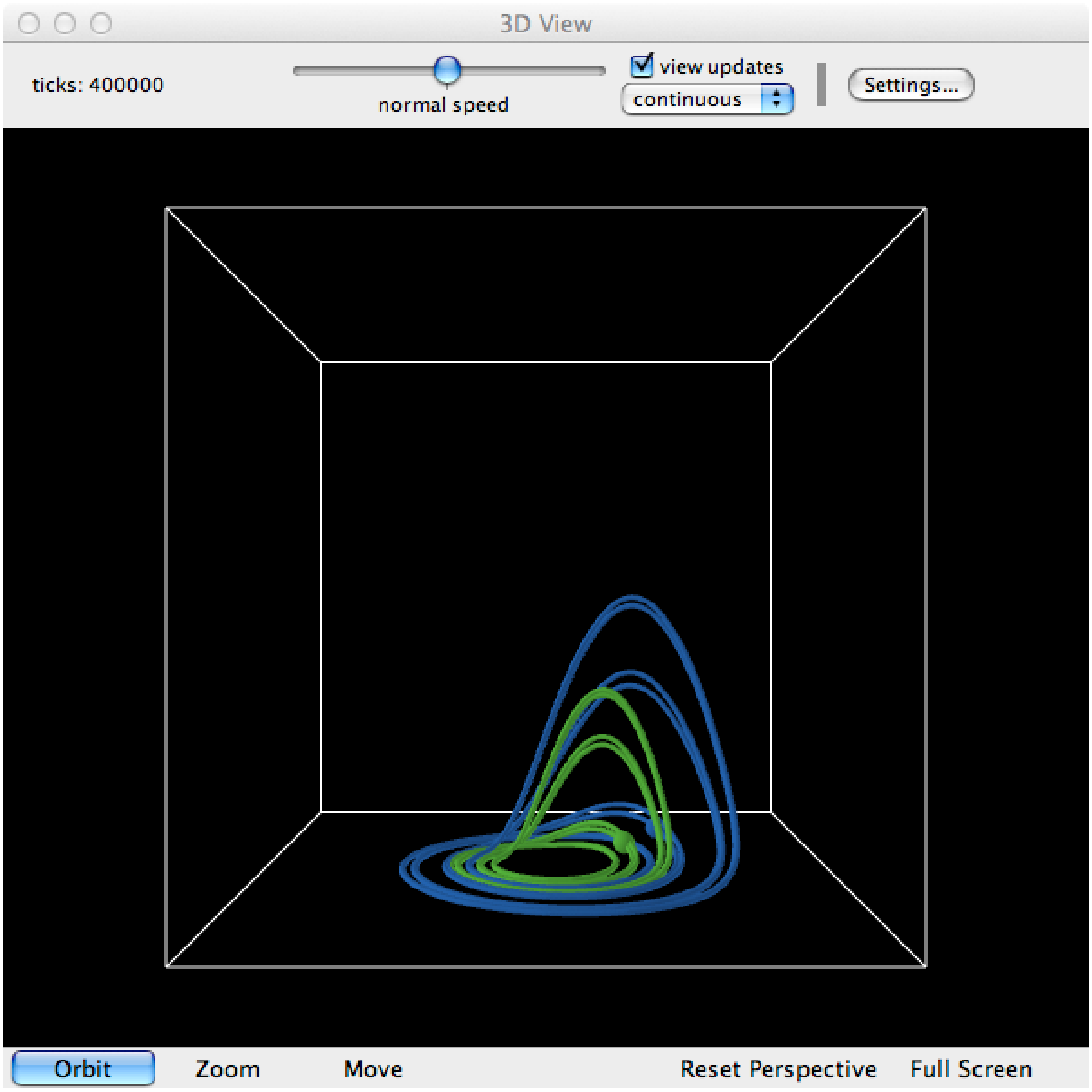

Figure 5: Two mutually coupled Rösslers, system 0 (green) and system 1 (blue), coupling constants 1.8, showing strong synchronization and periodicity. The simulation time shown is from 1500 to 2000.

