Some elements for a history of the dynamical systems theory

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I. INTRODUCTION

II. THE SMALE PROGRAM BY RALPH ABRAHAM

We begin with an outline of the development of dynamical systems theory along the lines of differential topology, that is, the Smale program, 1958-1968. This was contemporary with the parallel development of the chaos program of Ueda,[1] Lorenz,[2] Gumowski and Mira,[3] and Rössler,[4–6] 1961-1975.

Steve Smale finished his Ph.D. thesis in differential topology in 1956, working with Raoul Bott at the University of Michigan. At that time I was there in Ann Arbor, finishing my undergraduate program in Engineering Mathematics. I was introduced to differential topology in a course by Bott, on the way to my Ph.D. on general relativity in 1960, working with Nathaniel Coburn.

Solomon Lefschetz, the legendary algebraic topologist, was very altruistic. Noting that mathematics in Latin America was not well developed, he began devoting half of every year in Mexico City to build up a graduate program in the math department of the National Autonomous University of Mexico (UNAM). He had become interested in the Russian literature on dynamical systems theory, written an influential book in 1957,[7] and begun a series of summer conferences in Mexico City on topology and dynamical systems.

After his Ph.D. in 1956, Smale attended Lefschetz' summer conference in Mexico City.[8] (p. 147) There he met René Thom, Moe Hirsch, and Elon Lima, who played important roles in our story. Around 1958, Lima (Brazilian) finished his Ph.D. thesis on topology with Edwin Spanier in Chicago, and introduced Smale to Mauricio Peixoto. Peixoto was a Brazilian student of Lefschetz in Princeton, 1958-59. His theorem on the structural stability of flows in two dimensions,[9] based on work of G. F. De Baggis,[10] was an early breakthrough in dynamical systems theory.

I met Lefschetz at the UNAM in 1959, where I was writing my Ph.D. thesis and then in 1964, when I began at Princeton University, where he was chair of the Math department. During my time there, I saw the dynamical systems group attending a Summer meeting. Steve Smale was among them, but we did not meet.

In 1960, for my first academic job, I arrived at Berkeley University which, in the Fall of 1960, suddenly had a brand new staff of important new math professors and visitors. Steve Smale arrived from Princeton, along with the Ed Spanier (algebraic topology), Shing-Shen Chern (differential geometry), and Morris Hirsch (differential topology) from Chicago, René Thom (differential topology, Fields Medal, 1958) from Paris, Chris Zeeman (topology, expositor of catastrophe theory) from Warwick, Mauricio Peixoto from Rio, Bob Williams (knot theory), Dick Palais (nonlinear functional analysis) and others comprising a research group on dynamical systems theory based on differential topology. Moe Hirsch (a student of Spanier) and I were among the newbies in this group. The Smale program was focused on the stable manifolds, structural stability, and conjugacy of diffeomorphisms. At his time we devoted much time reading and discussing the works of Poincaré and Birkhoff, especially concerning the stable curves of surface transformations and their transversal intersections.

Smale had proved the existence of stable and unstable manifolds, his first major result in this field.[11] He developed the horseshoe map, his second major result, also in this year. After the publication of the stable manifold theorem, Thom proved that transversal intersection of stable manifolds is a generic property of diffeomorphisms.[12] In the Summer of 1961, Smale presented his stable manifold theorem in Urbino, and later that summer, in Bonn, I presented my own proof of Thom's transversality theorem.[13]

The Smale program was boosted into orbit by an influential survey article by Smale.[14] The first 10 pages set out the foundations of the program: conjugacy of diffeomorphisms, fixed and periodic points, stable and generic properties, the nonwandering set, hyperbolic fixed points, stable manifolds, his stable manifold theorem, the Morse inequalities, and a structural labelled diagram later called the Smale diagram. Already on page 10 we find drawings of homoclinic intersections of stable and unstable manifolds for surface transformations, discovered by Poincaré and analyzed in detail by Birkhoff and Smith.[15] Smale's ingenious simplification of the homoclinic tangle in the two-dimensional case, the horseshoe map, appears in page 25 (Fig. 1). The details of these ideas, and many other original concepts, may be found in the original publication, or its reprint in Smale's book.[8] In this epochal paper, Smale carefully credits his predecessors (in chronological order) — Poincaré[16], Birkhoff,[17], Morse,[18] Andronov and Pontrjagin,[19] Thom,[20] Elsgolts,[21] Reeb,[22] and Peixoto.[23]

