# Chaos Theory and the Tragedy of the Commons

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October 7, 2014

**Abstract**

Complex dynamical systems models have been used (and misused) in service of the sustainability (or tragedy) of a commons. Their misuse results from the widespread ignorance of chaos theory. Here we consider this problem in general, and study the special case of the tragedy of the oceans in detail. We go on then to relate the mathematical model for fisheries, due to Beverton and Holt in the 1940s, to the chaos revolution that followed. Finally, a potential role of education in commons management is proposed, in which participative simulation using NetLogo might be an integral part.

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# 1. Introduction

Many of the systems within which we live, such as ecosystems, are difficult to comprehend because their complexity strains our cognitive capacity. Mathematical modeling (and computer simulation) of a complex system as a complex dynamical system (CDS) is a powerful strategy to bring understanding of such complex systems within our grasp.[1] But the CDS modeling approach has been sometimes misused for making predictions, even though chaos theory has shown that CDS models are generally incapable of making predictions.

A crucial problem in the practical world of environmental management for sustainability is the conundrum known as the tragedy of the commons. In this article we will apply the CDS strategy to this problem, especially in the exemplary case of the tragedy of the oceans: overfishing.

After some math preliminaries we review the prehistory of the modern environmental movement, 1957-1962. Along the way we will review the parallel developments of the fisheries model in the domain of applied mathematics (Gerhardsen, 1952; Gordon, 1954; Schaefer, 1957, and Beverton and Holt, 1947-53, 1957), and the pure math of quadratic iterations (Myrberg, 1958-1965). These developments led the way to the chaos revolution (May, 1974), which is crucial to our story of the misuse of math modeling in the tragedy of the oceans.

We end with some suggestions for training students for stewardship of common-pool resources, using the participatory simulation feature of the NetLogo software package. The inclusion of systems theory in a school curriculum may enhance student understanding of the complexity and interconnectedness of the world around us. This strategy fits well into a school curriculum.[2]

# 2. Systems

A complex dynamical system is a formidable mathematical object. And yet, a simple and intuitive approach was provided in its early history.

[1] We distinguish *complex systems*, as found in nature, from *complex dynamical systems*, which are mathematical models for complex systems. Technical definition are relegated to a appendix.

[2] This has been abundantly proven at the Ross School of East Hampton, NY.

### Intuitive CDS

The state of a system is visualize as a set of *stocks*. A stock is a number representing an amount of fluid or grain in a tank or silo. The tank has an input pipe and an output pipe controlled by valves. This visual icon represents an ordinary differential equation.[3] A visual representation of such system is shown in Figure 1.

### Chaos theory

Chaos theory is a popular name (since the *chaos revolution* around 1974) for the branch of mathematics devoted to the behavior of dynamical schemes. This branch, known to mathematicians as *dynamical systems theory*, originated with the French mathematician, Henri Poincaré, around 1880. The theory is of special importance to the mathematical sciences due to its main implication for CDS models, namely, their unpredictability.

There are two reasons for this implication, both discovered by Poincaré, *sensitive dependence* and *bifurcation.* In both cases, a minor change in the system definition results in a major change in its behavior.[4]

## 3. The environmental movement

It is sometimes said that the history of the environmental movement began in 1962, with the publication of Rachel Carson's controversial and amazingly popular book on toxic pesticides and weed killers, *Silent Spring.* While that may be true, there was of course a prehistory that served as a springboard for her success.

### Environmental prehistory

1957. The gypsy moth was endangering the northeastern forests. The USDA began spraying millions of acres with DDT, killing birds, fish, and crops. Long Island

[3] Experience has shown that this concept is easily understood by young children, and CDS models have been taught in this fashion even in elementary schools.

[4] Again, these are usually learned by the study of special cases.

residents filed a suit, which was denied.[5]

1958. Fire ants in the American South were annoying farmers. The USDA began seeding millions of acres with pesticide pellets more toxic than DDT.

1959. The Secretary of Health, Education and Welfare announced that some Long Island cranberries were contaminated with aminotriazole, a weed killer thought to cause cancer, and had to be withdrawn from the market.

1961. Thalidomide, a sleeping pill supposed to be safe for pregnant women, was withdrawn from the market after it was implicated in a rash of birth defects.

## Rachel Carson

Rachel Carson (1907-1964) was born in Springdale, PA during the height of the US movement for women's' suffrage. Always interested in writing and nature, she switched from writing to biology in college, and earned a masters degree in zoology in 1932. Her thesis was a study of the larvae of catfish. She worked in the US Bureau of Fisheries from 1935. Her first sea story, *Undersea*, was published in *The Atlantic Monthly* of September, 1937.[6] There, in a scant eight pages, we are conducted on a guided tour of the whole watery system, including food chains, sensations, and the interconnection of all living things. The phenomenal literary style and beauty of these pages contributed to the success of the piece. In its final paragraph, she concludes:

> Thus we see the parts of the plan fall into place: the water receiving from earth and air the simple materials, storing them up until the gathering energy of the spring sun wakens the sleeping plants to a burst of dynamic activity, hungry swarms of planktonic animals growing and multiplying upon the abundant plants, and themselves falling prey to the shoals of fish; all, in the end, to be redissolved into their component substances when the inexorable laws of the sea demand it.

This success urged her on to write a trio of books on the sea – *Under the Sea Wind; a Naturalist's Picture of Ocean Life* (1941), *The Sea Around Us* (1951), and *The Edge of the Sea* (1955) – which became best-sellers. The success of *The Sea Around*

[5](Gartner, 1983; p. 86)

[6]Reprinted in (Brooks, 1972; ch. 3, pp. 22-29).

*Us* made her a literary celebrity, and she resigned from the Bureau of Fisheries in 1952.

In January 1958, at the height of the DDT battle on Long Island, Rachel Carson heard from her friend Olga Owens Huckins that the birds in her private refuge were dying from the DDT spraying. This set her on a two-year path of research and writing that culminated in the publication of *Silent Spring* in 1962, and the consequent explosion of the environmental movement. Suffering poor health all her life, she learned in 1960 that she had cancer, and ironically, finished her argument that DDT contributed to cancer as she was dying of the disease. She lived to see the advance of anti-pesticide legislation, received many honors, and passed away in 1964 at age 56.

Her early love of the birds and the sea lead directly to the regulation of pesticides on land. Like vertebrates, environmental sensitivity evolved from the sea to land, from fish to birds.

*Silent Spring* begins with a fable, in which she continues the strongly poetic voice of her sea trilogy. Some excerpts:

> I. A Fable for Tomorrow
>
> There once was a town in the heart of America where all life seemed to live in harmony with its surroundings. The town lay in the midst of a checkerboard of prosperous farms, with fields of grain and hillsides of orchards where, in spring, white clouds of bloom drifted above the green fields. In autumn, oak and maple and birch set up a blaze of color that flamed and flickered across a backdrop of pines. Then foxes barked in the hills and deer silently crossed the fields, half hidden in the mists of the fall mornings.
>
> .....
>
> Then a strange blight crept over the area and everything began to change. Some evil spell had settled on the community: mysterious maladies swept the flocks of chickens; the cattle and sheep sickened and died. Everywhere was a shadow of death.
>
> .........
>
> In the gutters under the eaves and between the shingles of the roofs, a white granular powder still showed a few patches; some weeks before it had fallen like snow upon the roofs and the lawns, the fields and streams.
>
> No witchcraft, no enemy action had silenced the rebirth of new life in this stricken world. The people had done it themselves.

> ........
>
> What has already silenced the voices of spring in countless towns in America? This book is an attempt to explain.

Following this poetic transition, she immediately changes her tone to a coldly scientific report, almost a legal brief, on the pollution of the oceans.

Due to her untimely demise, she missed the rapid evolution of the environmental movement around the world, and global condemnation of chemical pesticides, that she had spawned. Nevertheless, toxins are still seeping into the hydrosphere, where they will remain for centuries.

## Toxins in the sea today

In *Silent Spring*, Rachel Carson argued against chemical pesticides, and promoted integrated pest management as an alternative. One of her primary concerns was DDT. Its use on land leads to contamination of the oceans, the entire food chain of the sea, and through the fishing industry, to the human population at the top of the chain. Where does this stand today? According to Sea Web,[7]

> It is estimated that over 70,000 chemicals are currently in common use as industrial compounds, pesticides, pharmaceuticals, food additives, and other purposes, and that this number is increasing by approximately 1,000 each year. Of particular concern to the health of marine mammal populations are the halogenated hydrocarbons (HHCs) such as the PCBs, DDT, chlordane, dioxins and furans, and the chlorinated and brominated diphenyl ethers. Other chemical groups of concern include trace metals such as mercury and cadmium, organometals such as tributyltin, polycyclic aromatic hydrocarbons (PAHs), and radionuclides. Those coastal populations near intensive agriculture operations may be exposed to periodic pulses of carbamate and organophosphate pesticides.

We also have to consider the increasing concentrations of such chemicals as Prozak, Roundup, antibiotics, and CO2, as well as oil, sewage, noise pollution, and plastic debris. More information may be found online, for example, at Ocean Health Index[8] and Blue Voice.[9] The latter provides a list of the twelve worst Persistant Organic

[7] www.seaweb.org/resources/briefings/chempol_mammal.php

[8] www.oceanhealthindex.org/

[9] www.bluevoice.org/pdf/POPsFactSheet.pdf

Pollutants (POPs), including DDT, PCBs, and Dioxins.

# 4. Tragedy of the commons

A *commons*, also known as a common resource or common pool resource (CPR) is a resource shared by a community of people or agencies that exploit it for gain. Thus, a commons is a complex system composing a resource (CPR) and its exploiters.

## Examples

Here are some exemplary commons of current interest.

1. Atmosphere (Climate, Energy)
2. Hydrosphere (Oceans, Aquafers, Rivers)
3. Terrasphere (Topsoil, Minerals, Beaches)
4. Biosphere (Forests, Savannas, Flora, Fauna, Fisheries)
5. Infrastructure (Networks, Electromagnetic fields)

## Sustainability

A problem of sustainability arises if agents over-exploit the common pool resource, taking more than it can replenish by its natural functions. While this problem has been known since ancient times, it became a topic of academic discourse and research after the appearance in 1968 of an article by Garrett Hardin. It's title, *The tragedy of the commons*, gave a new name to the problem, as well as a pessimistic theory. For in Hardin's view, human greed would always lead to the destruction of the commons, that is the tragedy. From its abstract:

> The tragedy of the commons develops in this way. Picture a pasture open to all. It is to be expected that each herdsman will try to keep as many cattle as possible on the commons. Such an arrangement may work reasonably satisfactorily for centuries because tribal wars, poaching, and disease keep the numbers of both man and beast well below the carrying capacity of the land. Finally, however, comes the day of reckoning, that

> is, the day when the long-desired goal of social stability becomes a reality. At this point, the inherent logic of the commons remorselessly generates tragedy.

