Emergent Periodicity in a Field of Chaos

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Abstract

The synchronization of nonlinear oscillators is well-known and is a traditional topic in complex dynamical system theory. The synchronization of chaotic attractors is less well-known, but is of obvious interest in many applications to the sciences: physical, biological, and social.

In a recent experimental study of coupled lattices of Rössler attractors, (jointly with Michael Nivala) we were surprised to discover global periodic behavior in large regimes of the parameter space. This emergent periodicity in a field of chaos may be of significance in the origin of life, and in many life processes.

In this talk we will explore the emergence of global periodicity, and also the periodic windows in the bifurcation diagram of the Rössler attractor, which may be the local cause of this global behavior.

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Introduction: Periodicity and Life

Living organisms are complex systems, and have been modeled by complex dynamical systems. A biological organ, for example, may be simplified as a two- or three-dimensional lattice of cells or nodes, each modeled by identical dynamical schemes, with each node coupled mutually to its nearest neighbors.[1] Lattices of oscillators, for example, abound in the literature of biological modeling.

However, biological cells frequently exhibit chaotic behavior, so we have been motivated to explore two-dimensional lattices of Rössler schemes. An amazing natural phenomenon, crucial to life, is the emergence of global periodicity in such a complex system, that we call a field of chaos. For instance, we may cite swimming bacteria, respiration, the beating heart, and the regular rhythms of the brain.

In this talk I will demonstrate the ubiquity of periodic behavior in three related contexts.

1. One Rössler

This system is known as the simplest chaotic flow (continuous dynamical system), and exhibits an oscillation in the plane, together with spiking behavior in a third dimension.

The basic scheme

The scheme is defined by the equations,

$$\begin{aligned} x' &= -y - z \\ y' &= x + Ay \\ z' &= B - Cz + Mxz \end{aligned} \tag{1}$$

The usual values of the control parameters are, $A = B = 0.2, C = 5.7$, and $M = 1.0$. The attractor is shown in Figure 1, in which the speed along the trajectory is indicated by the colors, from blue (slowest) to red (fastest).

[1] A dynamical *scheme* is a family of dynamical systems (in this case, flows) parameterized by control variables.

Note the simple rotation in the (x, y) plane, and the spike in the z direction.

Bifurcations and Periodic Windows

As one of the four control parameters is varied while the other three are held constant, the behavior of the attractor changes through slightly different chaotic states, with occasional windows of periodic behavior. These are shown in abbreviated form in Figure 2. All four exhibit periodic windows, but note that they are most conspicuous in the B plot. There, as B is decreased from the right, a unit periodic attractor undergoes a period doubling bifurcation, and then another and another, as in the familiar route to chaos of the logistic family.

2. Two Rösslers

There are several ways of coupling to identical dynamical systems. We will be mostly interested in the /em direct coupling method, due to a geometric theory of synchronization. In this method, a proportion of the z-value of each trajectory is added to the z-component of the other vectorfield. This is expressed precisely in these equations, in which we have assumed identical values of the four control parameters in each of the coupled systems.

Synchronization

The 0-system now includes a z_1-dependent perturbation in the third component of its vectorfield, with coupling coefficient, D_0,

$$\begin{aligned} x_0' &= -y_0 - z_0 \\ y_0' &= x_0 + Ay_0 \\ z_0' &= (B + D_0 z_1) - Cz_0 + Mx_0 z_0 \end{aligned} \tag{2}$$

Similarly, the 1-system now includes a z_0-dependent perturbation in the third component of its vectorfield, with coupling coefficient, D_1,

$$x_1' = -y_1 - z_1$$

$$\begin{aligned} y_1' &= {}_1x + Ay_1 \\ z_1' &= (B + D_1 z_0) - Cz_1 + Mx_1 z_1 \end{aligned} \tag{3}$$

Note the addition of two additional control parameters, D_0 and D_1, the coupling coefficients. Also, we have grouped together the terms $(B + D_0 z_1)$ and $(B + D_1 z_0)$ to foreground the fact that the forcing terms effectively modify the B coefficients of each system. We call these terms *effective* B_0 and *effective* B_1 for the coupled Rössler systems.

We are interested in two special cases.

In the case $D_0 = 0$, the 0-system is called the master, and the 1-system is the slave. The master system forces the slave, while the master behaves as if the slave did not exist.

In the case $D_0 = D_1 \geq 0$, we say the two systems are mutually and symmetrically coupled.

In both of these special cases, increasing the coupling coefficients produces synchronization of the z-spikes, even though these spikes occur chaotically in time and in strength as well, as shown in Figure 3, for the master and slave.

