The Alhambra Theorem

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Abstract

The early evolution of geometric thinking produced geometric algebra in ancient Greece, rhetorical algebra in early Islam, and eventually a precursor of modern algebra in the repeating patterns of medieval textiles and wall decorations. This medieval craft mathematics culminated in the Alhambra in Islamic Spain. Here we tell the story of this milestone in the history of mathematics.

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1. Introduction

Since 1995 I have been involved with the Ross School – a private school in East Hampton, New York – as a curriculum consultant. Students of this school have the option of several school trips each winter. Recently, Courtney Ross, the founder of the school, decided on a special series of excursions concerning the diffusion and evolution of knowledge from Ancient Greece to the European Renaissance. Athens, Constantinople, Andalusia, and Florence were to be among the destinations.

As it happened, I was invited by Mrs. Ross to accompany the school excursion to Andalusia. This was to be my first school trip, and I would be helping the students to understand the migration and development of mathematics from Athens to Rome, through the Muslim world and across North Africa to Andalusia in the south of Spain. Special emphasis was to be given to the role of La Convivencia, the cooperation and harmony of the three cultures (Christian, Jewish, and Muslim) leading up to the explulsion of Muslims and Jews by Ferdinand and Isabella in 1492.

And so, in the last week of February, 2007, I left home for New York and Sevilla, the capital of Andalusia. Along with a score of students from the Ross School and a string of superb local guides, we explored Sevilla, Cordoba, and Granada for ten days, returning home via Toledo and Madrid. Although an ardent amateur of the history of mathematics, I knew nothing of the history of Islamic art. So, during these ten days, I was astonished repeatedly by the beauty and mathematics of the architecture and wall decorations of the palaces, mosques, cathedrals, and synagogues of Andalusia. And this progression culminated upon our arrival at the Alhambra in Granada, on the 4th of March.

Upon returning home, I was shown an excellent book (Abas, 1995) by my friend Larry Cuba, a pioneer of computer animated art. This book refers to the Alhambra as a mathematical theorem, and that set me on the track leading to this article. Here I will trace the history and prehistory of *the Alhambra Theorem* from dot to its discovery by Islamic craftsmen by around 1300 and its rediscovery by modern mathematicians some 600 years later.

It is a pleasure to thank Courtney Ross and Larry Cuba for inspiring this story.

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2. The long road

Our largest context is the growth of geometric thinking from the hand axe of *homo erectus*, around 1,500,000 years ago[1], to the birth of modern algebra around 1900 CE.

In previous writings,, I have described a path of cultural diffusion in which the earliest known occurrence of repeated geometric patterns, in the late Paleolithic painted caves in the south of France around 36,000 years ago[2] made their way to the early textiles (proto-kilims and kilims), wall paintings, and painted pottery of Neolithic Anatolia (southern Turkey) around 9,000 years ago, and on to the Alhambra in Andalusia (southern Spain) around 1300 CE, or roughly 700 years ago.[3]

Although the direct distance from the painted caves (for example, Chauvet) to the Alhambra is only 750 miles, the journey of the repeating patterns, by way of Anatolia, Mesopotamia, ancient Egypt, early Islam, and North Africa, is more than 5,000 miles and 30,000 years.[4] The milestones are shown in Figure 1.

As an anchor point for our story, we may take the cognitive revolution, in which *homo sapiens* acquired fictive language and dispersed out of Africa. Current estimates place this between 70,000 and 30,000 BP (years ago).[5] On reaching Europe, they experienced a further bifurcation, in the creative explosion which produced the first repeating patterns (50,000 BP) and the painted caves (36,000 BP).

We will be concerned with repeating patterns exhibiting one-dimensional and two-dimensional symmetries. These symmetries have been known to modern mathematics since the mid 19th century as crystallographic groups. The one-dimensional symmetries, occurring in the friezes (ribbon patterns) of early pottery, wall paintings, and carpets, are classified by seven frieze groups. Similar, the two-dimensional symmetries are classified by the 17 wallpaper groups. Our quest is to discover the emergence of the understanding of these symmetries in the geometric thinking of prehistory and history, leading up to the Alhambra, in which the full understanding of symmetry is concretely manifest in the wall tilings of the 14th century.