In the ensuing debates, a positive outcome was proposed, called the comedy of the commons, and thus evolved a whole spectrum of cases, called *the drama of the commons*. And this in fact is the title of an important book of 2002, in which the interdisciplinary field of commons studies, rapidly expanding since 1968, is summarized.[10] An introductory section provides an excellent history of commons studies.[11] The particular commons most frequently encountered in this literature are forests, agricultural fields, and fisheries. In these contexts, the tragedies are over-cutting, over-grazing, and over-fishing, respectively.

A formal theory of the commons preceded Hardin. An influential model of a fishery due to H. S. Gordon (1954) and M. B. Schaefer (1957) is summarized in Figure 6.[12] By the way, this image is suggestively similar to that of the logistic function, an icon of chaos theory since 1958.[13] More about this below.

The drama of the commons owes much to the pioneering work of the late political economist Elinor Ostrom, winner of a Swedish Bank Prize in Economics in 2009. She had faith in human nature to resolve the problem of sharing a common pool resource by spontaneously evolved agreements, called institutions, for sustainable management. She supported her ideas with extensive field work in actual commons in Africa and Nepal, and identified eight design principles for commons management institutions.

# 5. Tragedy of the oceans

We consider now an exemplary case of CPR mismanagement, the tragedy of the oceans. The oceans comprise the largest and most complex CPR on earth, including its water, living systems such as coral reefs, zooplankton, fishes of all sizes up to tuna, as well as sharks and marine mammals such as whales. Preying upon these systems are huge fleets of commercial fishing boats, some almost as large as aircraft carriers,

[10](Ostrom, 2002)

[11](Dietz, 2002).

[12]See (Dietz, 2002; p. 9-10). Also, (Gordon, 1954) contains a reference to (Gerhardsen, 1952), which may be the original source of this model.

[13]See (Myrberg, 1958-1965) and (Mira, 1987; ch. 3).

as well as millions of sport fishers world-wide. They remove from the oceans billions of fish and millions of metric tons of fish (the *catch*) every year, while egg producing fish try in vain to replenish the loss annually. As the larger fish are decimated, the fishers concentrate on smaller fish.

## Fishery defined

Here are some definitions of fishery from the expert literature.[14]

- A stock or stocks of fish and the enterprises that have the potential of exploiting them. (Anderson, 1977)
- A socioeconomic technological system in interaction with a marine ecosystem. (Spochr, 1980)
- Activities through which people link themselves with aquatic environments and renewable resources. (Anderson. 1982)

Thus, a *fishery* is a complex dynamical system consisting of two things, fish and humans, as a predator-prey system: fish as prey; humans as predators. Thus humans are at the top of a food chain, which is conventionally divided into discrete links, called *trophic levels*.

## Trophic levels

Fisheries are understood in terms of these levels. The biggest predators on top, with the small fry on the bottom. These are the levels in the language that fisheries scientists use for the sea.

- Level 5: including humans, marine mammals
- Level 4: large carnivores, predatory fish
- Level 3: small carnivores, fish
- Level 2: herbivores, jellyfish, herring, smelt, mollusks
- Level 1: plants, zooplankton

The large predators include:

[14] See (McGoodwin, 1990; p. 65).

- Sharks
- Tuna, billfish (swordfish, marlin, sailfish)
- Cod, halibut, groupers, sea bass
- Salmon

The large predators are prized for sport fishing as well as commercial (food) fishing. Overfishing leads to declining populations of these levels.

## The declining catch

For a given fish, the *catch* refers to that total mass – usually measured in thousands or millions of tonnes (metric tons) – of fish caught and taken ashore, to home or to market. The *by-catch* is the mass of fish killed in the fishing process and thrown overboard. *Fishing effort* refers to the magnitude of the collective fishing process, measured by the total number of hours of fishing, including all fishing vessels, normalized to a standard size and type of gear.[15] According to fishery scientists, ocean stocks of large fish are declining, and some species are in danger of collapse, that is, extinction or near-extinction. Catch sizes are declining despite ever-increasing effort. In Figure 2 we see a graph of declining catch versus time for large North Atlantic predators from 1950 to 2000.[16] This also shows the rising rate of fish mortality (catch plus by-catch) as a percentage of total fish stocks per year.

In Figure 3 we see the catch for some individual species, increasing in parallel from 1950 to about,1977, then declining. Note the especially precipitous decline of the North Atlantic cod fishery. It collapsed actually, in 1992.[17] The politics of this collapse will be discussed in the next section.

## The declining trophic level

As the larger fish are generally more valuable, their numbers and weight decline earlier, and the fishery becomes concentrated on a lower trophic level. This progres-

[15](Beverton and Holt, 2004; pp. 29-33).

[16]See the graph in (Pauly and Maclean, 2003; fig. 10). A similar plot in (Palumbi and Sotka, 2011; p. 90) shows the earlier collapse of the sardine industry in Monterey Bay, California, in 1946. At the time, this was the largest commercial fishery in the world.

[17](Pauly and Maclean, 2003; fig. 6)

sion, known as *fishing down*, is shown graphically in Figure 4. Here, fishing down is depicted on both sides of the North Atlantic, from 1950 to 1998.[18]

# 6. Codfish chaos

The collapse of the North Atlantic cod fishery in 1992 provided a significant shock to fishery management agencies worldwide. Its analysis reveals major problems in management politics, which are important to other fisheries, especially those of tuna and salmon.[19]

One of these major problems involves the misuse of mathematical models to intimidate and obfuscate those who are uninformed regarding the implications of chaos theory. We begin our fishy tale with the politics, following (Pilkey and Pilkey-Jarvis, 2007). Then we will examine the mathematical model of (Beverton and Holt, 1957), in relation to its contemporary chaos theory in some detail.

## The politics of decline

Orrin H. Pilkey (b. 1934) was Professor of Earth and Ocean Sciences at Duke University, and Founding Director of the Program for the Study of Developed Shorelines (PSDS). He was a master of the dynamics of shoreline movements due to coastal processes. He pioneered the understanding of the weaknesses of the mathematical models used in coastal geology.

The book, *Useless Arithmetic: Why Environmental Scientists Can't Predict the Future.* written by Pilkey and his daughter Linda Pilkey-Jarvis, a geologist in the State of Washington's Department of Ecology, is devoted to a number of cases in which mathematical models are abused by politicians to mislead the public and legislators, including the collapse of the cod fishery, the safety of nuclear material stored in Yucca Mountain, the rising sea level, toxicity from abandoned mines, and plant invasions. The cod collapse of 1992 is described in detail in the first chapter. Regarding this they have said:

> One example from our book is the "fig leaf" coverage provided by quantitative modeling in the Grand Banks fishery. The Canadian Grand

[18] (Pauly and Maclean, 2003; fig. 15)

[19] See (Safina, 1998) and (Clover, 2004/2006) for these.

> Banks fishery has been described as the greatest in the world. It provided cod to the Western world for 500 years. In our lifetime, we watched the wild and senseless overfishing lead to the demise of an industry that employed as many as 40,000 people. The models, which many realized were questionable, provided a fig leaf behind which politicians could hide to avoid making the unthinkable decision to halt fishing.[20]

We move on now to an oft-misused model for a fishery, and its evolution in parallel with that of chaos theory. This fishery model, originally due to Beverton and Holt in the interval, 1947-1953, was reported in the book (Beverton and Holt, 1957/1993/2004), still in print and much used by fishery scientists. Its cover, shown in Figure 5, features the graph of their key equation, to which we now turn.

## Definition of the fishery model

The dynamic model of Beverton and Holt consists of the iteration of a one-dimensional function. Its graph is highly reminiscent of the logistic function much studied in chaos theory since 1954. They give some credit to Michael Graham, director of the Fisheries Research laboratory at Lowestoft, 1945-1958.[21] Figure 6 shows a prequel of the key graph, from (Gordon, 1954) and (Schaefer, 1957). This graph shows the catch (here called landings, L) and the cost (C) as functions of effort (E), all supposed to be totals for a given fishing fleet for an entire season.

We must pause here to give definitions of the basic variables, according to Beverton and Holt. The fleet may consist of vessels of different size, speed, fuel usage, gear, and so on. The *catch* of a vessel or fleet is the total weight of fish caught and taken ashore. This is sometimes indicated by $Y_W$, the yield by weight, in contrast to $Y_N$, the yield by number.

Each vessel is to be assigned a factor called its *fishing power*, $PF$, the catch per unit fishing time, relative to a standard vessel. The *fishing effort* of a fleet, $F$, is the total hours of fishing time for the whole fleet, measured in standard vessels, for a year.

The basic equation and key graph of Beverton and Holt, the foundation of their discrete dynamical model, is shown in Figure 7. Here three cases are shown. In each

[20] Found online at https://cup.columbia.edu/static/Interview-pilkey-orrin, September 2014).

[21] See (Graham, 1952). The papers (Gerhardsen, 1952) and (Gordon, 1954) also belong to the early literature of this model.

case, the upper part shows the graph of the basic function, and below, a time series of values obtained by iteration, as we shall explain.

The basic function here shows recruitment, $R$, as a function of egg production, $E$. Both $R$ and $E$ are measured in numbers of individual fish and eggs, respectively.

The reproductive cycle of a fish begins with egg production (spawning) by a mature fish. The eggs settle to the bottom, produce larvae, which feed on zooplankton, and hang out in a nursery area. This is called the *pre-recruitment phase*. At a certain age (or time) $t$, all surviving members of this brood or *year-class* enter the region in which capture may take place. This is called *recruitment*, and the number entering this *post-recruitment phase* is denoted, $R$. Somewhat later, at time $t'$, a marginally reduced number, $R'$, become eligible for capture. This is the *exploited phase*. A natural mortality is assumed with exponential rate of decline, $M$.

The recruitment, as determined by egg-production and growth in the function $R(E)$, is depicted in Figure 7. The nonlinearity of this function, shown by its single-humped graph, is due to the inclusion of a nonlinear density-dependent growth function due to von Bertalanffy, The basic equation, ascribed to (Becking, 1946), is the *compound exponential* [22]

$$R(E) = \alpha E e^{-\beta E}$$

where $\alpha$ and $\beta$ are positive constants. This is very similar to the iteration studied in (May and Oster, 1976).[23]

## Iteration of the fishery model

The iteration of this discrete dynamical system is obtained by assuming a constant egg production from each individual in the post-recruitment phase. This constant is shown as the diagonal line labelled $\gamma$ in the upper graphs. This is the line through the origin, $(0, 0)$, with slope $1/\gamma$.

Each year sees recruitment, eggs, and growth. Next year, new recruitment, new eggs. new growth, and so on. The three graphs differ only in the slope of the $\gamma$ line, which varies among the three cases, decreasing from left to right.

[22]See (Beverton and Holt, 2004; eqn. 6.16, p. 56).
[23]See Fig. 4 on P. 578.

The iteration of the discrete dynamical system goes like this.[24] Our explanation refers to the enlargement of the graph in the upper left corner of Figure 7, which is shown in Figure 8.

The red point labeled $fp$, at the intersection of the humped curve and the diagonal line, is a fixed point of the process. Its coordinates are $(E_0, R_0)$.