Bifurcations and Periodic Windows

We now consider the double Rössler system in the second case of symmetrical coupling. A simulation with NetLogo 3D, showing the two trajectories side-by-side, one green, the other blue, reveals the chaotic synchronization. The trajectories may be clarified by indicating a Poincaré cross-section. For this we have chosen the positive half of the $X - Z$ plane. When the green trajectory pierces this half-plane it leaves a red drop. And when the blue transits the section, it leaves a yellow drop. A bifurcation movie of this system, as $D = D_0 = D_1$ increases from 0.0 to 4.0 reveals an extensive periodic window around $D = 2.0$. This emergent periodicity, seen from the positive Y-axis, is illustrated in Figure 4.

3. Many Rösslers

Finally we consider a regular lattice of $160,000$ usual Rösslers, in a 400 by 400 square grid, on a two-dimensional torus. Each node is mutually and symmetrically coupled to each of its four nearest neighbors. Thus, at each node, we have,

$$\begin{aligned} x' &= -y - z \\ y' &= x + Ay \\ z' &= (B + Dz_s) - Cz + Mxz \end{aligned} \qquad (4)$$

where D is the common coupling constant, and z_s denotes the sum of the z-coordinates of the four neighbors.

Synchronization

Extensive simulations of this system, the $2D$-toral Rössler lattice, by Michael Nivala of the UCLA have been recorded as movies, with each frame representing an instantaneous state revealing the z of each node as a color, from blue (0) to red (25). Three frames of such a movie[2] are shown in Figure 5. The colors, especially in the third frame, reveal islands of z-synchronization, which move around with advancing simulation time.

Global Periodicity

A surprising feature of these simulations is a robust global periodicity. This may be observed by averaging the z-values of all the nodes, and plotting as a function of time. As we see in Figure 6, there is a periodic fluctuation in this average value.

Conclusion: Future Work

Our interest in the emergence of global periodicity in a field of chaos is heightened by the crucial role of periodicity in life processes, and we feel justified in thinking that nature has selected for attractors with shapes that facilitate synchronization, and bifurcation diagrams with periodic windows. And yet, these periodic windows are quite surprising from the point of view of pure mathematics. We began our investigation with the idea of observing patterns of chaotic synchronization, and were astonished to discover global periodicity by accident.

[2] Nivala's a14

We have by now a large number of related simulations, and will be filing more progress reports as time goes on. But at this point, we may say that global periodicity is ubiquitous for Rössler lattices. In the future, we plan to explore other fields of chaos, such as Lorenz and Ueda lattices, to discover their secrets as well.