We may now consider the long road from the Chauvet cave (34,000 BCE) to the

[1](Abraham, 2014)
[2](abraham, 2011a)
[3](Abraham, 2011b)
[4](Abraham, 2013)
[5](Harari, 2015; p. 21)

Alhambra (1300 CE) in four stages: Cantabria, Anatolia, Damascus, and Andalusia, as shown on the map, Figure 1.

2.0 Cantabria: Late Paleolithic painted caves

Perhaps we have language to thank for the earliest stages of geometric thinking. Alternatively, language and mathematics may be concurrent developments of a common semiotic stimulus. In any case, geometric signs first appear in Late Paleolithic caves, from 50,000 BP at Blombos, in southern Africa, and further in the painted caves of Cantabria (Location 0 in Figure 1), in southwestern Europe.

2.1 Anatolia: Early Neolithic woven textiles

Along the path from Cantabria to Anatolia (Location 1), the geometric signs of a widely agreed visual language evolved, perhaps in the context of Neolithic painted pottery, into the repeating patterns that gave rise eventually to the mathematics of symmetry. An example of the Anatolian repeating pattern is shown in Figure 2. A similar pattern in painted pottery is shown in Figure 3.

These repeating patterns are very compatible with the techniques of early textiles..[6] We propose that the repeating patterns evolved in Anatolia in the tradition of woven textiles around 8,000 BCE. These textiles, which I will call *proto-kilims*, evolved at some uncertain time into the traditional form known today as a *kilim*.

A kilim is a flat-woven carpet thought to originate in Anatolia. The symmetric repeating patterns of these textiles are still woven in Anatolia today. As these patterns eventually found expression in Islamic tiling, including the Alhambra, their origin is of particular interest to the history of mathematics. Regarding the origin of the kilim, there are two main theories: the Goddess theory and the Türkmen theory. Both are highly hypothetical.

My favorite is the Goddess Theory, in the version of James Mellaart, the pioneer excavator of Çatal Hüyük, a Neolithic village in Anatolia. He reported wall paintings suggestive of kilim patterns, and proposed that these weavings were hung from pegs on the walls of shrines of the Mother Goddess cult in the village around 6,000 BCE.

[6]The earliest textile known thus far (5600 BCE) is the linen burial cloth found by James Mellaart at Çatal Hüyük in Anatolia in 1962. See (Barber, 1991; pp. 10-11) regarding the identification of the fiber as flax.

There are two problems for this theory. Firstly, there is no material evidence to support it. Secondly, there is some evidence that contradicts it.[7]

In the other theory, the kilim tradition is supposed to have arrived in Anatolia later, with the Türkmen migrations from Central Asia after 1,000 CE.

I am going to interpolate between these two theories of the origin of the kilim patterns by proposing the existence of an Anatolian proto-kilim tradition in which early Neolithic techniques — such as basketry, linen cloth, or painted pottery (as shown in Figure 3) — supported a design tradition on a continuous trajectory from the Paleolithic painted caves of Cantabria.

- From 5,000 BCE, a fragment of coarse linen, along with a spindle whorl and flax seeds, has been found at Faiyum in Egypt.[8]
- Around 3000 BCE, the slit-weave technique entered Egypt from Syria.[9] The slit-weave technique provides sharp lines between different colored yarns.
- By 2000 BCE, tons of wool were produced by farmers in Sumer, sufficient to manufacture lots of kilims.[10]
- By 1100 BCE, flat-weave tapestries were found at Fostat, Egypt.[11]
- By 1000 BCE, simple geometric designs in slit-weave tapestries are attested in surviving fragments, in support of my proposal.[12]
- From 600 BCE, actual carpet remains have been found in Siberia.

This Siberian carpet shows frieze symmetry, and two-dimensional symmetries as well, as shown in Figure 4.