Now choose a starting value for egg production, $E_1$, at time, $t_0$. This would have been the egg production of an initial recruitment, $R_1$. This event is represented in our figure by the blue point labelled $a$, with coordinates $(E_1, R_1)$, which is on the diagonal line directly above the value $E_1$ on the horizontal ($E$) coordinate axis.

After an epoch ($\Delta$, the duration of the growth phase from eggs into recruits) we obtain a new recruitment, $R_2 = R(E_1)$, at time, $t_1 = t_0 + \Delta$. This is represented in Figure 8 by the blue point labelled $b$, with coordinates $(E_1, R_2)$, which is on the humped curve directly above the point, $a$. At this time, there is a new egg production, $E_2 = \gamma R_2$. This is presented in Figure 8 by the blue point labelled $c$, with coordinates $(E_2, R_2)$, which is again on the diagonal line.

We may think of our dynamic operating in two steps, in which the point $a$ on the diagonal moves to the point $c$ also on the diagonal. The two steps are: vertical to the humped graph at $b$, then horizontal to the diagonal.

After another epoch, a recruitment $R_3 = R(E_2)$ is produced at time $t_2 = t_0 + 2\Delta$, and so on. Thus:

- The point $a = (E_1 R_1)$ is on the diagonal, the graph of the linear function, $E \rightarrow E/\gamma$.
- The point $b = (E_1, R_2)$ is on the humped curve, the graph of the function, $E \rightarrow R(E)$.
- The point $c = (E_2, R_2)$ is on the diagonal line.
- The sequence of points, $a, b, c, \ldots$, define the *cobweb* of horizontal and vertical line segments in the upper three plots of Figure 7.
- The sequence of points $(R_1, t_0)$, $(R_2, t_1), \ldots$, define the solid lines in the lower plots of Figure 7.
- The sequence of points $(E_1, t_0)$, $(E_2, t_1), \ldots$, define the dashed lines in the lower plots of Figure 7.

[24] (Beverton and Holt, 2004; p. 52)

Beneath the key graph in each case, the time series showing $R$ and $E$ as functions of time exhibits, respectively, static, periodic, and chaotic behavior. In the first case, the fixed point, $fp$, is an attractor, the cobweb converges to it.

## Bifurcations of the Berverton and Holt model

Note that the decline of the egg-production rate per fish, $\gamma$, is the bifurcation parameter, creating the changes in behavior from fixed, to periodic, to chaotic behavior. This parameter, for fish in the ocean, is highly sensitive to environmental factors that are unpredictable. Hence, the predictions of the model are highly unreliable, according to chaos theory.

Note that if the egg-production, $\gamma$, decreases, the diagonal line, having slope $1/\gamma$, becomes steeper, and the attractive fixed point moves to the left. At a certain critical value of egg-production, this fixed point arrives at the origin. This represents the death of the fishery. It occurs when $\gamma = 1/\alpha$, the reciprocal of the lead coefficient of the compound exponential Becking equation above. This parameter represents the survival rate of larvae into recruits.

On the other hand, as $\gamma$ increases, the diagonal line drops, and the fixed point moves to the right. Eventually, we will find the period doubling sequence leading to chaotic behavior. This is characteristic of a healthy fishery.

This is the basic setup of chaos theory, as it developed for the iteration of one-dimensional functions, from 1954 in the work of Myrberg.[25] Understandably, this work would not be known to Beverton and Holt during their evolution of the fishery model. But a generation later, we have experienced the chaos revolution following the publications of (May, 1974) and (Li and Yorke, 1975). May, a theoretical ecologist, mentions population models even in the title of his 1974 paper. And this becomes even more explicit in the joint paper (May and Oster, 1976). The graph of the function studied therein is shown in Figure 9. Note the similarity to Beverton and Holt, shown in Figure 5.

But in the work of May and Oster, the full paraphernalia of chaos theory is applied to the model. So it seems that fisheries scientists should be well aware of the chaotic behavior of their fundamental model well before the demise of the North Atlantic cod fishery in 1992.

[25] See the wonderful exposition of this work in (Mira, 1987; Ch. 3.).

# 7. The early days of chaos theory, 1959-1974

In *The Chaos Avant-Garde* a scheme for the history of chaos theory was proposed, based on the idea that chaos theory began with a revolution, the "chaos revolution", in which the word "chaos" was attached to the branch of mathematics known as dynamical systems theory, as a popular name. All the prior background, from Poincaré up to this historical bifurcation, is regarded there as the prehistory of chaos theory, also known as "the early days of chaos theory." The book comprises the memories of the pioneers of the early days, beginning with Steve Smale in 1959.[26] One of the early threads of chaos theory concerns the iteration of a one-dimensional function, as above.[27]

## The chaos revolution

This bifurcation moment is placed in early 1974 by Li and Yorke:

> Each academic year, the Math Department of the University of Maryland routinely organized a special year program. The topic of the program for the academic year of 1973-74 was mathematical biology. Robert May, who was trained as a physicist but had become a professor of biology at Princeton University, was one of the distinguished invited speakers of the program. During his visit in the first week of May, 1974, Professor May delivered five lectures, one per day. The subject of his fifth talk was the Logistic Model,
>
> $$T_a(x) = ax(1-x), x \in [0,1], a \in [0,4]$$
>
> He described his discovery, the now well-known *doubling period bifurcations* as $a$ varies. He did not know what happened in the *chaotic region*, the region beyond the main cascade of period doubling! ...
>
> In the summer of 1974, Professor R. May was invited to give talks by many institutions in different countries of Europe. He adopted our use of "chaos" as a mathematical term, and *Period three implies chaos* therefore began to attract considerable worldwide attention by his strong advocacy in talks and papers.[28]

[26]See the preface in (Abraham and Ueda, 2000).
[27]The several texts of Robert Devaney provide elementary explanations.
[28]See (Li and Yorke, 2000; pp. 204-205).

### The work of Myrberg

At that time, in 1974, these pioneers did not know of the earlier work of P. J. Myrberg of Helsinki, who published much of what is known today of the logistic map family in a series of six papers in German and French, in the interval 1958 to 1963.[29] By 1976, May was crediting the priority of Myrberg's papers of 1958 and 1963.[30]

The results of Myrberg concern the periodic attractors of the logistic family of quadratic real mappings, and their convergent cascade to a limit point in the parameter interval (see $a$ in the equation above), beyond which the chaotic behavior of the model was discovered later, perhaps by Metropolsky, Stein and Stein in 1973. In any case, the bifurcation cascade was known to May, and independently to Li and Yorke, in early 1974. And the chaotic behavior was known to Li and Yorke, and communicated by them to May, in 1974.

## 8. The shape of the Beverton-Holt function

The function,

$$f(x) = \alpha x e^{-\beta x}$$

where $\alpha$ and $\beta$ are positive constants, and $x$ represents the current fecundity of the fish (average annual egg production per fish).[31]

So given a time $t_0$, an increment $\Delta t$ (e.g., equal to one year), and the time sequence, $\{t_0, t_1 = t_0 + \Delta t, t_2 = t_0 + 2\Delta t, , \ldots\}$, we have a corresponding sequence of egg counts, $\{E_0, E_1, \ldots\}$, with $E_1 = f(E_0)$, and so on. This is the basic iteration of Beverton and Holt, which we now wish to study.

First, we consider the graph of this function, restricted to the unit interval, $[0, 1]$. From the first derivative,

$$f'(x) = \alpha(1 - \beta x)e^{-\beta x}$$

[29]The results of Myrberg are covered in detail in (Mira, 1987; Ch. 3), including the rediscovery in part by Metropolsky, Stein and Stein in 1973, and by Milnor and Thurston, 1977, and others.
[30](May, 1976)
[31](Beverton and Holt, 1957; p. 56)

we see that there is a unique critical point at $x = x_c = 1/\beta$, and this will be within the unit interval providing $\beta > 1$. The value of the function at this point is $\alpha/\beta e$, which will be within the unit interval as long as $\alpha < \beta e$. Recall that $e = 2.72$, approximately.

Evaluating the second derivative,

$$f"(x) = -\beta(\alpha + 1)e^{-\beta x}$$

at the critical point, $x_c = 1/\beta$, we find

$$f"(x_c) = -\beta(\alpha + 1)/e$$

which is negative. Thus, the unique critical point is a maximum, and the function has a single hump, at the point,

$$(1/\beta, \alpha/\beta e)$$

.

This is the sort of function which is well studied in chaos theory. If is known, for example, that the iteration of this function on the unit interval has a single basin, and a single attractor.

In summary, we have an endomorphism of the unit interval (that is a mapping of this interval into itself) defined by the function,

$$f(x) = \alpha x e^{-\beta x}$$

wherein the two positive constants are restricted, $\beta > 1$ and $\alpha < \beta e$ . There is a unique local maximum at the point, $(1/\beta, \alpha/\beta e)$. This point is indicated on a plot of the function in Figure 10, with $\alpha = 10$ and $\beta = 5$.

# 9. Bifurcations of the Beverton and Holt iteration

Increasing the amplitude, $\alpha$, for a fixed rate of decay, say $\beta = 8.0$, we observe:

- first a fixed point attractor, with $\alpha = 6.0$, shown in Figure 11,
- then a two-periodic attractor, with $\alpha = 9.0$, Figure 12,
- and a period doubling sequence leading to chaos, with $\alpha = 15.0$, Figure 13,

all shown for $\beta = 5$. A full sequence is shown in the response diagram, Figure 14, as $\alpha$ is increased from 0 to 20 and $\beta = 8.0$.[32]

Here the periodic doubling sequence, familiar from the much studied logistic system, leads to chaos around $\alpha = 14$. Note the prominent five-periodic window around $\alpha = 18.5$.

# 10. Interpretation of the response diagram

Regarding the relation between the model data and real fish data, compare the recruitment of North Sea haddock (the main focus of Berton and Hold in the 1950s) show in Figure 15 (dashed line) with the model data of Beverton and Holt (lower right of Fig. 7, solid line). The similarity is striking. Note that the variation defies linear thinking, in that prior to the overfishing of the 1990s, recruitment jumps from one extreme to the other are typical.

Our amplitude coefficient, $\alpha$, is a combination of the two factors, $\alpha'$ and $\gamma$, in the original equation of Beverton and Holt, where $\alpha'$ is a fixed constant, and $\gamma$ is the relative fecundity, or average egg production of a single fish in some units. That is, $\alpha = \alpha'\gamma$. Thus if we follow the behavior of the unique attractor as $\gamma$ decreases from a healthy value of $\gamma = 20$ down to $\gamma = 0$, we observe the behavior declining from chaotic behavior over a full range of total egg populations, to a fixed-point attractor declining to zero (collapse of the fishery).

Further, as $\gamma$ declines, the complexity declines as well. First healthy chaos, then worrisome periodicity, and finally stasis, declining to zero. This is evident in the Berverton-Holt model, Figure 7, in real data, Figure 15.