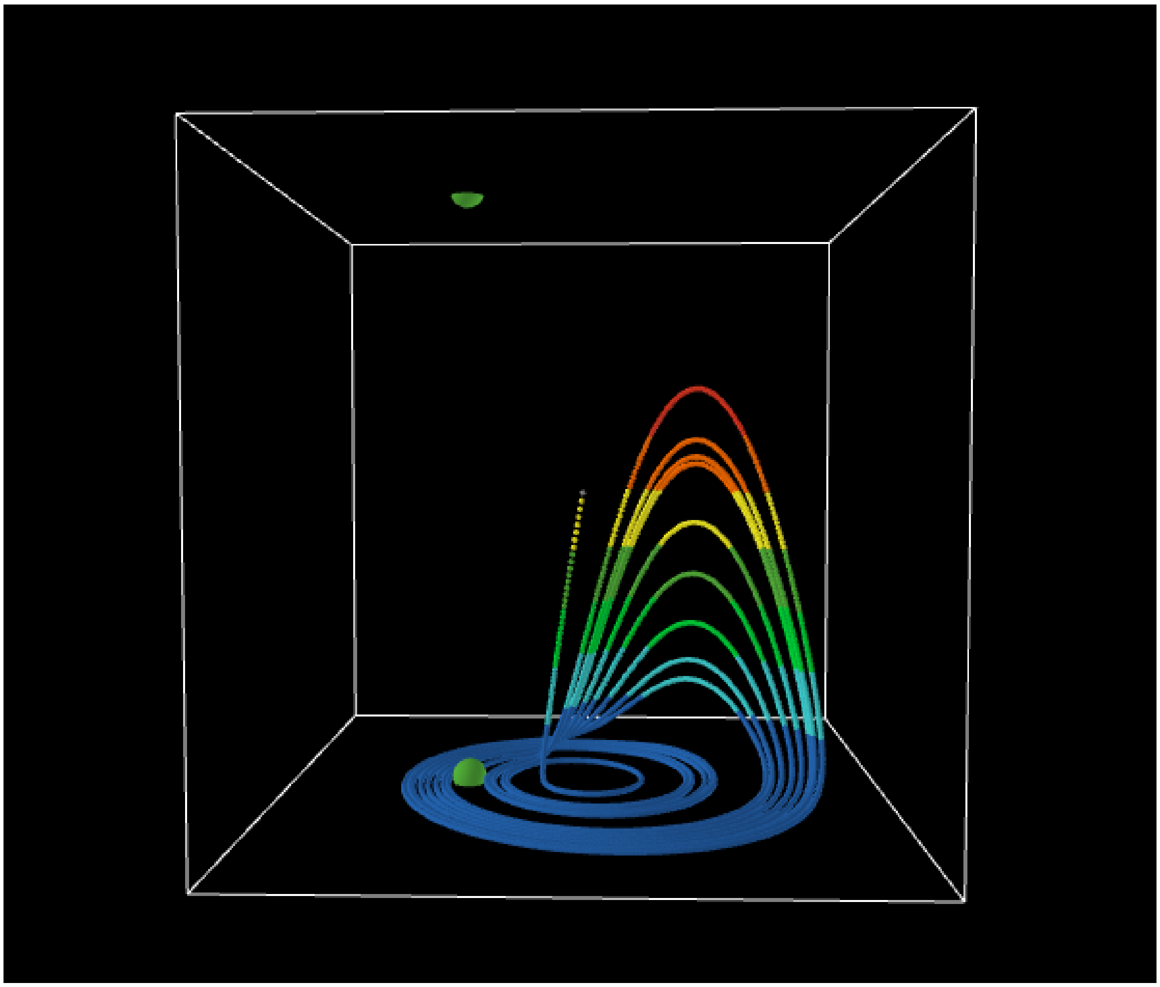

Figure 6: Color code projected onto a single Rössler attractor.