> Let us take, for example, the celebrated knotted carpet from Pazyryk (fig. 95), the oldest carpet in existence and one of the most stunning archeological finds of the past half century. Of its six zones of decoration, two are nonmimetic, representing figures without external referent, for which only geometry-derived descriptions are immediately reasonable,

[7] See (Davies, 1993; pp. 71–83).
[8] (Barber, 1991; p. 10)
[9] See (Davies, 1993), (Barber, 1991), and (*www.kilim.com/about_kilims/origins.asp*).
[10] (Kramer, 1963)
[11] (Hull, 1993; p. 12)
[12] See (Davies, 1993; p. 70).

like squares, bordered or not, with designs arranged around two axes of symmetry.[13]

2.2 Damascus: Early Islamic glazed tiles

I further propose that the kilim carried the repeating patterns from Anatolia to ancient Egypt in fabric. Glazed tiles appeared in ancient Egypt around 3700 BCE. They are later known from Mesopotamia, but thence disappeared, and were resurrected in early Islam.[14] Early Islamic culture encountered the geometric design tradition in Syria and Iraq, and then evolved it into high art, especially in glazed tiles. Thence began the spectacular artistic tradition of architectural ceramics unique to Islam.

We may begin with a drastically compressed history of Early Islam, that is, the first half of its history, from roughly 600 to 1300 CE.[15]

- A. 610–632, Direction of the prophet Muhammad, 22 years.
- B. 632–657, The early caliphates, 4 caliphates, 29 years.
- C. 661–749, The Umayyad dynasty, 10 caliphates, 88 years.
- D. 750–932, The Abbasid dynasty, 18 caliphates, 290 years.
- E. 912–941, Kingdom of al-Andalus in Spain, Caliph Abd al-Rahman III, 30 years.

After the nomadic invasions into Anatolia of the 11th century, some design elements from Central Asian shamanism may have been incorporated into Islamic decorations

The expansion of Islam across North Africa and into Andalusia (Southern Spain) in 711 CE finally terminated the long road, in the neighborhood of the Alhambra, built around 1350 CE.

The repeated geometric ornaments, meanwhile, made the transition from fabric to glazed tiles. Early steps occurred during the Umayyad dynasty (epoch C above), as the fourth caliph, Walid ibn Abd al-Malik (C8, 685-705), created mosques in Mecca,

[13](Grabar, 1992; p. 125)
[14](Degeorge, 2002; p. 13)
[15]Extracted from (Armstrong 2000; pp. xiii–xxxiv).

Medina, and Jerusalem with assistance from the Byzantine emperor.[16] Caliph Abd al-Malik built the Dome of the Rock in Jerusalem between 685 and 692, one of the oldest works of Islamic architecture. It exhibits design elements from Byzantine churches and palaces. Frieze symmetries are notable in some of the mosaics, one is shown in Figure 5.

Another early step occurred during the Umayyad dynasty (epoch C above). In 661, Muawiyyah ibn Abi Sufyan I, the governor of Syria, became the first caliph of this dynasty, and the fifth caliph overall (C5, 661-680 CE). Damascus, which had been taken by the Muslim Arabs under Caliph Omar (C2) in 635, was an ancient Greco-Roman and Byzantine city, prosperous and sophisticated, primarily Christian and Jewish. Caliph Muawiyyah moved the capital from Medina to Damascus. A large church stood in the center of the city.[17] The fifth Umayyad caliph, al-Walid (C9, 705-717), rebuilt this as a mosque only a century after the prophet Muhammad began preaching.

> The first Islamic monument to appear in Damascus was the Great Mosque, rebuilt by Caliph al-Walid in 706. It remains one of the most splendid Islamic buildings ever erected, yet it dates from a period when Islamic art and architecture were in their infancy. As a result some historians still refer to the mosaics that decorated al-Walid's mosque as Byzantine rather than Islamic art.[18]

As in the Dome of the Rock, frieze symmetries appear in some of the mosaics of the Great Mosque. One such is shown in Figure 6. Two-dimensional symmetries begin to emerge in these early monuments. One example, from the Mosque of Ahmad ibn Tulun of the Abbasid court in Cairo, is shown in Figure 7.

> The Dome of the Rock at Jerusalem, which was built between 688 and 692 A.D., about 60 years after the Prophet's death, and is the most ancient Muslim monument surviving in a state of complete preservation, still belongs to Byzantine art, at the same time as being a work of Muslim art in the choice of its constituent elements. The Great Mosque of Damascus, started in 706 and finished in 715 is, on the other hand, *a priori* a work of Islamic art, at least in its major forms if not in the details. After this date, and more precisely towards the middle of the eighth century A.D., the new art was to expand very rapidly and on a broad front;

[16] See (Papadopuolo, 1979; p. 60).
[17] See (Nicolle, 2003; pp. 52-53).
[18] (Nicolle, 2003; p. 53).