After the first year of the Smale program in Berkeley (1961), Smale moved to Columbia University, where also there was a cadre of leading mathematicians, including Paul Smith, Sammy Eilenberg, and Serge Lang. In the September of 1961, he visited the Soviet Union and met the dynamics community there, especially Anosov. The next year, I also moved to Columbia. At this time, Smale began applying the

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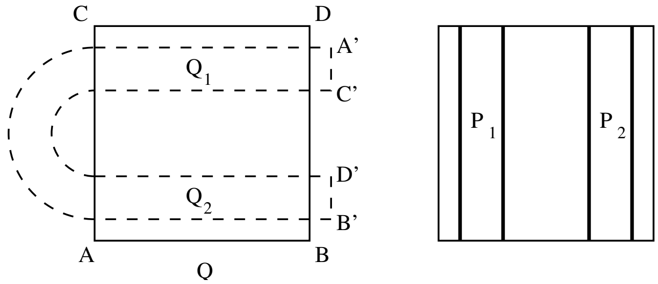


FIG. 1. Smale's horseshoe map. A global diffeomorphism maps the square Q into the region bounded by dotted lines with $G(\mathrm{A}) = \mathrm{A}'$, etc. Each component P_1 and P_2 of $g^{-1}(g(\mathrm{Q}) \cap \mathrm{Q})$ is such that g is a linear map with $g(\mathrm{P}_i) = \mathrm{Q}_i$ $(i = 1, 2)$.

methods of differential topology to the calculus of variations – the beginning of a new field which became known as global analysis. This comprised rewriting of all of classical analysis using differential topology and geometry. The calculus of variations, the theory of minimal surfaces, and mathematical physics (classical mechanics, Maxwell theory, and general relativity, in particular) were particularly extended using this new approach. At this time Mike Shub and Charlie Pugh joined our group.

In the Fall, Smale organized a seminar on global analysis. But early on, in October, the Cuban Missile Crisis intervened. For a few weeks, I took over the seminar, lecturing primarily on my proof of the Thom transversality theorem for global manifolds in the context of infinite-dimensional manifolds and mappings. I published the notes as a preprint.[24]

A year after Smale's return to Berkeley, I moved on to Princeton University, where Lefschetz still had huge influence, and other greats such as Jack Milnor were rising. Besides the University, there was the Institute of Advanced Studies, with its own math department, including permanent members like Armand Borel, and visitors including Dick Palais, Bernard Morin, and Al Kelley. I was able to teach graduate courses and seminars in dynamical systems and global analysis. In one of these courses, I rewrote much of celestial mechanics with the new language and technology of global analysis.[25] In another, I treated the transversality of stable manifolds in the global context of (infinite-dimensional) manifolds of mappings.[26] Jerry Marsden and Joel Robbin were important collaborators in these courses and the related books. In 1966, Smale was in Moscow to accept his Fields Medal at the International Congress. In my fourth year, 1967-68, I decided to accept a position at UC Santa Cruz.