According to Carl Safina (1998; p. 355),

> Overfishing not only directly depletes fish, it can rob other fish of food, lowering overall abundance across many species. Fishing can lead to shifts in the community, rearranging the neighborhood by pulling out

[32] This may be regarded as an extension of Fig. 8 in (May and Oster, 1976).

key species, causing cascading effects as the rest of the species adjust. This is called "ecosystem overfishing."

This is supported by the "fishing down" data, showing declining trophic levels of the annual catch, Figure 4. Complexity, and thus sustainabiity, of the entire ocean ecosystem at large, is on the way down.

We have seen that the earliest application of chaos theory, in the case of one-dimensional iterations, began in the interval 1947-1953, in the fishery model of the applied mathematicians Beverton and Holt, working at the Fisheries Research Laboratory in Lowestoft, on the North Sea coast, north-east of London. Meanwhile, the pure mathematics of this field was undergoing independent development by Myrberg in Helsinki, about 1000 miles away.

Beverton and Holt were aware of the bifurcations of the model from point attractor, to periodic attractor, and ultimately to chaotic attractor. Yet it was up to Myrberg to map out the full bifurcation sequence of the model for the first time, including the cascade of period doubling bifurcations in the runup to chaos.

It is understandable that the full implications of chaos theory were unknown to the early workers in fisheries science. But by the time of (May, 1974) and (May and Oster, 1976), there was really no excuse. So the use of this model for setting the total allowable catch (TAC) for fisheries at risk of collapse, which was (and continues to be) the case for the cod and tuna fisheries in the North Atlantic, must be due to inexcusable ignorance (or duplicity) on the part of the scientists and/or the politicians involved.

# 11. Systems thinking

Part of the problem here is the widespread ignorance – on the part of all the actors in the drama of the commons – of the basic concepts of complex dynamical systems theory. This might be remedied by changes in our school curriculum. adding an extensive thread on systems thinking.

Another part of the dilemma concerns the state of morals in our global culture. A giant bluefin tuna, its catch forbidden by law, might be worth $180,000 in Tokyo.[33] What to do? This is a strong case of markets versus morals. Mathematical economist Daniel Friedman, in his excellent book, *Morals and Markets*, devotes a chapter to

[33] See (Safina, 1998; p. 114).

"Environmental markets and morals," in which the North Atlantic cod collapse of 1992 is carefully analyzed. While we might hope for morals to be improved in our educational system, Friedman concludes:

> The point should now be clear. Environmentalists were naive to believe that a moral crusade will, by itself, secure a sound and durable environmental policy. Laws may be passed and bureaucracies built, but inevitably, they will be eroded when they conflict with market imperatives.[34]

A system dynamics model for the tragedy of the commons is included in the *NetLogo Model Library* that is freely distributed along with the NetLogo computer language system. This model runs in *participatory simulation mode* (using a NetLogo feature called HubNet) in which students with laptops may participate in a simulation of the management of a commons. A screen shot of this model is shown in Figure 16.

A more realistic and informative model might be built upon the NetLogo model, be enclosing it within a larger system, combining the commons with political and environmental-activist nodes. Such a scheme is indicated in Figure 17.

# 12. Conclusion

The plight of the ocean ecosystem is a classic example of the tragedy of the commons. The best hope for recovery seems to be conservation institutions made by local interest groups. The development of those agreements require new levels of morality, which are difficult to create, and/or new levels of understanding of ecosystems as complex systems, which may be addressed through education. In this direction, we have proposed an educational thread on systems thinking for K-12 schools using complex dynamical systems and chaos theory.[35]

[34]See (Friedman, 2008; p. 164).

[35]This thread has been implemented successfully at the Ross School, of East Hampton, New York.

# Appendix: CDS Definitions

We begin with some necessary background definitions from mathematics. These need not be understood in detail, but must at least be recognized.

Simple dynamical systems occur in three flavors: flows, cascades, and iterations.

A *flow* is a continuous-time dynamical system defined by a vectorfield, that is, a system of ordinary differential equations on a space, the *state space*, state space.

A *cascade* is a discrete-time system defined by the iteration of an invertible function (a diffeomorphism) from a space onto itself.

An *iteration* is a discrete-time system defined by the iteration of a noninvertible function, (an endomorphism) from a space into itself.

A *dynamical scheme* is a family of simple dynamical systems depending on parameters, that is, control variables.

A *complex dynamical system* comprises a number of simple dynamical systems (iterations or flows) linked by functions from the state space of one to the control parameters of another.[36] The dynamical behavior of a CDS may be studied in terms of its *attractors* and their *basins*. A *bifurcation* of a CDS occurs when a small change in a control parameter results in a significant change in its configuration of attractors and basins.[37]

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[36] A synonym for CDS is *system dynamics*, introduced by Jay W. Forrester around 1968.

[37] All these may be learned from a textbook, such as (Abraham and Shaw, 1992). They are usually learned by the study of special cases.

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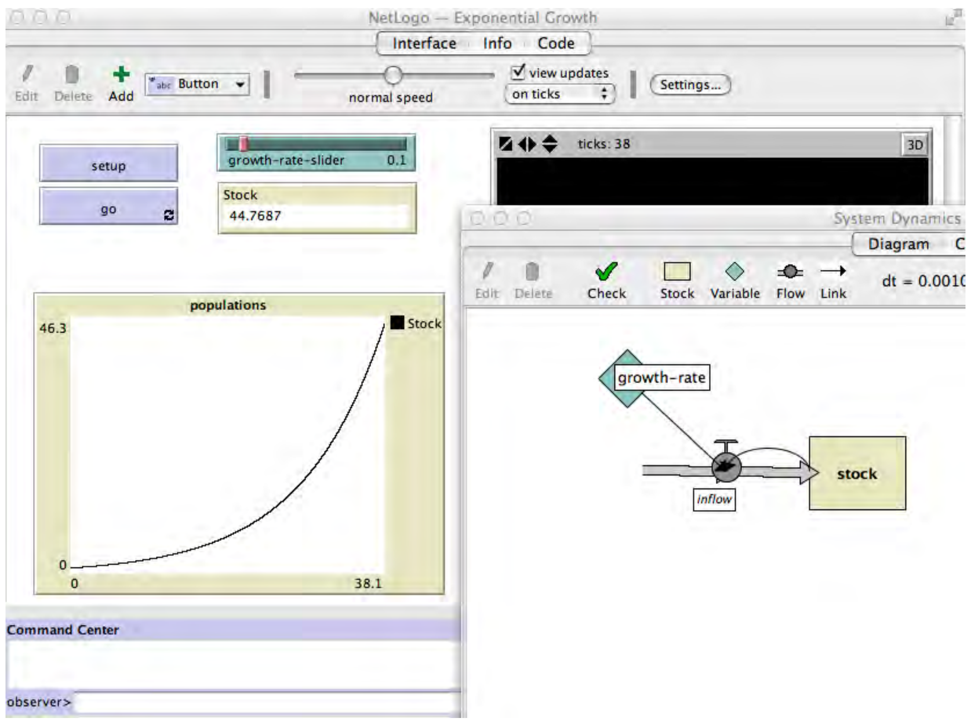
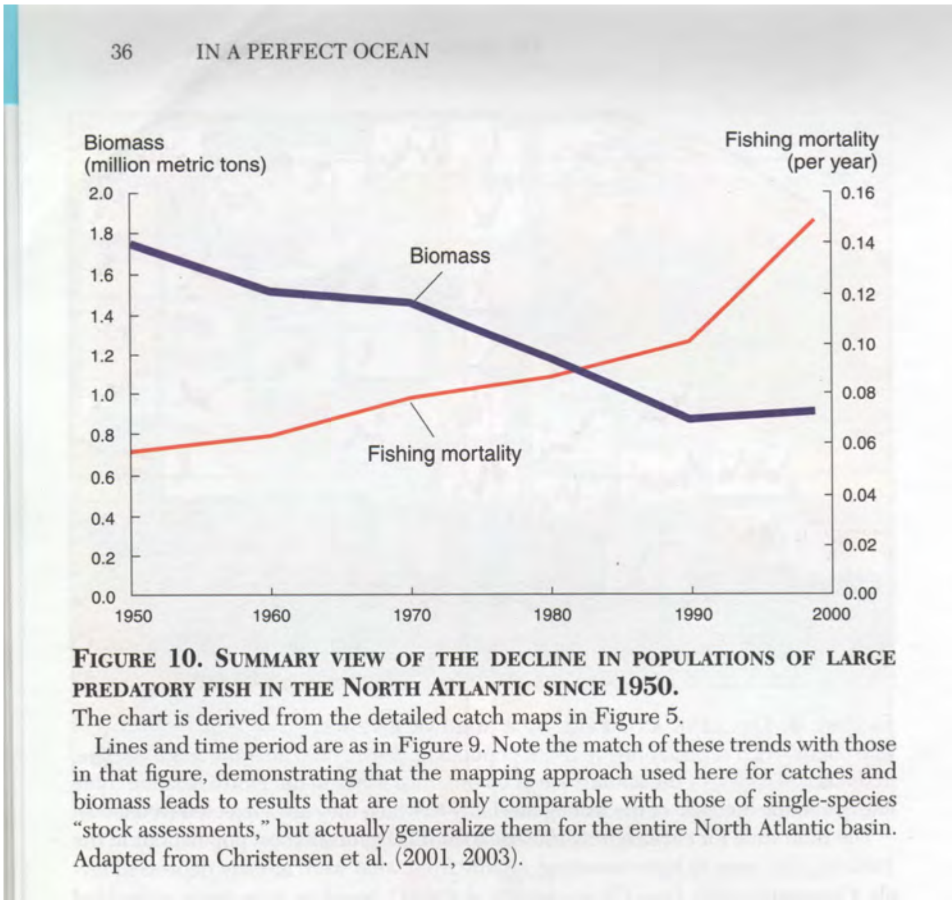



Figure 1: A one-dimensional flow in the NetLogo Systems Dynamics Modeler



36 IN A PERFECT OCEAN



**FIGURE 10. SUMMARY VIEW OF THE DECLINE IN POPULATIONS OF LARGE PREDATORY FISH IN THE NORTH ATLANTIC SINCE 1950.**
The chart is derived from the detailed catch maps in Figure 5.

Lines and time period are as in Figure 9. Note the match of these trends with those in that figure, demonstrating that the mapping approach used here for catches and biomass leads to results that are not only comparable with those of single-species "stock assessments," but actually generalize them for the entire North Atlantic basin. Adapted from Christensen et al. (2001, 2003).

Figure 2: Fish decline, from Pauly and Maclean, 2003, Fig. 10, p. 36.