> the great works which then emerge out of the darkness of time, such as the Great Mosque of Cordova, founded in 785, and that of Ibn Tūlūn in Cairo, finished in 879, no longer represent phases in a still tentative evolution but are, in their quality as art, unsurpassable masterpieces. This means that Islamic art had, by the middle of the second century of the Hegira, found its own language.[19]

2.3. Andalusia: Symmetric tiling

Our long road now becomes the story of Islamic art in Umayyad Andalusia (al-Andalus). The final stages are listed in this chronology.

- 680-705, Islamic conquest of North Africa.[20]
- 711-716, Islamic expansion into Spain.
- 716, capital of al-Andalus shifted from Toledo to Cordoba, subordinate to the Umayyad caliphate of Damascus.
- 750, Caliph Abu al-Abbas al-Saffah, (C15, 750-754, the butcher) massacres the entire Umayyad family, excepting Abd al-Rahman, who escapes to Spain. Al-Saffah moved the capital from Damascus to Baghdad.
- 756, Spain secedes from Abbasid dynasty, is ruled by an Umayyad emir, Abd al-Rahman I.
- 912-941, kingdom of al-Andalus, caliph Abd al-Rahman III.
- 969–1027, Dominance of Cordoba in al-Andalus, 58 years.
- 1236, Christian *reconquista* takes Cordoba.[21]
- 1248, Christian *reconquista* takes Sevilla.
- 1492, Christian *reconquista* takes Granada, including the Alhambra.

Abd al-Rahman, the last survivor of the 750 massacre in Damascus, arrived in Spain in 755. He gathered an army and conquered the capital, Cordoba, in 756, and created a kingdom that survived for nearly 300 years.[22] The period from 780 to

[19](Burckhardt, 1976; p. 9)
[20]See (Nicolle, 2003; pp. 92-93) for details and maps.
[21](Armstrong, 2000; p. 105)
[22]For the full story beautiful told, see (Lane-Poole, 1967).

1492 is known as *la convivencia, the coexistence*, a period of creative cooperation of Muslims, Christians, and Jews, who are known collectively as the *Moors*.[23]

Cordoba was once the largest and most sophisticated city of Western Europe. During its heyday, it was a center for the translation and study of the Greek corpus, when (it is reported) there were some 60 libraries in the city, the largest of which – holding perhaps 600,000 books – rivaled the Alexandrian Library, destroyed just a century before the golden age of Islamic Cordoba began.

Many Moorish monuments were built, spectacular mosques, churches, and synagogues, The first of note was the Mezquita, or Great Mosque of Cordoba. Originally a Christian church, it was rebuilt as a mosque by Abd al-Rahman I, beginning in 784. It reopened as a mosque in 795. And of course, after the Reconquest of 1236, it again became a Christian church. Of it, Stanley Lane-Poole wrote in 1886,

> Among the great architectural beauties of Cordova, the principle mosque held, and still holds, the first place. It was begun in 784 by the first Abd-er-Rahmān, who spent 80,000 pieces of gold upon it, which he got from the spoils of the Goths. Hishām, his pious son, complete it, in 793, with the proceeds of the sacking of Narbonne. Each succeeding Sultan added some new beauty to the building, which is one of the finest examples of early Saracenic art in the world.[24]

Arab tribal movements carried the carpet and tile crafts from Anatolia and the Near East across North Africa into Spain from 700 on.[25]

A new innovation along the way was the *zellij* technique,

> Tiles cut and arranged into a mosaic (or marquetry). The term is the equivalent in the Maghreb of the *kshi-ye mo'arraq* (inlaid tilework) of the Persian-speaking East. [26]

This development is thought to have occurred in Egypt or Morocco. Sevilla was a center of zellij craft.

One milestone was the Mezquita of Cordoba, completed in 793, from which followed an amazing evolution of the Islamic decorative arts. Another was the Koutoubia Mosque in Marrakesh, completed in 1157.