Around 1966, I began to receive letters from René Thom in which he reported regular progress in his creation of catastrophe theory. In this simple context of gradient (non-chaotic) dynamical systems, he made crucial use of the language of attractors, basins, and bifurcations, which became fundamental in the further evolution of dynamical systems theory, and later, chaos theory. He popularized a style of application of these notions, introduced earlier by Poincaré and his Russian followers. It has been observed by the chaos pioneer, Christian Mira, that the ideas had occurred before Thom in the Russian literature, from 1963.[27] (p. 175)

Thom's difficult ideas appeared eventually in book form. The French original of 1972 was updated by Thom and translated to English by David Fowler in 1975.[28] The impact on the mathematical community was further facilitated by a series of exemplary articles by Chris Zeeman, beginning in 1976, and collected in book form in 1977.[29] The word *attractor* is indexed in Thom's book on 19 pages, and 37 pages in the index of Zeeman's book. The word *bifurcation* is indexed by Thom on 18 pages, and by Zeeman on 67 pages. Poincaré is mentioned on four pages by Thom. These words and ideas were burned into the working vocabulary of the entire scientific community from this time on, dominating all the applications of chaos theory which followed.

In 1968, the *American Mathematical Society* had organized an four-week conference on global analysis, July 1-26. The proceedings were edited by Chern and Smale, in 1970. Fifty-three papers were included.[30] Dynamical systems theory was only one of several topics covered. Members of our community were nearly all there. This was the moment, I believe, at which our group finally became aware of the experimental work and simulations on chaotic attractors. Yoshisuke Ueda, discovered the first clearly chaotic attractor in analog simulation, the *Japanese attractor*,[31] for which he accurately drew the homoclinic tangle of inset and outset curves for the forced Duffing equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu\left(1 - \gamma x^2\right) y - x^3 + B \cos \nu t \end{cases} \tag{1}$$

obtained at Kyoto University, on November, 1961 (Fig. 2),[1] Edward Lorenz, discovered his chaotic attractor, at MIT, on 1963,[2] and Christian Mira, discovered his chaotic attractor in the iterated quadratic map of the plane,

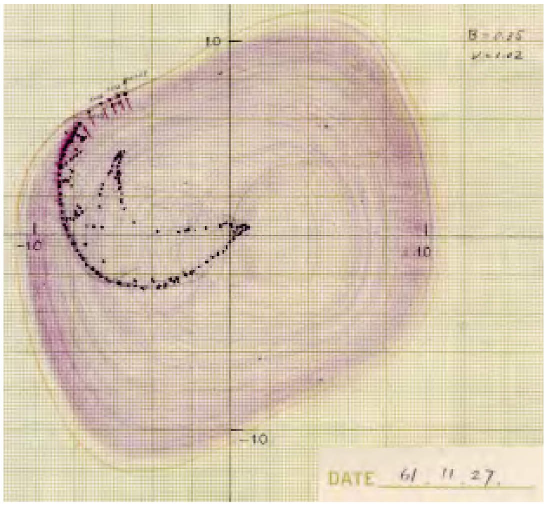
$$\begin{cases} x_{n+1} = (1-\lambda)x_n + y_n \\ y_{n+1} = y_n + f(x_n) \end{cases} \tag{2}$$

where

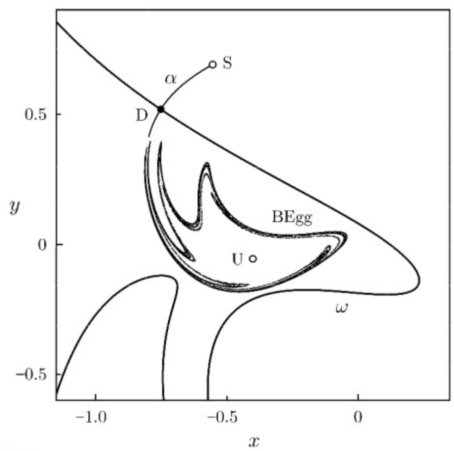
$$f(x_n) = \left| \begin{array}{lll} -2\lambda\, x_n - 0.9\lambda & & x_n < -0.5 \\ \qquad -\dfrac{\lambda\, x_n}{5} & \text{if} & |x_n| < -0.5 \\ -2\lambda\, x_n + 0.9\lambda & & x_n > 0.5 \end{array} \right. \tag{3}$$

creating the theory of critical curves for iterated maps, Toulouse, 1968.[32,33] These discoveries sounded the death knell for our approach based on differential topology. There were no presentations of experimental work in this conference. The summer ended in shock for our whole group.