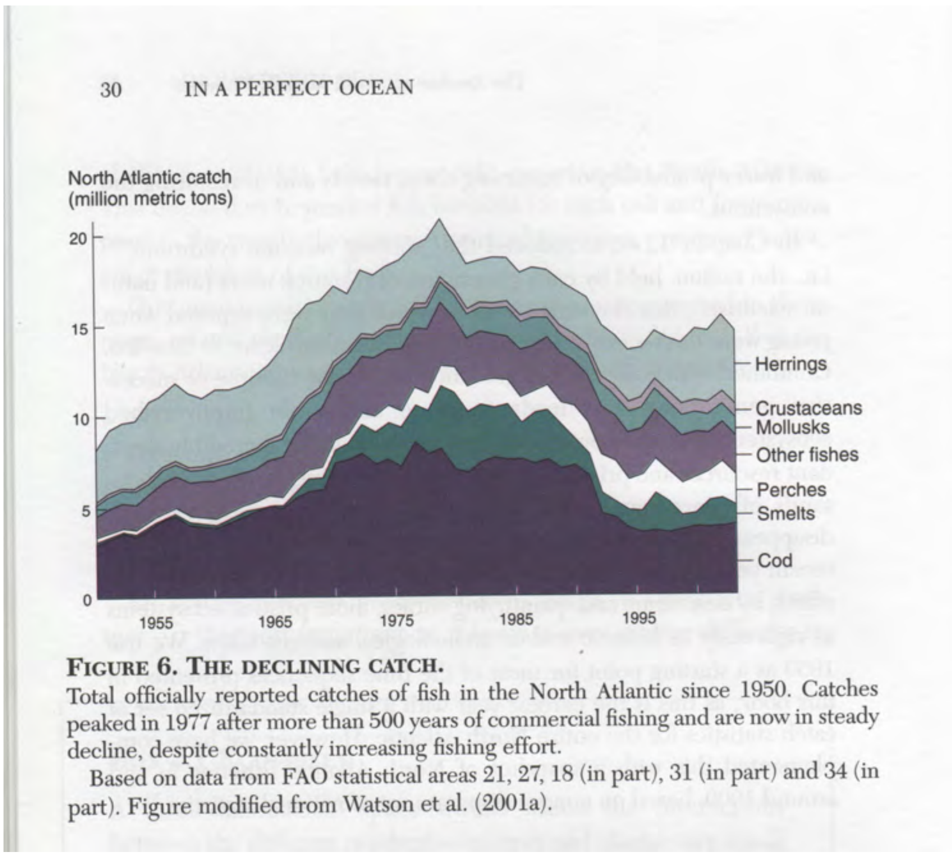

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**Figure 6. The declining catch.**
Total officially reported catches of fish in the North Atlantic since 1950. Catches peaked in 1977 after more than 500 years of commercial fishing and are now in steady decline, despite constantly increasing fishing effort.

Based on data from FAO statistical areas 21, 27, 18 (in part), 31 (in part) and 34 (in part). Figure modified from Watson et al. (2001a).

Figure 3: Fish decline by species, from Pauly and Maclean, 2003, Fig. 6, p. 30.

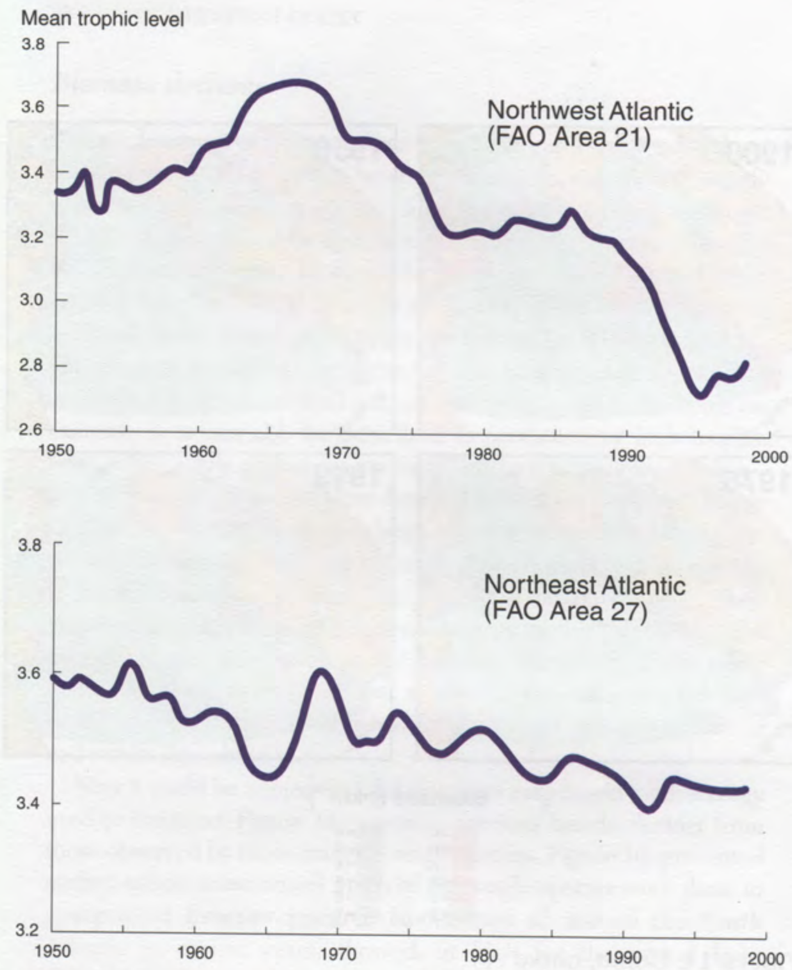



**FIGURE 15. "FISHING DOWN" NORTH ATLANTIC FOOD WEBS.**
Trophic level trends in the North Atlantic, 1950 to 1998, indicating the rapid (in the West) or gradual (in the East) replacement of large predators in fisheries catches by small fishes and invertebrates.

Various studies have shown that such changes in catch composition indicate changes in relative abundance in the underlying ecosystems, and thus reflect the collapse of large populations (when trophic level declines are rapid) or the serial depletion of a number of smaller populations (when declines are gradual, but continuous over longer periods). Modified from Pauly et al. (1998c)

Figure 4: Fish decline by trophic level, from Pauly and Maclean, 2003, Fig. 15, p. xx.

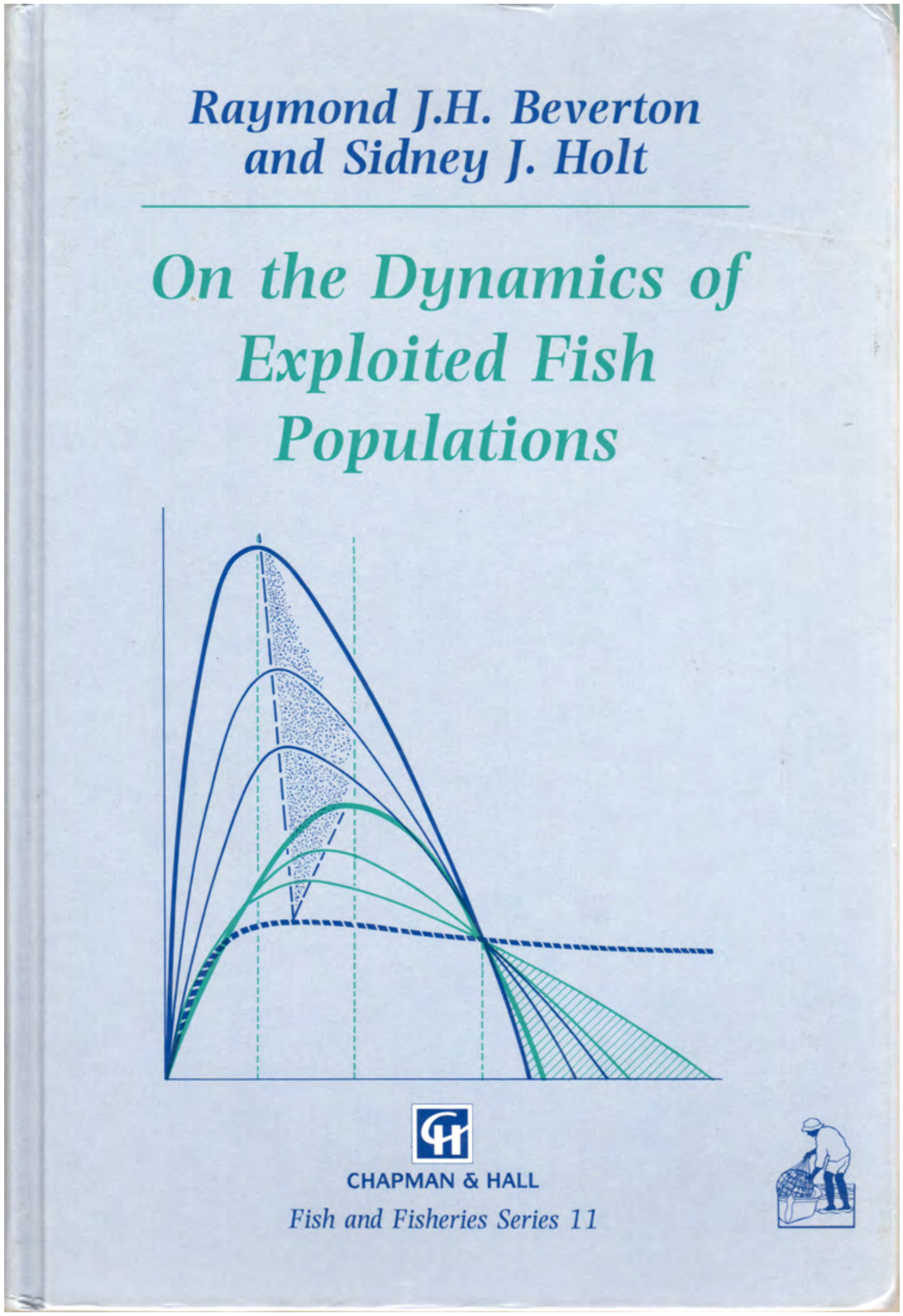



Figure 5: Beverton and Holt, 1957; Cover.