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[23] See (Menocal, 2002; ch. 2).
[24] (Lane-Poole, 1967; p. 136)
[25] See the detailed map in (Hull, 1993; p. 13).
[26] (Degeorges, 2002; p. 279)

This evolution culminated in the Alhambra, the most spectacular of the Andalusian monuments to survive. Following the conquest of Sevilla, in 1248, the Almohades palace called al-Muwarak was rebuilt as a cathedral, the Alcazar of Sevilla. This began soon after the creation of the Alhambra. The details of the evolution, comprising the 500-year long road from 800 to almost 1300, remain obscure. We may summarize this last portion of the long road in this chronology.

- 700, development of zellij tiles in Egypt or Morocco
- development of Sevilla as a center of the zellij craft
- 793, the Mezquita of Cordoba
- 1157, the Koutoubia Mosque in Marrakesh
- 1248, the conquest of Sevilla, the Alcazar

We may suppose that the practical understanding of the frieze groups was complete in early Neolithic times, between Anatolia and Siberia, and that a few wallpaper groups became known between Siberia and Damascus. The final elaboration of this knowledge took place within the zellij craft community in Morocco or Sevilla, and was firs manifest in the Alhambra.

3. The Alhambra

The first king of the Moors in Grenada, Alhamar, reigned from 1238. To protect Grenada, he made a deal with the Christian King Ferdinand the Saint. In exchange for helping Ferdinand with the reconquest of Seville in 1248, Alhamar would be allowed to remain in charge of Grenada.. Following the recovery of Sevilla, the Moorish kingdom of al-Andalus (Andalusia) was reduced to Granada alone. And there, Alhamar began the construction of the Alhambra. It was completed in 1348 by his successor, Yusef Abul Hagig, who ruled from 1333 to 1354. This story is movingly told in Washington Irving's *The Alhambra* of 1852.

Washington Irving

Irving (1783–1859) was an American writer who served as the U.S. ambassador to Spain from 1842 to 1846. In 1829, as a middle-aged man on holiday in Spain, he made an adventurous excursion from Seville to Grenada. Arriving there, and following an

initial tour of the Alhambra, he had the good fortune to be offered lodging within the palace itself.[27] There he resided for several weeks, making notes which eventually became his book. He describes the palace itself thus.

> NOTE ON MORISCO ARCHITECTURE
>
> To an unpracticed eye the light relievos and fanciful arabesques which cover the walls of the Alhambra appear to have been sculptured by the hand, with a minute and patient labor, an inexhaustible variety of detail, yet a general uniformity and harmony of design truly astonishing; and this may especially be said of the vaults and cupolas, which are wrought like honey-combs, or frostwork, with stalactites and pendants which confound the beholder with the seeming intricacy of their patterns. The astonishment ceases, however, when it is discovered that this is all stucco-work; plates of plaster of Paris, cast in moulds and skillfully joined so as to form patterns of every size and form. This mode of diapering walls with arabesques and stuccoing the vaults with grotto-work, was invented in Damascus; but highly improved by the Moors in Morocco, to whom Saracenic architecture owes its most graceful and fanciful details.[28]

Commenting on the writing of other authors on Spain's past, British historian Robert Irwin wrote,

> WASHINGTON IRVING,
> THE MAN WHO DISCOVERED THE ALHAMBRA
>
> This slack and self-indulgent way of writing about Spain's Islamic past is traceable ultimately to Washington Irving, though when Irving wrote in similar terms about the indolent and vanished joys of Moorish culture, he was at least original.[29]

A more favorable assessment may be found in Oleg Grabar's excellent *The Alhambra* (1978).

Plane symmetries of patterns

An example of tiling, showing several plane symmetries, is shown in Figure 8. In this context, *symmetry* or *isometry*, has a meaning which is careful defined in math-

[27](Irving, 1852; p. 42)
[28](Irving, 1852; p. 40)
[29](Irwin, 2004,; p. 135)

ematics. They may be regarded as rigid motions of a flat plane onto itself. The basic examples are:

- translation (a rigid motiion in a fixed direction),
- reflection (through a given straight line),
- rotations (around a fixed point in the plane), and
- glide reflection (being a combination of a reflection and a parallel translation).