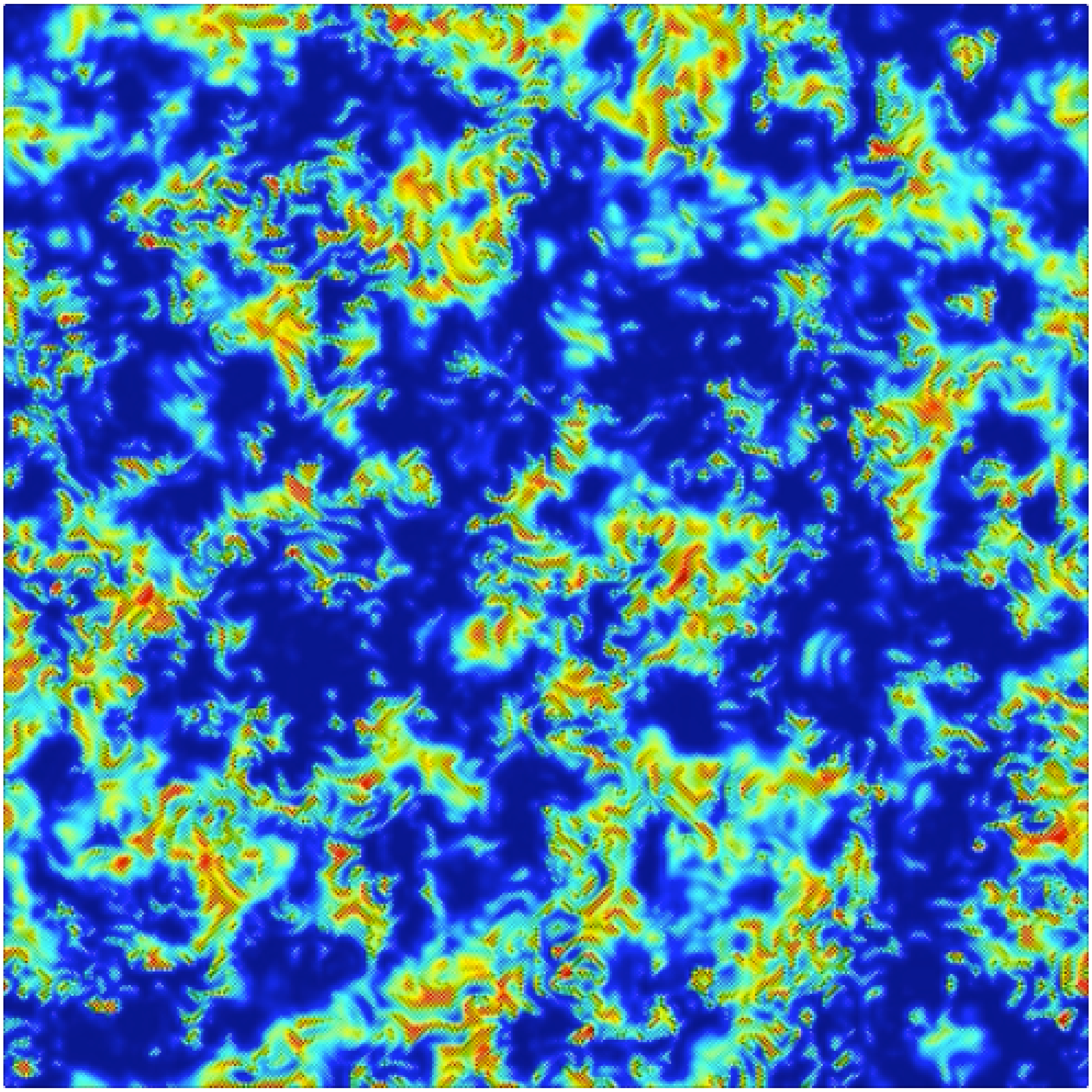

Figure 7: Single frame, 400 x 400 lattice of Rössler attractors, $D_z = 4.0$.

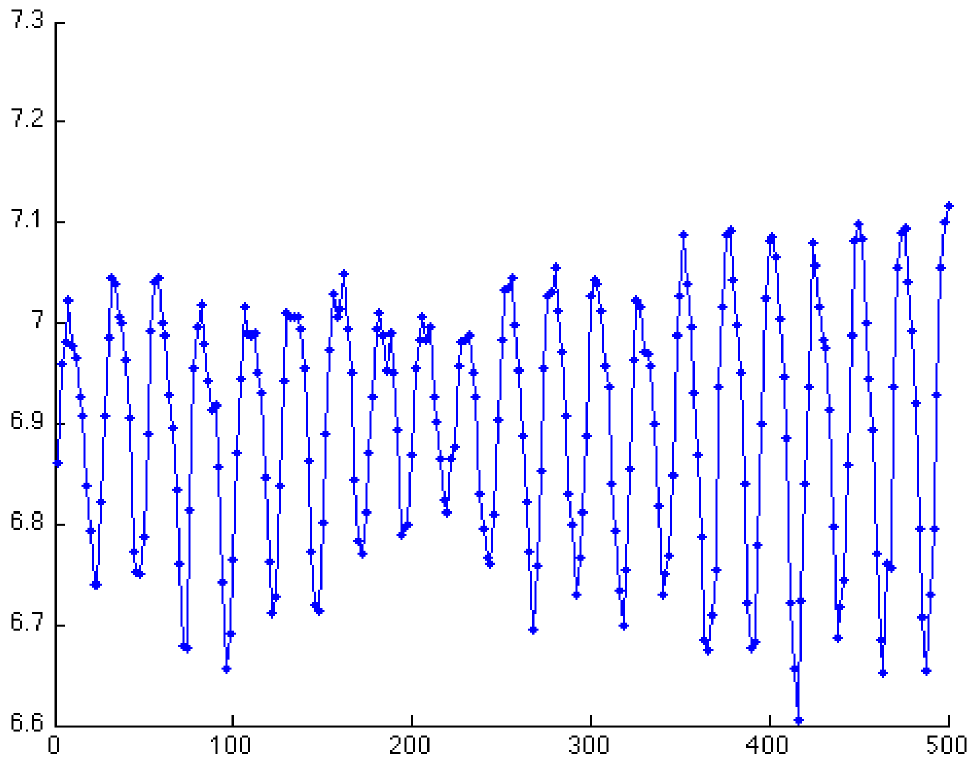

Figure 8: Image entropy of frames of the simulation movie.