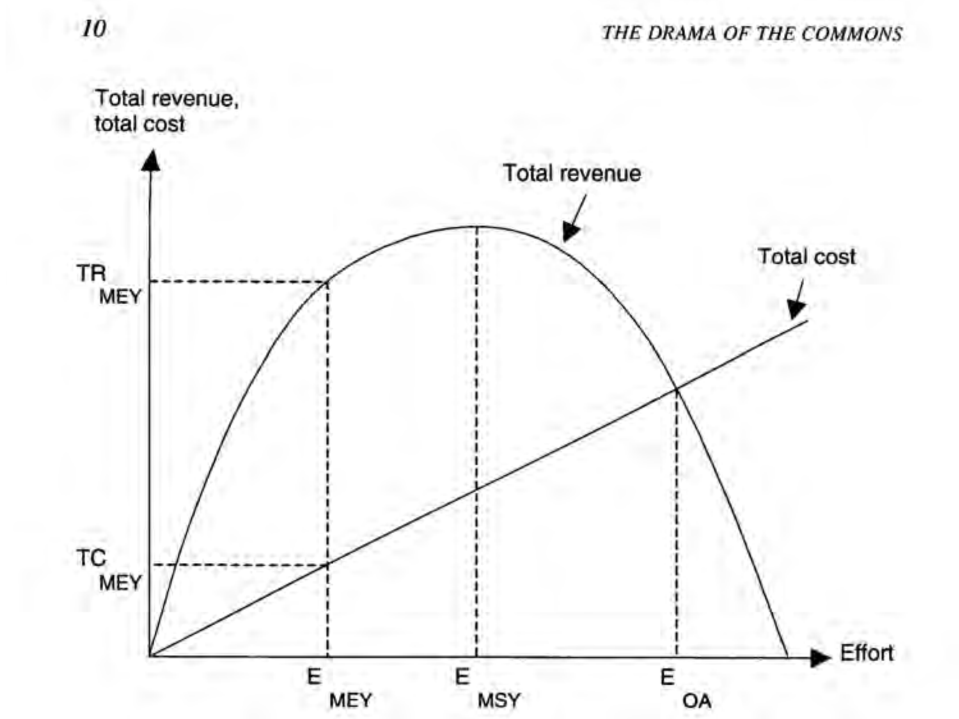
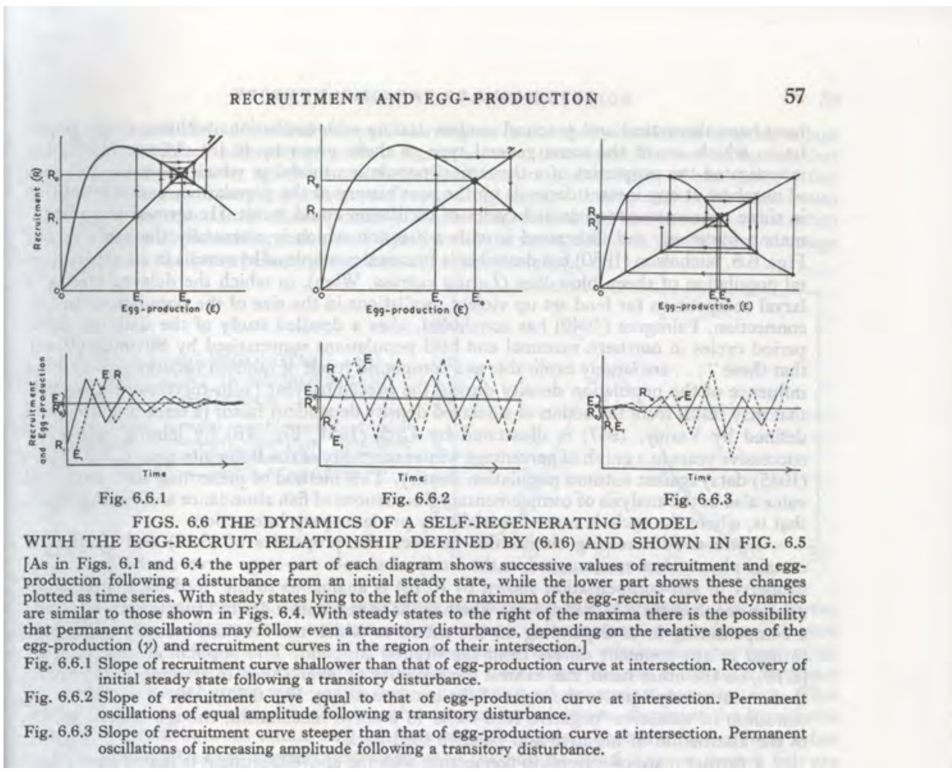



Figure 6: Summary of Gordon and Schaefer, from (Dietz, 2002; p. 10).



RECRUITMENT AND EGG-PRODUCTION 57

Fig. 6.6.1 Fig. 6.6.2 Fig. 6.6.3

FIGS. 6.6 THE DYNAMICS OF A SELF-REGENERATING MODEL WITH THE EGG-RECRUIT RELATIONSHIP DEFINED BY (6.16) AND SHOWN IN FIG. 6.5

[As in Figs. 6.1 and 6.4 the upper part of each diagram shows successive values of recruitment and egg-production following a disturbance from an initial steady state, while the lower part shows these changes plotted as time series. With steady states lying to the left of the maximum of the egg-recruit curve the dynamics are similar to those shown in Figs. 6.4. With steady states to the right of the maxima there is the possibility that permanent oscillations may follow even a transitory disturbance, depending on the relative slopes of the egg-production ($\gamma$) and recruitment curves in the region of their intersection.]

Fig. 6.6.1 Slope of recruitment curve shallower than that of egg-production curve at intersection. Recovery of initial steady state following a transitory disturbance.

Fig. 6.6.2 Slope of recruitment curve equal to that of egg-production curve at intersection. Permanent oscillations of equal amplitude following a transitory disturbance.

Fig. 6.6.3 Slope of recruitment curve steeper than that of egg-production curve at intersection. Permanent oscillations of increasing amplitude following a transitory disturbance.

Figure 7: Beverton and Holt, 1957; p. 57

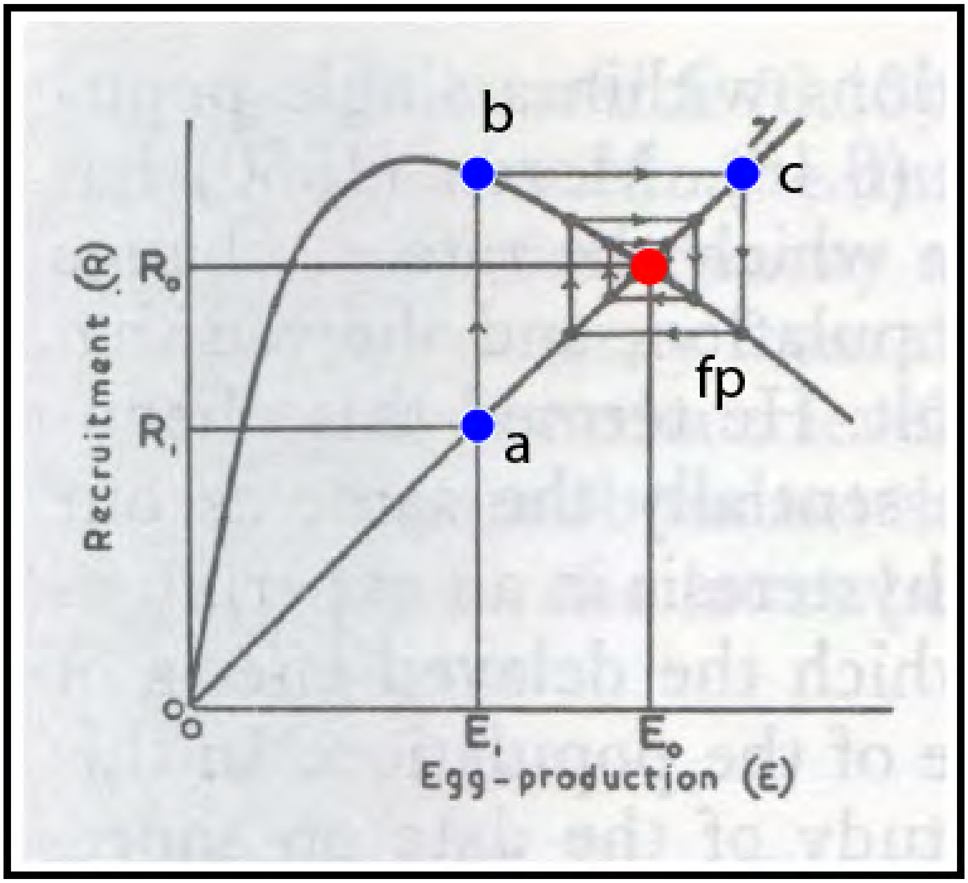



Figure 8: Beverton and Holt, 1957; p. 57, Fig. 6.6.1, upper, enlarged

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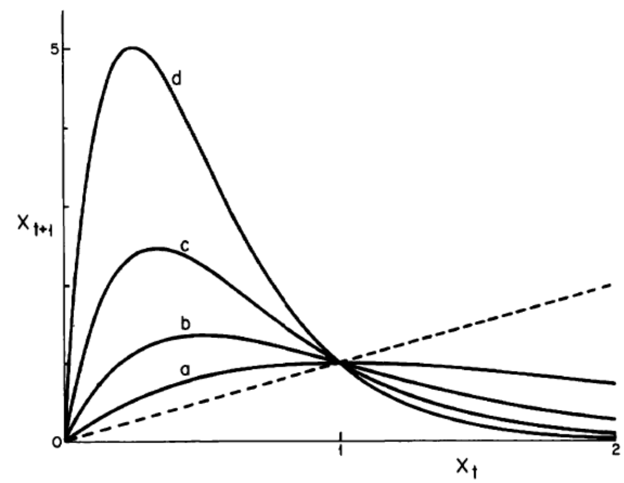


FIG. 4.—The equation $X_{t+1} = X_t \exp[r(1 - X_t)]$ for various values of $r$: $a$, $r = 1$; $b$, $r = 2$; $c$, $r = 3$; $d$, $r = 4$.

Figure 9: The key graph, from May and Oster, 1976; fig. 4, p. 578.

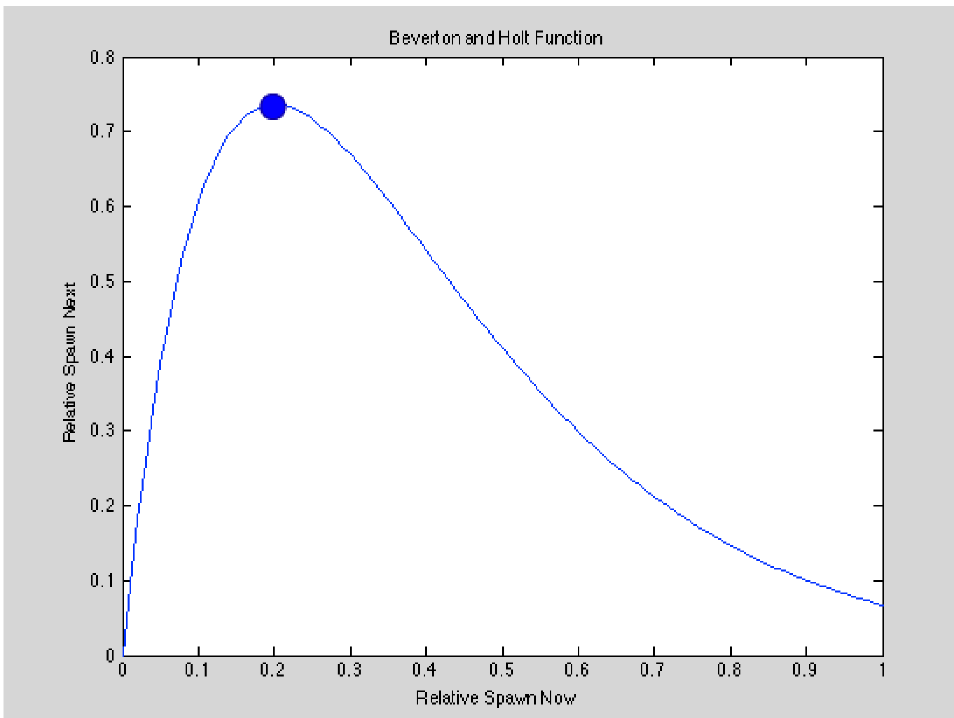


Figure 10: Plot of Beverton-Holt function, $\beta = 5$.

