Kepler's Music — a Computer Simulation

Ralph Herman Abraham

May 27, 2019

Abstract

This article is a supplement to my presentation at the conference, *The Harmony of the World* at the Kepler Archives, Saint Petersburg, Russia, 21-23 June, 2019. It is a report of joint work with Pablo Viotti. We presented stereo-audio and video recordings of our computer simulation of Kepler's planetary music, as described in Book V of his *Harmonice mundi.* The video synchronizes the combined sounds of the planets with a visual display of the planetary disks, as seen by angels watching from a window in the Sun, similar to the view from the moon described in Kepler's *Somnium: the dream, or posthumous work on lunar astronomy.*

Contents

1. Introduction

In Chapter 9 of Book 5 of his text, *Five Books on the Harmony of the World,* Kepler described his method to associate a musical pitch to each planet at two points on its elliptical orbit, the perihelion and the aphelion. Together with Pablo Viotti, a computer simulation of the chorus of the six planets was created in 2002, and performed at the San Francisco Art Institute in 2002 in surround sound. Astronomical data from 1983 was used to calculate the pitches of the planets.[1] The warbling sounds of the six planets – with the pitch of each scaled to its solar angular velocity according to the second law (equal areas in equal times) – were played separately and together. At the Saint Petersburg conference, we played recordings of our computer simulation, and explained the procedure used to create the recordings.

2. The pitches, Kepler's method

Kepler's method is to scale the pitch of a planet to the path velocity of the planet's motion at two points on its orbit: perihelion (closest approach to the Sun) and aphelion (furthest point). See Figure 1. The path velocity is the angular velocity (rate of change of the true anomaly) multiplied by the radius, that is, the distance from the Sin to the planet. The pitch then would vary between the extremes as the planet rounds its elliptical orbit. Kepler did not speculate on this variation.

In outline, his method proceeds in these steps:

- A. Determine extreme values of path velocity for each planet. See table in Figure 2.
- B. Transpose the group of all six velocities upward buy a constant number of octaves, See Figures 3.
- C. Transpose each interval separately by an appropriate number of octaves to place all six within a single octave. See Figures 4 and 8.

A. Extreme values

The angle ν is the true anomaly of the planetary position, as shown in Figure 1. Kepler computed the path velocity, or *diurnal motion, M*, of each planet by the formula,

$$M = K * r * D\nu \tag{1}$$

[1]See (Moore, 1990) for the 1983 data.

where r denotes the mean distance from the sun to the planet (the semi-major axis of its elliptical orbit), $D\nu$ is the diurnal angular motion of the planet, and K is some constant, depending on the units used for the data. Thus M is the approximate geometric distance traversed by the planet in the interval of a single Earth day. These he called the *extreme motions* of the planet. Kepler obtained his data from the records of Tycho Brahe.

B. Transpose the group

He then scaled these 12 data to obtain the maximum (at the perihelion) and minimum (at aphelion) velocities of each of the six planets. His results are listed in the table in Figure 2, shown graphically along a piano keyboard in Figure 3, recalculated in Figure 5, and shown in staff notation in Figure 6.

C. Transpose each interval

Kepler's scaling by octaves is shown in the table of Figure 4, and graphically in Figure 8. For example, Mercury's perihelion motion is reduced by seven octaves, to arrive within the octave of Saturn's extreme motions.

Harmony of the extreme values

Kepler was impressed by the pitch ratios (perihelion versus aphelion) which approximate harmonic ratios, or musical intervals. Let P_{high} be the pitch of the planet at perihelion, and P_{low} that at aphelion. The the pitch ratio, P_{high}/P_{low} is given by $(1 + e/1 - e)^2$, where e is the eccentricity of the planet. Kepler's ratios, modern values, and errors for these are shown in the table of Figure 7.

For example, in the case of Mercury, Kepler's extreme values, $P_{high}/P_{low} = 133.02428/87.0984 = 1.5274$, very close to a perfect fifth. Using modern values for the extreme velocities yields 1.518, an error of only about half a percent.

Accuracy of Kepler's diurnal motions.