An isometry of the plane, in general, is a combination of these basic elements. A *group* of isometries is generated by the various combinations of some of the elements. These are known as *plane crystallographic groups*, or wallpaper groups. In 1891, the mathematician Fedorov proved that; with reasonable assumptions, there are exactly 17 essentially different such groups. In addition, there are seven symmetries of one-dimensional friezes. These are called the *frieze groups.*

The Alhambra theorem

Among the decorative tiles of the Alhambra, as it survives today, all 17 of these two-dimensional symmetry groups have been identified. Further, this is the only historical site, previous to Fedorov's theorem, in which all 17 have been found. This amazing fact has been called *the Alhambra theorem.*[30] In 1944, Edith Muller discovered 11 of the symmetry groups at the Alhambra. In 1986, B. Grünbaum, Z. Grünbaum, and G. C. Shephard announced they had found two more. Finally, in 1987, R. Pérez-Gómez and J. Montesínos announced the discovery of the remaining four, proving the Alhambra theorem![31] Splendid photos and explanations of the 17 crucial tilings from the Alhambra may be found online at the University of Granada, thanks to Francisco Martin, Professor of Mathematics. [32]

4. Conclusion

We have traced the emergence of geometric thinking over 50,000 years, from Paleolithic caves to Neolithic carpets, and eventually into glazed tiles, and the spectacular

[30] F. Martin, preprint

[31] (Abas and Salman, 1995; p. 137)

[32] Additional images of the patterns may be found in (Gomez, 1969) and (Casals, 2000).

wall decorations of Islamic art. Perhaps the frieze groups were fully understood in Neolithic Anatolia. And in the interval 900-1300 CE, the concrete understanding of the 17 plane wallpaper groups, in other words, the basic concepts of modern theory of groups, became the common knowledge of Moorish craftsmen in Morocco or Andalusia. We are thus challenged to imagine the process of mathematical creativity, or discovery, underlying these earliest theorems of symmetry groups.

A future project for historians might be the discovery of the first occurrence of each of the symmetry groups, in surviving carpets and tilings. We surmise that most of these finds will be within the corpus of Islamic art.

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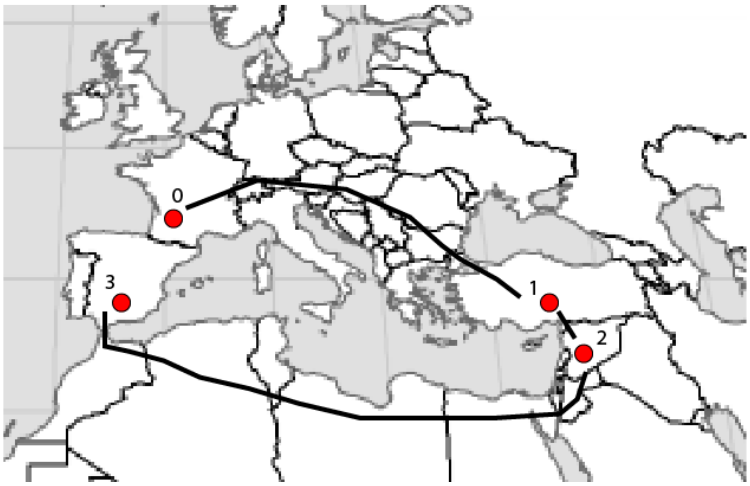

Figure 1: Key locations around the Mediterranean. From Cantabria (0) to Anatolia (1), Damascus (2), and Granada (3). Base map courtesy of the BYU Geography Department.

Figure 2: Grooved and painted kilim pattern from the west wall of shrine VII.21, Çatal Hüyük, 5,600 BCE. From (Mellaart, 1965; Plate VIII, p. 72).

Figure 3: Earliest painted pottery, from Tepe Guran Level 5, 6,700 BCE. From (Mellaart, 1967; Figure 27, p. 51).

Figure 4: Siberian carpet, 600 BCE. From Wikipedia. See also (Grabar, 1992; Fig. 95 on p. 125.)

Figure 5: Jerusalem, Dome of the Rock. One of the earliest Islamic mosaics, 691692. From (Papadopoulo, 1979; Plate 14).

Figure 6: Damascus, the Great Mosque. Another of the earliest Islamic mosaics, 710–715. From (Papadopoulo, 1979; Plate 16).