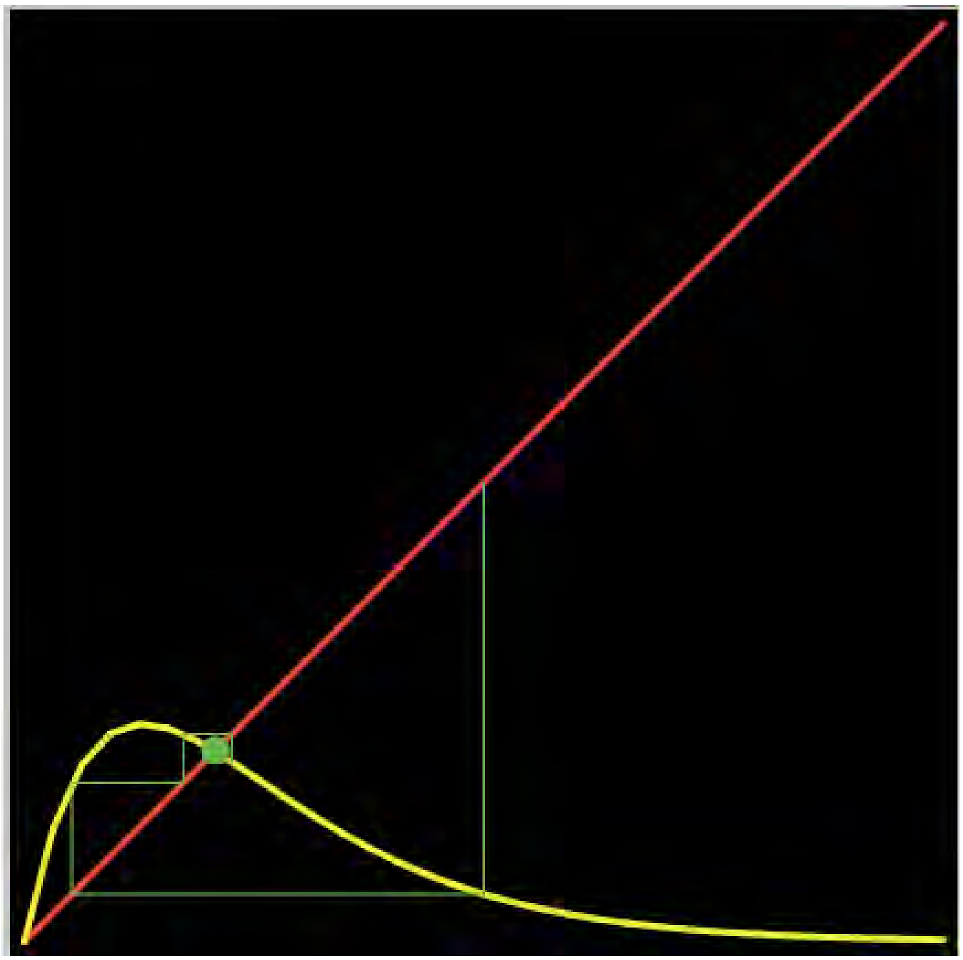

Figure 11: Beverton-Holt iteration, $\beta = 5$, $\alpha = 6.0$. Compare (Beverton and Holt, 1957; Fig 6.6.1, p. 57).

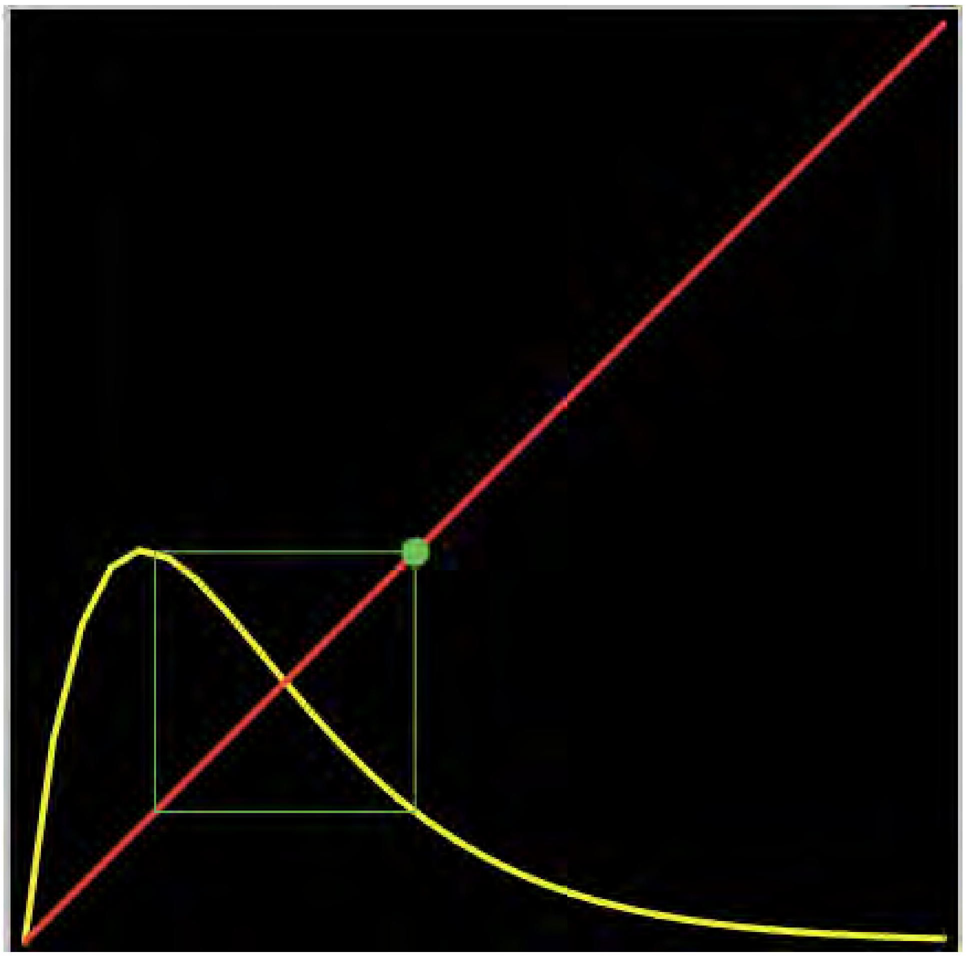

Figure 12: Beverton-Holt iteration, $\beta = 5$, $\alpha = 9.0$. Compare (Beverton and Holt, 1957; Fig 6.6.2, p. 57).

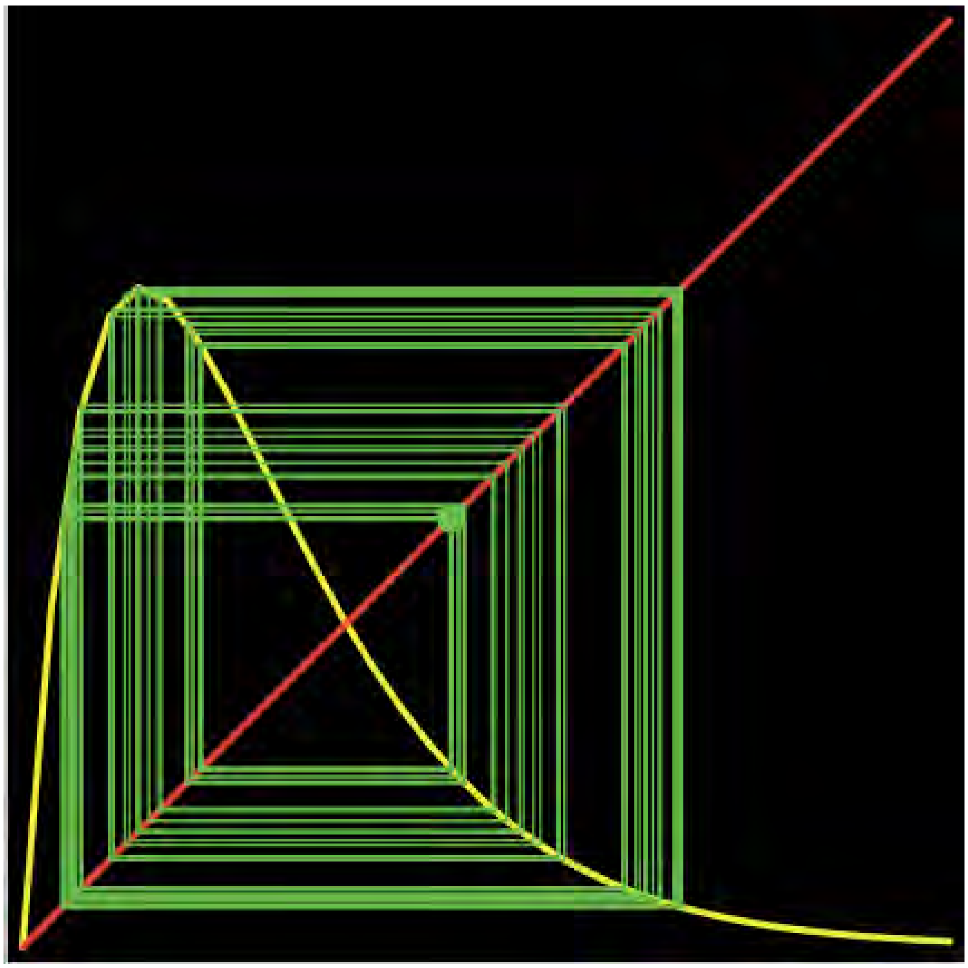

Figure 13: Beverton-Holt iteration, $\beta = 5$, $\alpha = 15.0$. Compare (Beverton and Holt, 1957; Fig 6.6.3, p. 57.

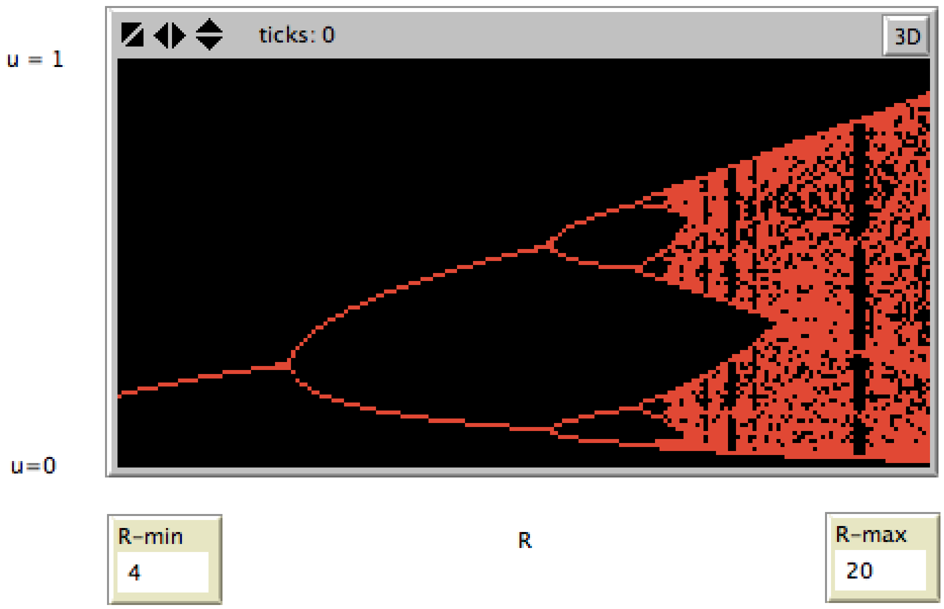


Figure 14: Beverton-Holt iteration, $\beta = 8$. Here, $R = \alpha$ (on the horizontal axis) increases from 4 to 20, while the domain, the unit interval [0.1], is shown as the vertical axis. For each value of $R$, the corresponding attractor is shown in red directly above.