As shown in Figure 2, the diurnal motions (column 4) are are obtained by multiplying the diurnal angles (column 2, changed to arc seconds) by the mean distance (column 3.) The diurnal angles were obtained by Tycho Brahe by amazingly accurate visual observation using his telescopes without lenses. But whence came the mean distances? Presumably, he used the method of Eratosthenes, who obtained the value of the mean distance of the Earth in 240 BCE.[2]

The pitches of the planets

According to Kepler, the pitches are to be scaled to the angular velocities, but the scaling is arbitrary. Kepler chooses two tunings. In the first, the lowest pitch of Saturn (at aphelion) is chosen to be 20 cycles per second. In the second, tuning, the highest pitch of Saturn (at perihelion) is chosen to be 20 cps. And, by the way, in the modern tuning of A4 = 440 cps, the frequency of 20 cps is D0#, while G0 is 25 cps. Were A4 taken to be 350 cps, we would have Kepler's association of 20 cps with G0.

Finally, Kepler adjusted each pitch range up or down by octaves, as described above, to obtain pitch ranges within a single octave, G2 (lowest line of the bass clef) and G3 (just below middle C, or C4).

The accuracy of Kepler's pitches

Kepler's three laws seem miraculous, as the data he inherited from Tycho Brahe was based on naked-eye observations. Nevertheless, his laws have withstood the test of time.

The periods of the planets were remarkably accurate. And from the the third law, the the mean distance (that is, the semi-major axis of the elliptical orbit) may be calculated from the periods, at least, up to a multiplicative constant. As the pitches are to be shifted such that the pitch of the aphelion of Saturn is 20 cps, only the relative values of the mean distance are needed . It is evident from the table in Figure 2, taken directly from Kepler (1619), that the Earth, at 1,000, is the reference point for all the other distances. Today, the Earth's mean distance defines the Astronomical Unit, or AU, of about 149.60 Mkm (million kilometers.)

In Tycho's time, this distance was only approximately known by the method of Eratosthenes.[3] However, Kepler, thanks to his third law (which appeared in his work, *Five Books on World Harmony, 1619*) was able to improve on Tycho's data.

Comparing Kepler's pitches with those we have computed from modern data, we find an amazing concordance, as shown in Figure 9. There is no significant difference excepting the pitches for Mercury. Both ends of Kepler's range for Mercury are about

[2]See Wikipedia, https://www.aps.org/publications/apsnews/200606/history.cfm.

[3]See the Wikipedia entry for Astronomical Unit.

5 per cent high, that is, about a semi-tone sharp. The pitches for Jupiter are off by about 3 percent.

In any case, our simulation of Kepler's music of the planets is based on the modern data, conforming to Kepler's original intent: pitch is proportional to velocity.

A note on other tunings

Changing the root pitch of Saturn shifts all of the pitches by a constant number of semitones, without changing their frequency ratios, so musical harmonies are preserved. The frequency choice of A4 as 440 cps became a world standard in 1939.[4] But in earlier times, other values were used. For the year 1700, values from 374 to 567 have been mentioned.[5]

The sidereal period of rotation of the Earth around its orbit corresponds (after raising the tone by many octaves) to a pitch of 194.711 cps. This pitch will be an octave of C if A is tuned to 437 cps.[6]

Similarly, the tropical period (Earth year) corresponds to a pitch of 136.10 cps. This pitch may be useful for human health, and is much used in sound healing.[7]

The period of a mean solar Earth day corresponds to a pitch of 388.36 cps, which will be an octave of G if A is tuned to 435 cps.[8]

These tunings are based on Earth periods, rather than (as with Kepler) on angular velocities.

3. The pitches, our method

We follow Kepler exactly, except for substituting modern values of the extreme velocities for those of Tycho Brahe. The calculations are shown in Figure 10. All six pitch ranges are within normal hearing. Changing the scaling would shift all the pitch ranges up or down by a constant distance.

We have used the modern equations of motions to vary the pitch of each planet around its orbit, connecting the extreme values. The sinusoid of pitch variation for Saturn is shown in Figure 11. The sinusoid for Mercury in the same time period (one Saturn year, is shown in Figure 12.

[4](Cousto, 2000; p. 9)
[5](Ellis, 1877/1968)
[6](Cousto, 2000; p. 104)
[7](de Muynck, 2015; p. 16)
[8](Cousto, 2000; p. 103)

The difference between the pitch variation from the equations of motion and the sinusoidal approximation is shown in Figure 13. These curves have been computed for Mercury. As the most eccentric planet, the discrepancy between the two curves is the greatest of all the planets.