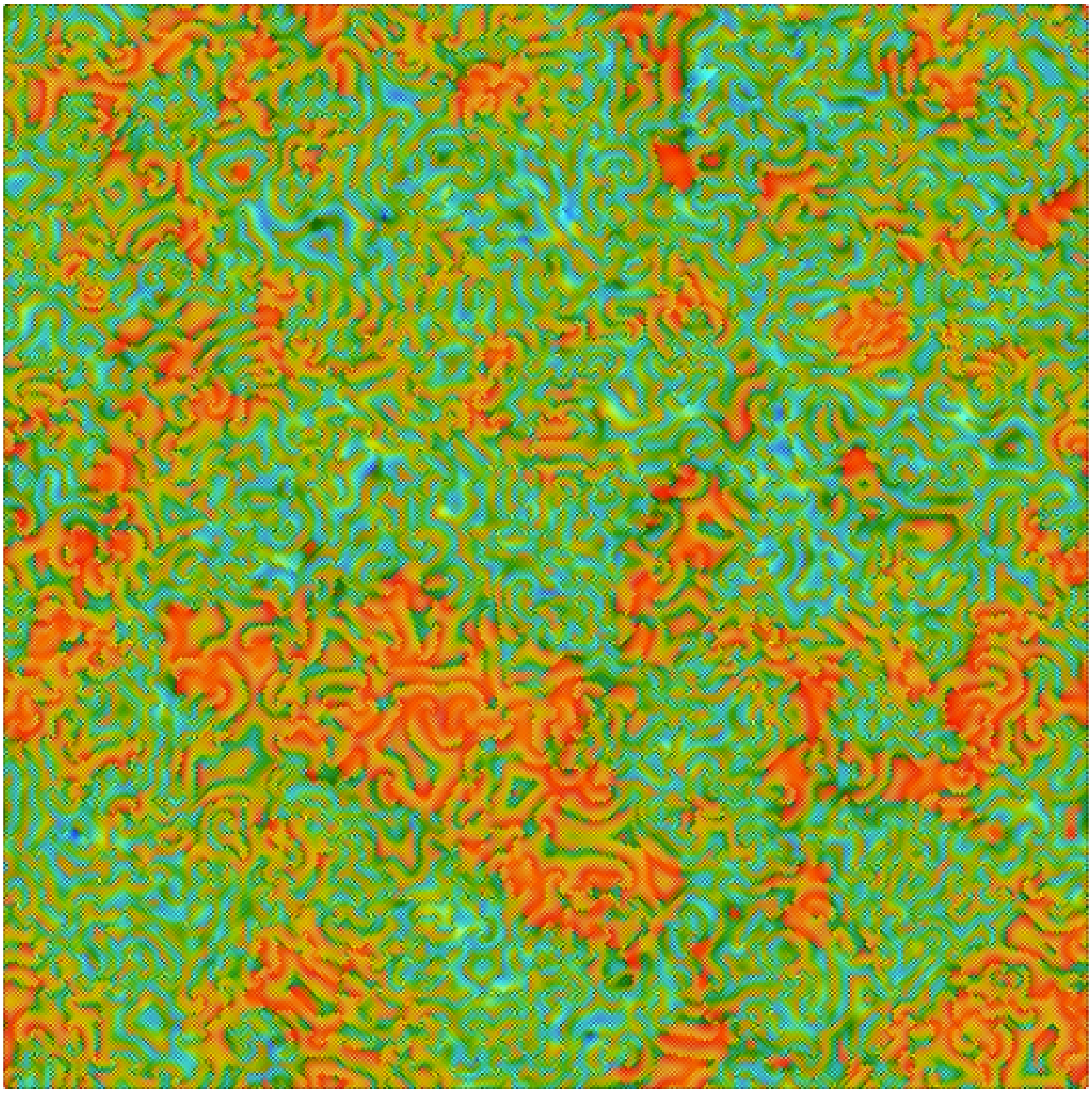

Figure 9: Horizontal phase of Rössler nodes of the simulation movie. The coupling is of z spikes only, with $D_z = 4.0$.