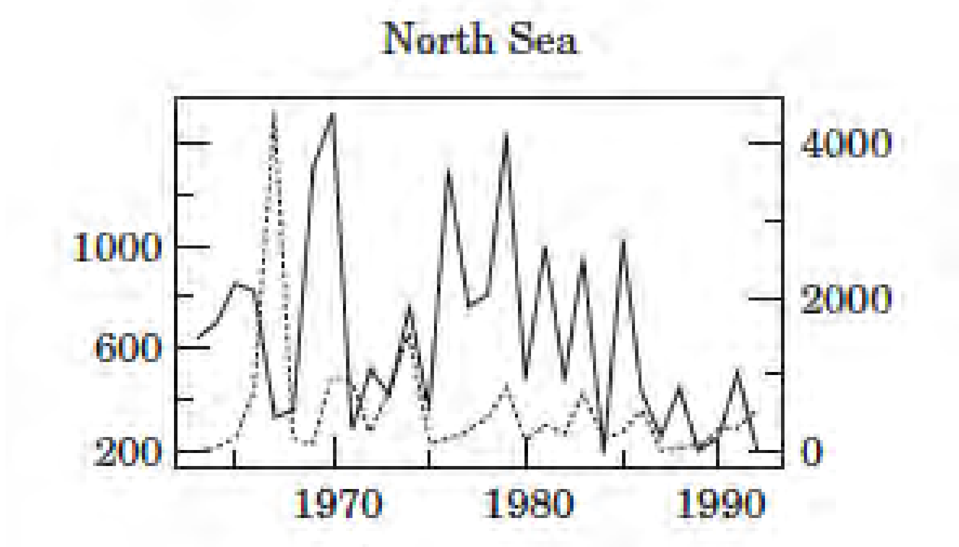


Figure 15: Recruitment variability of North Sea cod (solid) and haddock (dashed). Annual recruitment (one-year old fish) in millions of tonnes from 1950 to 1995. From (Fogarty, 2001; Fig. 1).

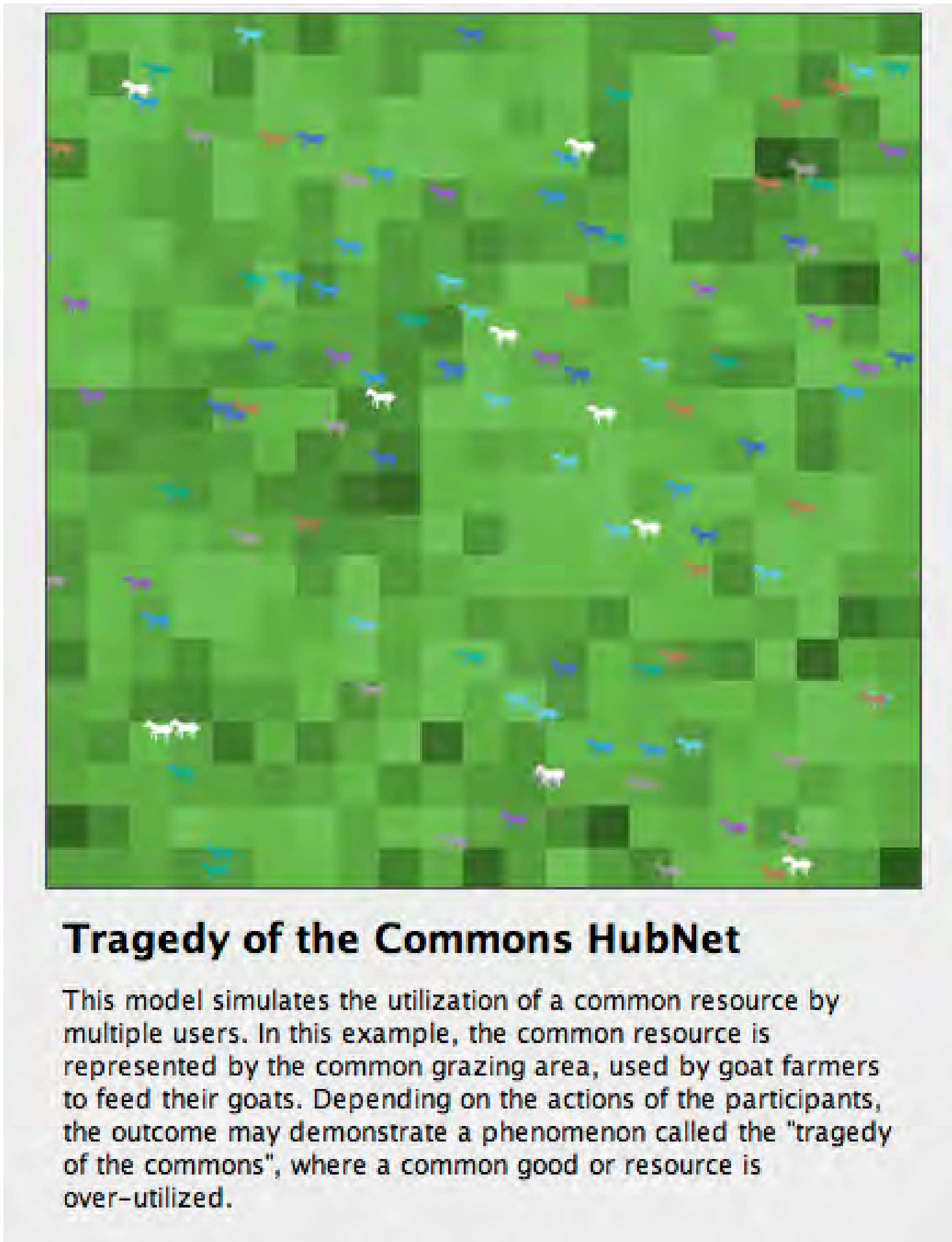


Figure 16: A NetLogo model for the tragedy of the commons.

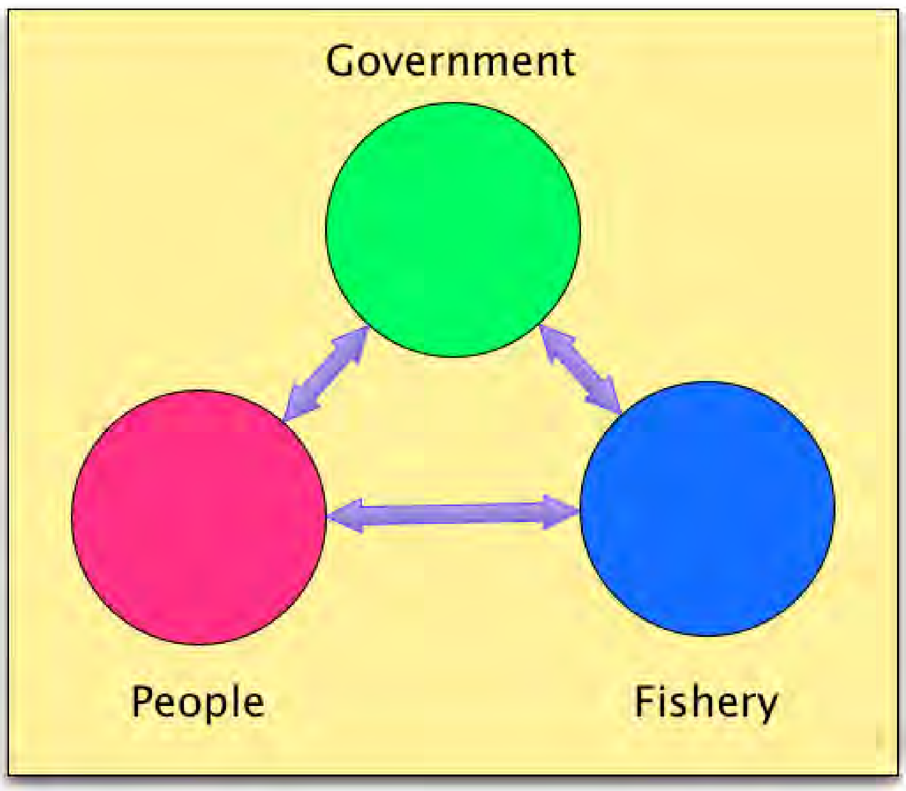


Figure 17: A scheme for an all-encompassing model.

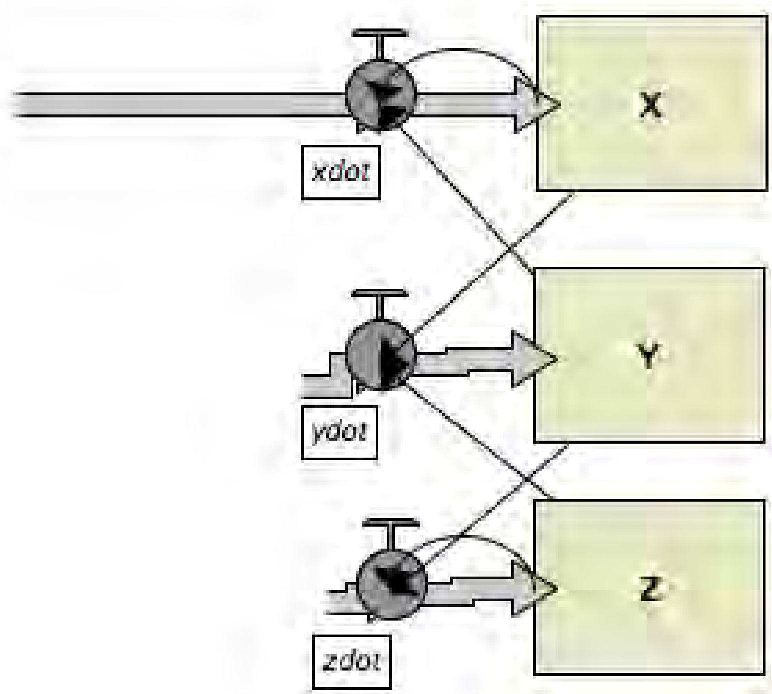


Figure 18: The systems dynamics representation of the three-node model, from NetLogo. Here X, Y, and Z represent people, fisheries, and government, respectively.