4. Conclusion

Finally, it is time to stand back from all these details, and bring to the foreground the actual harmonies intended by Kepler in his title for Book Five,

> The fifth is astronomical and mathematical, on the most perfect harmonies of the celestial motions, and the origin of the eccentricities in the harmonic proportions.[9]

While he appreciated the ratios of small integers as musical intervals, he fully understood that most of them (for example, the extreme pitches of a single planet) could not be observed by people, nor even angels watching the solar system from the Sun.

However, when the pitch of one planet and the pitch of another momentarily struck an harmonic musical interval, a resonance might ring out like a bell. For one such case, consider Mars and Mercury. During one Mars year (of 687 Earth days) Mercury moves by almost 12 Mercury years (of 58 days, 15 hours, and 30 seconds of Earth time.)

And the pitch range of Mars (49-59 cps), adjusted upward by a single octave, fits within the pitch range of Mercury (87-133 cps.) Thus within a Mars year, there would occur 24 octave resonances of these planets. It is conceivable that such events could have astrological significance.

In addition, the passing of any planet through its perihelion could be significant. For example, the Earth passed closest to the Sun on January 2, 2019, as it does every year about two weeks after the Winter solstice.

These harmonies of the planets might have more than astrological interest, as perhaps in furthering harmonies on Earth such as church and state, as Kepler hoped.[10]

Our simulation of the chorus of the planets carries these ideas much better than all these numbers and diagrams.

[9] (Aiton, 1997; p. 1)
[10] See (Rothman, 2017).

REFERENCES

Books

- Abraham, Ralph (1967). *Foundations of Mechanics.* New York, NY: Benjamin.
- Cousto, Hans ((2000). *The Cosmic Octave: Origin of Harmony.* Mendocino, CA: LifeRhythm.
- de Muynck, Marjorie (2015). *Sound Healing: Vibrational Healing with Ohm Tuning Forks. Secoond Edition.* Santa Fe, NM: Sound Universe.
- Ellis, Alexander (1877). *The History of Musical Pitch.* Boston, MA: Da Capo Press, Reprint, 1968.
- Moore, Patrick, Garry Hunt, Iain Nicolson, and Peter Cattermole (1990). *The Atlas of the Solar System, Revised and Updated.* New York: Crescent Books.
- Rothman, Aviva (2017). *The Pursuit of Harmony: Kepler on Cosmos, Confession, and Community.* Chicago: University of Chicago Press.
- Stephenson, Bruce (1994). *The Music of the Heavens: Kepler's Harmonic Astronomy.* Princeton, NJ: Princeton University Press.

Websites,

- Abraham, Ralph (2002). *Kepler's music of the planets, pitch ranges.* http://www.visual-kepler.org
- Viotti, Pablo (2002). *Kepler's music of the planets, Computer simulation.* http://www.viotti.com
- Planet Facts (5/05/2019). http://planetfacts.org/orbital-speed-of-planets-in-order/

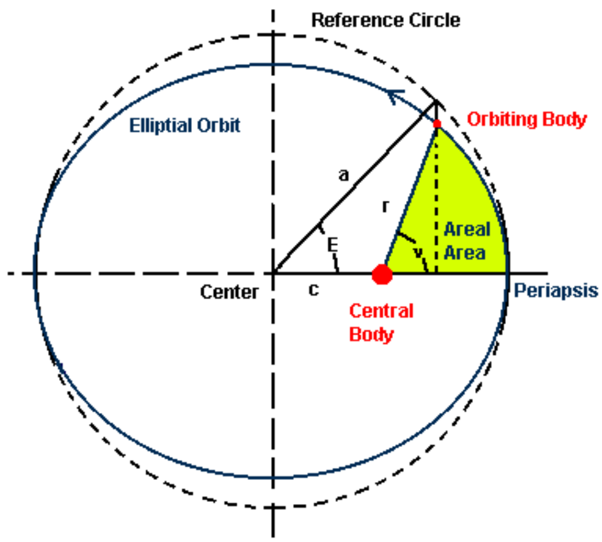

Figure 1: The *true anomaly* is the angle, ν, at the sun at the right-hand focus, measured counterclockwise from the periapsis (or perihelion, where the orbit is closest to the sun) to the planet.