Figures

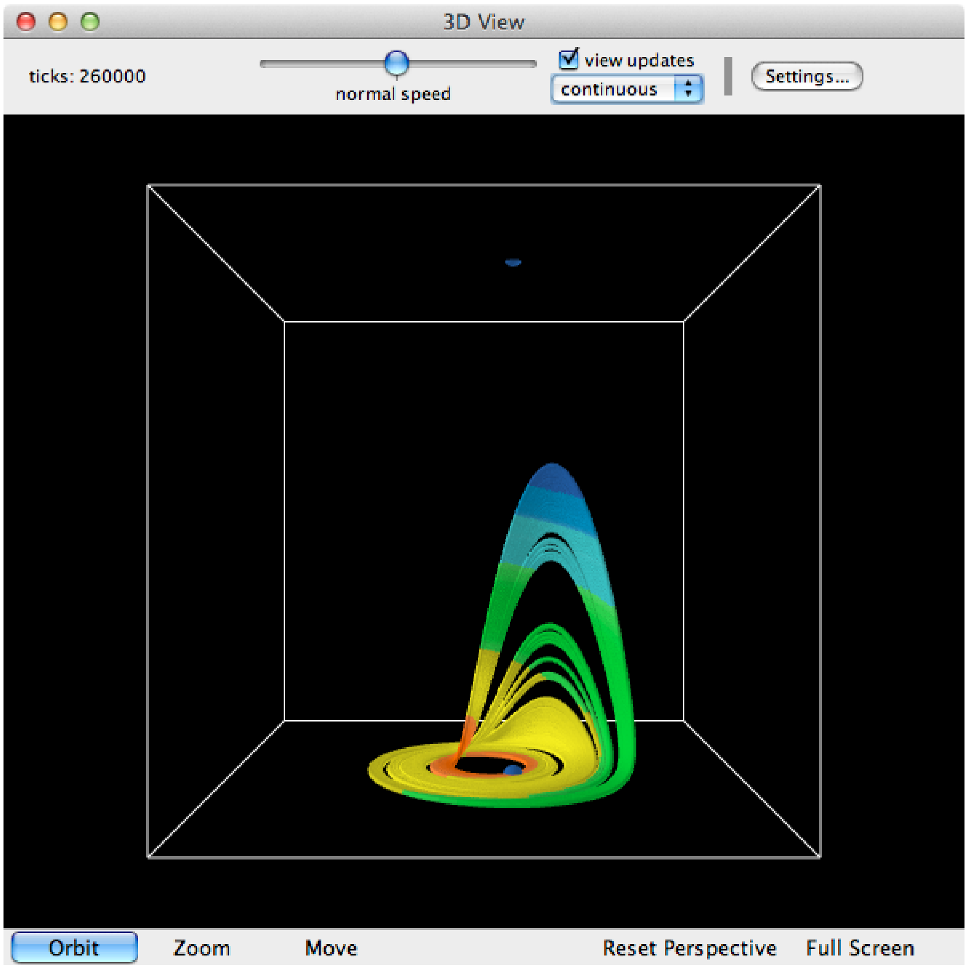

Figure 1: The usual Rössler.

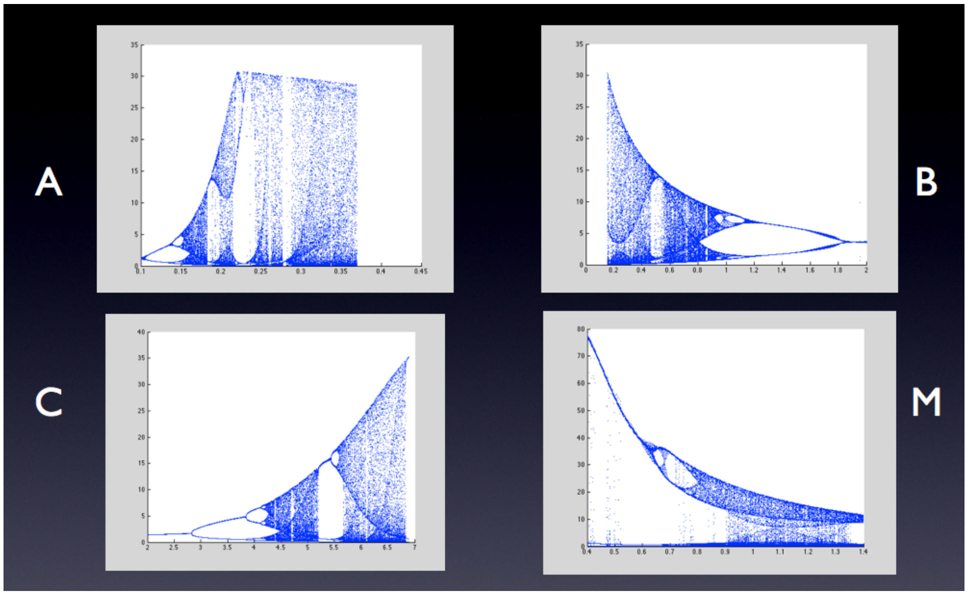

Figure 2: The four bifurcation plots: Poincaré-z vs A, B, C, M.

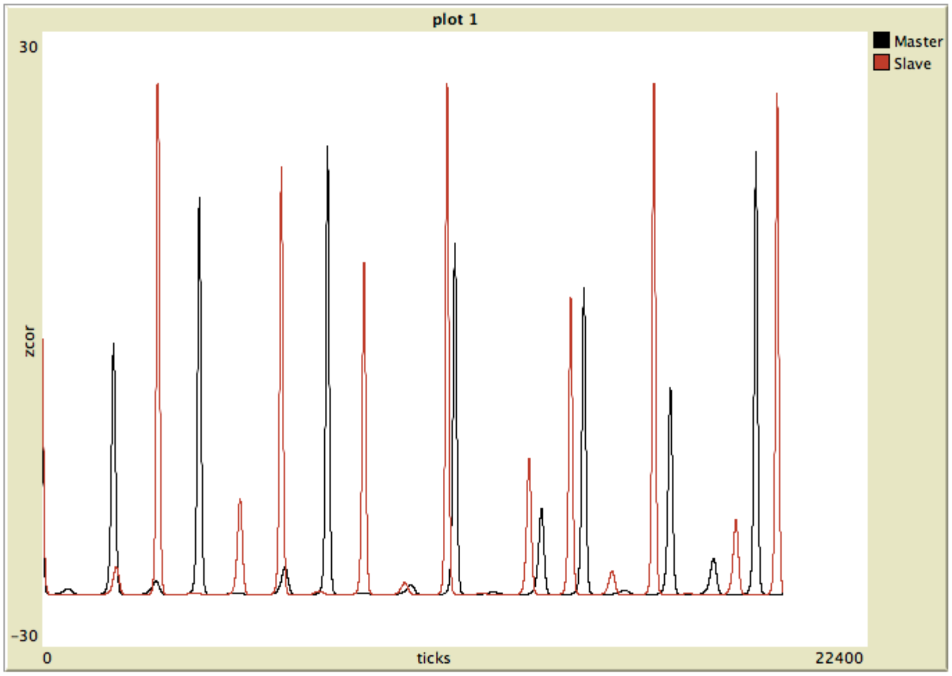

Figure 3: The master (black) and slave (red) z's vs time.