Our group reconvened at the Math Institute newly created at Warwick University by Chris Zeeman, for the academic year, and in a Summer conference, 1969. Experimental work (not chaotic) was presented by a single attendee, Minoru Urabe (1912-1975) of the Math Department, Kyoto University.[34] Ironically, he was a chaos disbeliever. In 1961, Ueda, then still a graduate student, had presented his chaotic data in Urabe's seminar. Urabe discouraged Ueda, whose results were not published until 1970. Chihiro Hayashi, Ueda's thesis advisor,



(a) Original plot obtained on November 27, 1961



(b) Poincaré section of the Duffing attractor

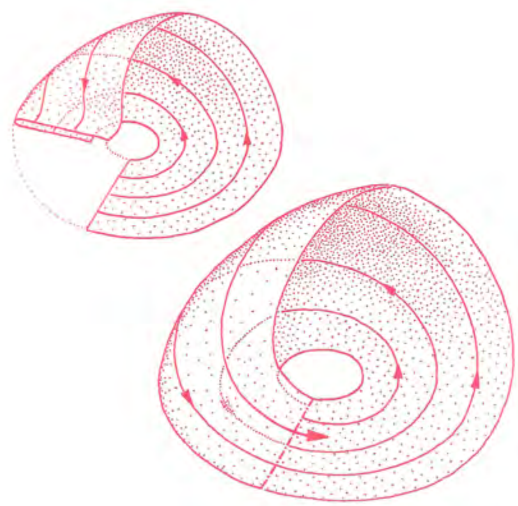
FIG. 2. With amazing prescience, shortly after our study in Berkeley of homoclinic intersections for surface diffeomorphisms, Yoshisuke Ueda, then a graduate student at Kyoto University, discovered his chaotic attractor in a Poincaré section of the forced Duffing equation. He correctly obtained the homoclinic intersection. BEgg = Broken eggs.

also rejected the work and Ueda was forced to submit another thesis on electrical machines.[35] (pp. 23-80)

In 1971-73, new people had arrived in my department, including John Guckenheimer, a recent Ph.D. with Steve Smale, and veterans of our group, Dick Palais and Mike Shub. Dick Palais collaborated in creating a computational program using a digital mainframe and a primitive graphics terminal. We were able to recreate the attractors of Ueda, Lorenz, and Mira, with assistance of a talented group go undergraduates. After a couple years we also studied the Rössler attractor and other new developments. Extensive graphics of this work are on display in my website.[36] John Guckenheimer was also active in computational dynamics at UCSC in the 1970s.

Our experimental work with chaotic dynamical systems, begun in the summer of 1974, was soon matched by related but independent activity in the Physics Department of UCSC. And around 1974 I was invited to Stuttgart by Werner Guttinger to speak on dynamical systems theory. This year saw the first connection of the word chaos with dynamical systems theory, in a paper of Robert May,[42] beginning the chaos revolution. At this time, I abandoned the Smale program, and commenced a new career in experimental dynamics and chaotic behavior. And there in Stuttgart, I met Otto, already a professor at age 25, and also Gottfried Mayer-Kress, then a graduate student. All these – Yoshi, Christian, Otto, and Gottfried – became friends and collaborators. They were especially gifted in 3D visual intuition. Rarely, people meet and become close friends instantly. This happened to Otto and me in 1974, before his famous attractor and the chaos revolution. Meeting Otto, and spending time with him in front of his analog computer (Fig. 3), excited me to change my line of work from topology to simulation in 1974.

(a) Otto in the front of his computed in 1976



(b) Sketch of the Rössler attractor as drawn by Chris Shaw

FIG. 3. Otto E. Rössler found his eponym attractor by performing numerous simulation with his analog computer, a Dornier 240. The structure of his attractor was investigated by Abraham and Shaw.[38]

Over the years since 1974 we have maintained contact. I have greatly admired him for his moral code, and his readiness to take action on behalf of a variety of issues, as well as his highly original new directions in mathematical physics. Otto has been extraordinarily prolific and innovative. More than 300 papers and 8 books in some 20 fields in less than 50 years. My own span is ordinarily narrow, just a small part of chaos theory. So I will comment here only an that small fraction of Otto's output. In chaos theory, Otto had a meteoric rise, from nowhere in 1975, just two years after his habilitation in theoretical biochemistry, to a commanding position in chaos theory the next year.