BOOK FIVE OF THE *HARMONICE MUNDI*

TABLE 9.2.
Extreme Linear Velocities

1 Motion	*2* Diurnal Minutes Seconds	*3* Mean Distance	*4* Diurnal motion
Saturn A	1′53″	9,510	1,075
Saturn P	2′ 7″		1,208
Jupiter A	4′44″	5,200	1,477
Jupiter P	5′15″		1,638
Mars A	28′44″	1,524	2,627
Mars P	34′34″		3,161
Earth A	58′ 6″	1,000	3,486
Earth P	60′13″		3,613
Venus A	95′29″	724	4,148
Venus P	96′50″		4,207
Mercury A	201′ 0″	388	4,680
Mercury P	307′ 3″		7,148

Figure 2: The pitches of the planets from Kepler. The three columns are values for ν (Diurnal Minutes Seconds), r (Mean Distance), and ν' (Diurnal Motion.) For example, 1' 53" is 113 arc seconds. Multiplying by the mean distance, 9510, and dividing by 1000, yields 1075. See Stephanson (1994); p, 149.

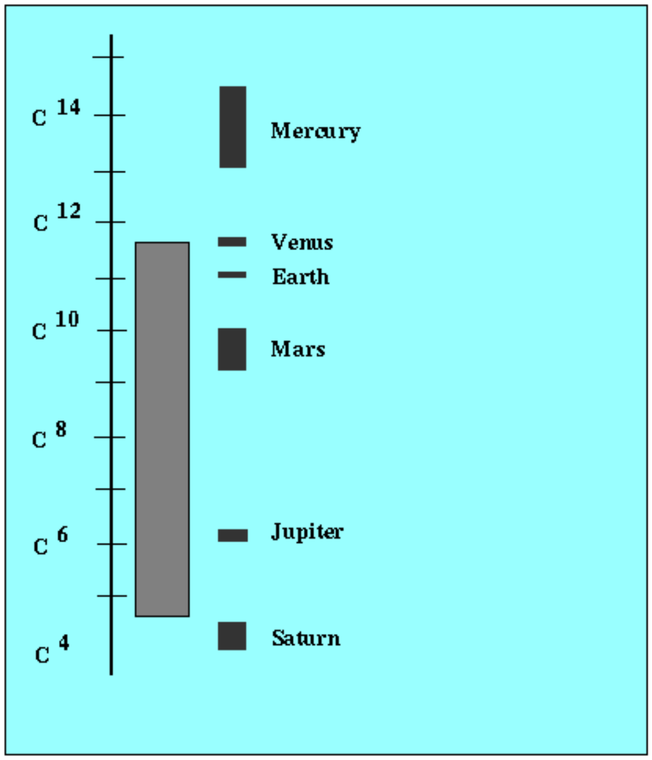

Figure 3: The pitches of the planets scaled such that the mean pitch of Saturn is 20 cps. Note that c^4 denotes 16 cps, c^5 is 32 cps, and so on. Thus c^8 is 256 cps, $C4$ or middle C on the piano keyboard. The range of human hearing is about c^4 to c^{14}.

TABLE 9.4.
Reduced Extreme Motions

1 *Motion*	*2* *Octaves reduced*	*3* *Divisor*	*4* *Reduced motion*
Mercury P	7	128	3′0″
Mercury A	6	64	2′34″ −
Venus P	5	32	3′3″ +
Venus A	5	32	2′58″ −
Earth P	5	32	1′55″ −
Earth A	5	32	1′47″ −
Mars P	4	16	2′23″ −
Mars A	3	8	3′17″ −
Jupiter P	1	2	2′45″
Jupiter A	1	2	2′15″
Saturn P	0	1	2′15″
Saturn A	0	1	1′46″

Figure 4: Kepler's table of octave adjustments. From Stephanson (1994); p. 155.