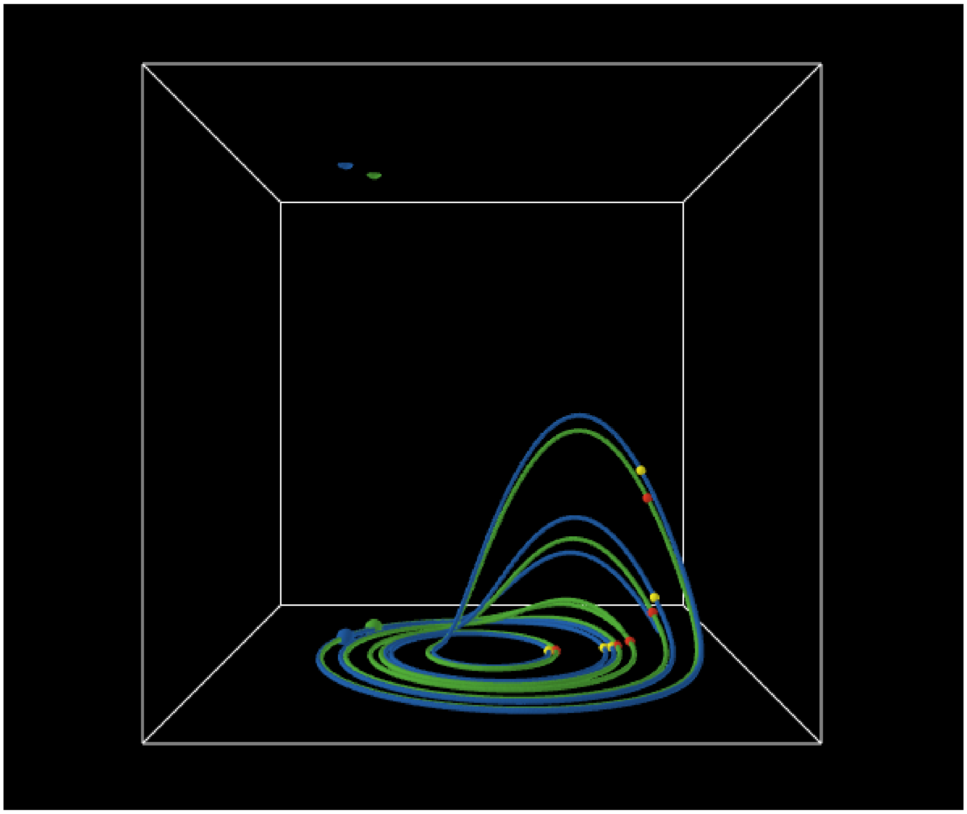

Figure 4: Visualizing the Poincaré cross-section.

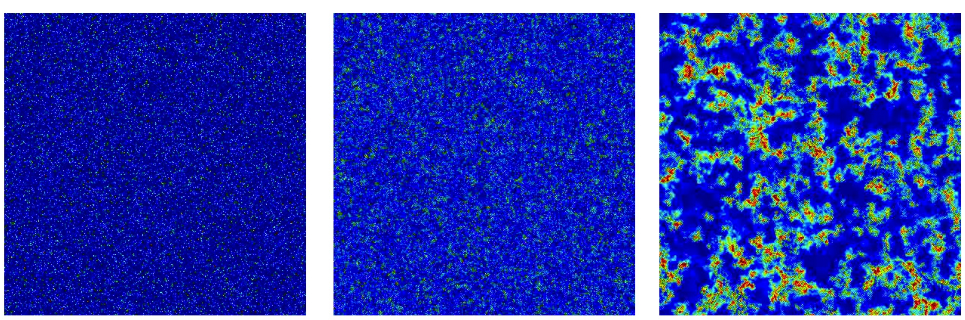

Figure 5: Three frames of the Rössler lattice.

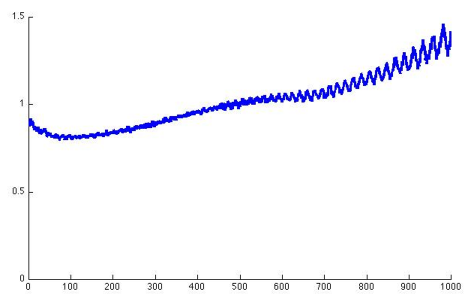

Figure 6: Periodic temporal fluctuations in the z average.