In 1974 or 1975, he received a crucial gift of chaos reprints from the late Art Winfree.[43] And in 1976 he published his wonderful papers on the attractor he discovered while searching for chaos in the simplest possible set of equations,[5] as well as a video illustrating its dynamics.[44] In the interim he must have spent uncountable hours in front of his analog computer. And by 1977, he had ascended to the leadership team of the historic conference on chaos theory organized by the New York Academy of Sciences.

Heinz Pagels, a far-sighted physicist and then president of the NY Academy of Sciences, arranged for a major conference in New York in 1977. Under the direction of Okan Gurel and Otto Rössler, a profoundly diverse collection of math, engineering, and science researchers were brought together for sharing new ideas from the radical frontier. Chaos, fractals, and complex systems were connected in new ways, and the chaos revolution was exploding. An entire section of the meeting was devoted to experimental results.

This meeting, entitled "*Bifurcation theory and applications in scientific disciplines*," October 31 to November 4, 1977, was jointly sponsored by the New York Academy of Sciences and the University of Tübingen. The proceedings were organized in eight parts:[37] Mathematics, Biology, Chemistry, Physics, Ecology, Economics, Engineering, and finally, Experimentation and Simulation. The last paper in the final section was the first announcement of my simulation of chaos using digital computer graphics. This work evolved into the graphic introduction for chaos theory in my book, written jointly with the artist, Christopher Shaw.[38] I met again Otto in in Tübingen in 1981, and perhaps also in Guelph (Canada) in 1981.

Around 1978, a group of students, primarily Rob Shaw, Doyne Farmer, Norman Packard, and Jim Crutchfield, later known as the *Santa Cruz Chaos Cabal* (after Gleick's best-seller[39]) began a literature seminar and chaos program on chaotic dynamics. This resulted in an audacious article in the *Scientific American* of December, 1986, in which chaos theory reached a wide popular audience for the first time.[40]

In the early 1980s, Rob Shaw, working with UCSC Physics Professor Peter Scott, began a careful experimental project on the chaotic dynamics of a dripping faucet. This was the subject of a short video in which the attractor reconstruction method was applied to physical data, as well as computer simulation of a dripping faucet. A complete report was published as a book.[41] Rob Shaw credits Otto with the idea for this project.

This 1977 conference was the occasion of first meetings of many from the chaos community. Ranking these attendees according to the number of entrees in the Author Index of the published proceedings, we find: Gurel (on singular points, 16 entries), Nicolis (chemical bifurcations, 14), May (bifurcations in ecology, 13), Marsden (bifurcations in engineering, 12), Yorke (bifurcations in fluid flow, 12), Haken (synergetics, 11), Rössler (with graphics of his famous equations, 11), and Ruelle (turbulence, 11). This meeting gave an enormous boost to chaos theory world-wide.

Thus, in only two years, Otto had ascended into the top ten percent of the attendees, ahead of some of the pioneers of the field such as Smale (bifurcations in economics, 9), Joseph (eigenvalue branching, 7), Williams (Lorenz attractor, 6), Gollub (rotating and convecting fluids, 4), and Devaney (homoclinic orbits, 3). In the following four decades Otto has continued to contribute to the field, comprising about 10 percent of his output. Chaos theory is greatly in his debt.

III. CONCLUSION

[1] Y. Ueda, "Some problems in the theory of nonlinear oscillations (PH.D. thesis, 1965)," in *The road to chaos* (Aerial Press, 1992).

[2] E. N. Lorenz, "Deterministic nonperiodic flow," Journal of the Atmospheric Sciences **20**, 130–141 (1963).