Motion	Diurnal Mins Secs	Diurnal Secs	Mean Distance	Diurnal motion Kepler, DM	Pitch, Kepler #1 DM * 20 / 1075 Hz = DM * 0.01861	Rounded, Kepler Pitch #1	Kepler #1 Notes A440 scale	Kepler #1 Notes A350 scale	Kepler's #1 Notes adjusted
Saturn A	1'53	113		1075	20.00575	20	D#0	G0	G2
mean			9,510	1142	21.25262	21	E0		
Saturn P	2' 7"	127		1208	22.48088	22	F0	A0	B2
Jupiter A	4'44"	288		1477	27.48697	27	A0	C1#	B2
mean			5,200	1343	24.99323	25	G0		
Jupiter P	5'15"	315		1558	28.99438	29	A#0	D1	D3
Mars A	28'44"	1724		2627	48.88847	49	G1	B1	F3#
mean			1,524	2894	53.85734	54	A1		
Mars P	34'34"	2074		3161	58.82621	59	A#1	D2	C3
Earth A	58' 6"	3486		3486	64.87446	65	C2	E2	G3
mean			1,000	3550	66.0655	66	C2		
Earth P	60'13"	3613		3613	67.23793	67	C2	E2	G3
Venus A	95'29"	5729		4148	77.19428	77	D#2	G2	E3
mean			72	4178	77.75258	78	D#2		
Venus P	96'50"	5810		4207	78.29227	78	D#2	G2	E3
Mercury A	201' 0"	12060		4680	87.0948	87	F2	A2	C3#
mean			388	5914	110.05954	110	A2		
Mercury P	307' 3"	18423		7148	133.02428	133	C3	E3	E3

Figure 5: Spread-sheet of Kepler's calculations.

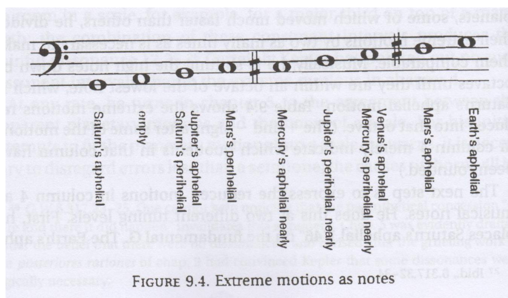

FIGURE 9.4. Extreme motions as notes

Figure 6: The extreme pitches of the planets in Kepler's staff notation. From Stephanson, p. 156.

Motion	Kepler pitch ratios from Kepler pitches	Modern pitch ratios	Kepler Error +/-
Saturn P	1.124	1.118	0.53%
Jupiter P	1.055	1.102	-4.26%
Mars P	1.203	1.206	-0.25%
Earth P	1.036	1.020	1.57%
Venus P	1.014	1.014	0
Mercury P	1.527	1.518	0.60%

Figure 7: The pitch ratios, from Kepler and modern data, compared.

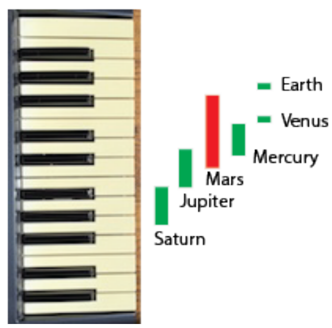

Figure 8: Graphic representation of the extreme pitches of the planets, transposed to fit into a single octave. .

Motion	Pitch, Kepler #1 DM * 20 / 1075 Hz = DM * 0.01861	Modern pitch	Kepler Error +/-
Saturn A	20.00575	20.00	0.03%
Saturn P	22.48088	22.36	0.54%
Jupiter A	27.48697	27.18	1.13%
Jupiter P	28.99438	29.95	-3.19%
Mars A	48.88847	47.71	2.47%
Mars P	58.82621	57.53	2.25%
Earth A	64.87446	64.00	1.37%
Earth P	67.23793	65.28	3.00%
Venus A	77.19428	76.02	1.54%
Venus P	78.29227	77.06	1.60%
Mercury A	87.0948	83.11	4.79%
Mercury P	133.02428	126.14	5.46%

Figure 9: Comparison of Kepler's pitches with those based on modern data.