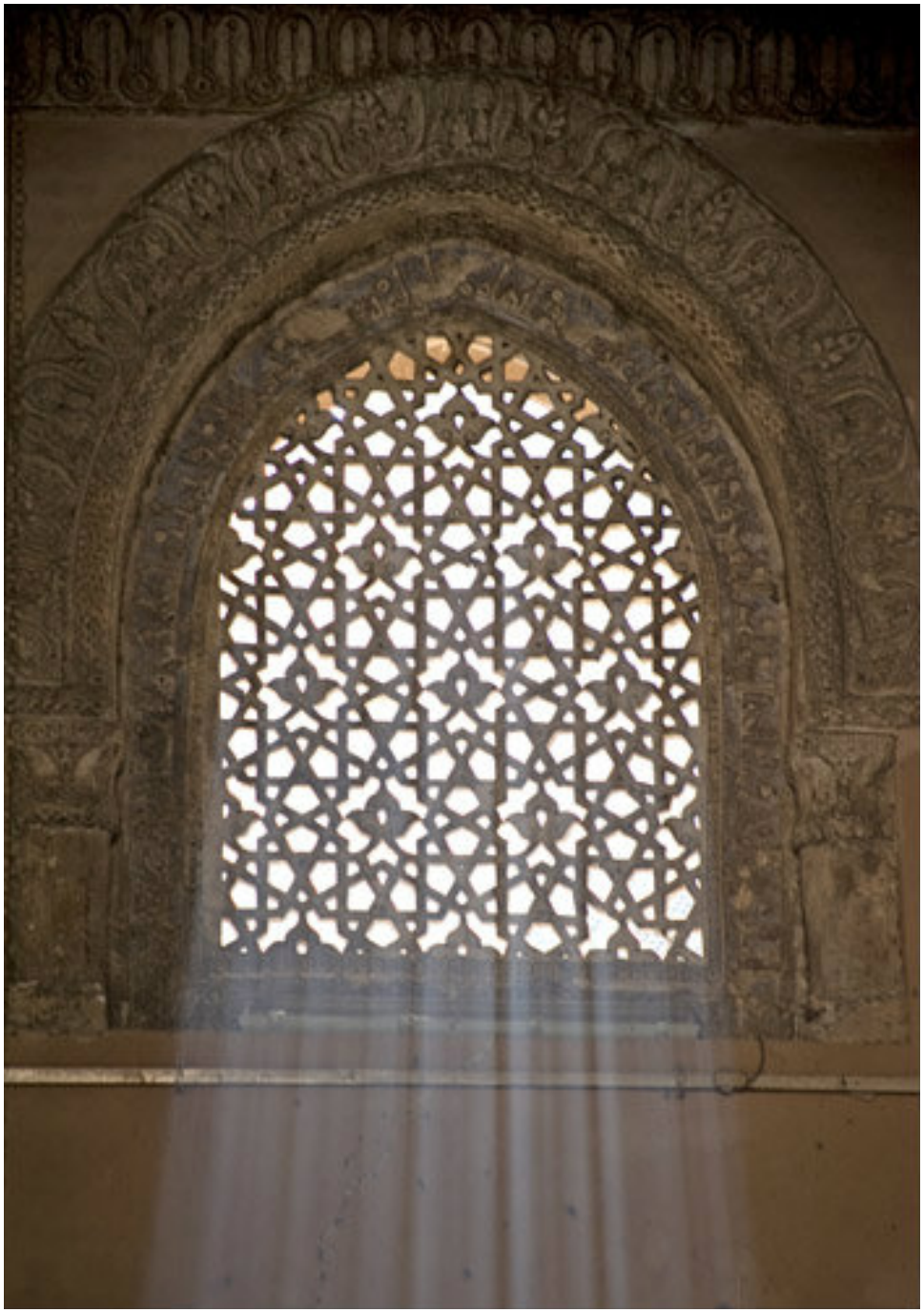

Figure 7: Cairo, the Mosque of ibn Tulun. An early example of two-dimensional symmetry, 879. From Wikipedia.

Figure 8: Alhambra, details of wall tiling from the Comares Room, ca 1300. From (Casals, 2000; p. 78).