Planet	Mean orbital velocity, km/s	Eccentricity, 1+e, e, and 1-e	Velocities, V	Pitch, 2.156 * V	Pitch round
Saturn, min		0.9443875	9.151	20.00	20
Saturn, mean	9.69	0.0556125	9.69		
Saturn, max		1.0556125	10.229	22.36	22
Jupiter, min		0.9515319	12.437	27.18	27
Jupiter, mean	13.07	0.0484681	13.07		
Jupiter, max		1.0484681	13.703	29.95	30
Mars, min		0.9066135	21.829	47.71	48
Mars, mean	24.077	0.0933865	24.077		
Mars, max		1.0933865	26.325	57.53	58
Earth, min		0.9832825	29.282	64.00	64
Earth, mean	29.78	0.0167175	29.78		
Earth, max		1.0167175	30.278	65.28	65
Venus, min		0.9932174	34.782	76.02	76
Venus, mean	35.02	0.0067826	35.02		
Venus, max		1.0067826	35.258	77.06	77
Mercury, min		0.7943694	38.026	83.11	83
Mercury, mean	47.87	0.2056306	47.87		
Mercury, max		1.2056306	57.714	126.14	126

Figure 10: Calculations for pitches from modern data.

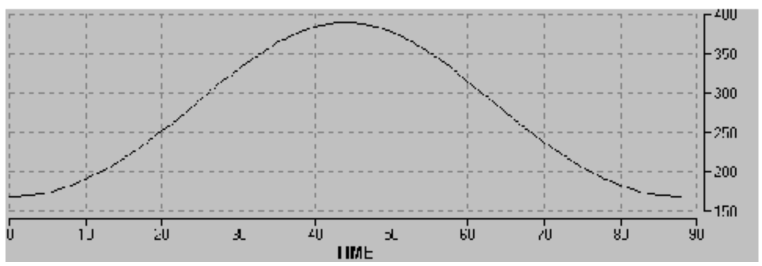

Figure 11: Sinusoid for Saturn, one Saturn year.

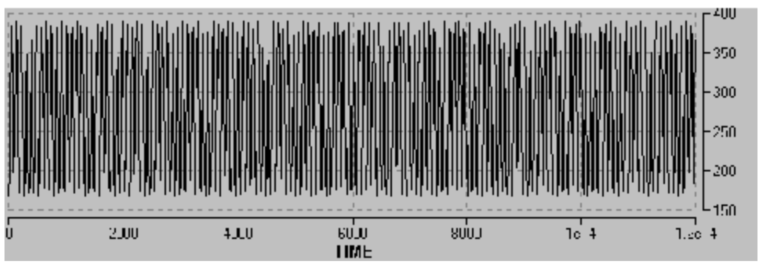

Figure 12: Sinusoid for Mercury, one Saturn year.

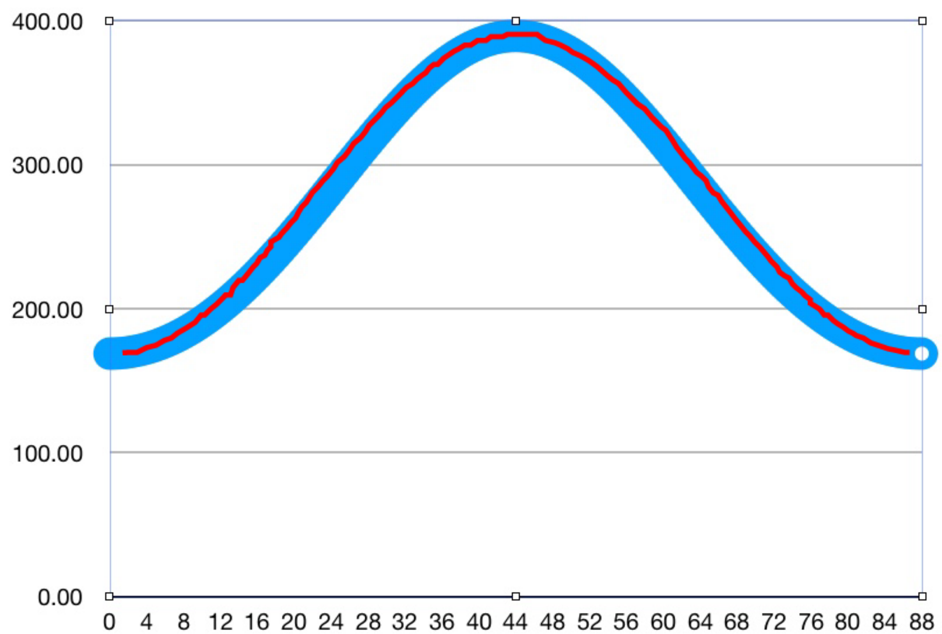

Figure 13: True pitch variation (blue) compared to the sinusoidal approximation (red) for Mercury.