[3] I. Gumowski and C. Mira, *Dynamique chaotique: Transformations ponctuelles, transition ordre-désordre* (Cepadues Editions, 1980).

[4] O. E. Rössler, "Chaotic behavior in simple reaction system," Zeitschrift für Naturforschung A **31**, 259–264 (1976).

[5] O. E. Rössler, "An equation for continuous chaos," Physics Letters A **57**, 397–398 (1976).

[6] H. Haken, ed., *Continuous chaos*, Springer Series in Synergetics, Vol. 2 (Springer-Verlag, Berlin, Germany, 1977).

[7] S. Lefschetz, *Lectures on Differential Equations* (Princeton University Press, 1946).

[8] S. Smale, *The mathematics of time: Essays on dynamical systems, economic processes, and related topics* (Springer-Verlag, New York, 1980).

[9] M. M. Peixoto, "Structural stability on two-dimensional manifolds," Topology **1**, 101–120 (1962).

[10] G. F. De Baggis, "Dynamical systems with stable structure," Uspekhi Matematicheskikh Nauk **10**, 101–126 (1955).

[11] S. Smale, "Generalized poincaré's conjecture in dimensions greater than four," Annals of Mathematics **74**, 391–406 (1961).

[12] R. Thom, "Un lemme sur les applications différentiables," Boletín de la Sociedad Matemática Mexicana , 59–71 (1956).

[13] R. H. Abraham, "Transversality in manifolds of mappings," Bulletin of the American Mathematical Society **69**, 470–474 (1963).

[14] S. Smale, "Differentiable dynamical systems. I diffeormorphisms," Bulletin of American Mathematical Society **73**, 747–817 (1967).

[15] G. D. Birkhoff and P. A. Smith, "Structure analysis of surface transformations," Journal de Mathématiques Pures et Appliquées 9, **7**, 345–380 (1928).

[16] H. Poincaré, "Sur les courbes définies par les équations différentielles (3)," Journal de Mathématiques Pures et Appliquées, IV **1**, 167–244 (1885).

[17] G. D. Birkhoff, "Surface transformations and their dynamical applications," Acta Mathematica **43**, 1–119 (1922).

[18] M. Morse, "The critical points of a function of *n* variables," Transactions of the American Mathematical Society **33**, 72–91 (1931).

[19] A. A. Andronov and L. S. Pontryagin, "Rough systems (in russian)," Doklady Akademii Nauk SSSR **14**, 247–250 (1937).

[20] R. Thom, "Sur une partition en cellules associée à une function sur une variété," Comptes-Rendus de l'Académie des Sciences **228**, 973–975 (1949).

[21] L. E. Èl'sgol'c, "An estimate for the number of singular points of a dynamical system defined on a manifold," Matematicheskii Sbornik **26**, 215–223 (1950), translated for the American Mathematical Society, 1952.

[22]G. Reeb, "Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques," Mémoires de l'Académie Royale de Belgique: Classe des Sciences **27**, 9 (1952).
[23]M. M. Peixoto, "On structural stability," Annals of Mathematics **69**, 199–222 (1959).
[24]R. Abraham, "Lectures of smale on differential topology," Tech. Rep. (Columbia University, 1962) available at https://www.math.cuhk.edu.hk/course_builder/1415/math5070/.
[25]R. H. Abraham and J. E. Marsden, *Foundations of Mechanics: a mathematical exposition of classical mechanics* (Benjamin, Reading, MA, 1967).
[26]R. Abraham and J. Robbin, *Transversal mappings and flows* (W. A. Benjamin, Inc., New York-Amsterdam, 1967).
[27]R. H. Abraham, L. Gardini, and C. Mira, *Chaos in discrete dynamical systems: A visual introduction in 2 dimensions* (Springer-Verlag, New York, 1997).
[28]R. Thom, *Structural stability and morphogenesis: An outline of a general theory of models* (W. A. Benjamin, New York, 1973) translated from the French by David Fowler.
[29]C. E. Zeeman, *Catastrophe theory: Selected papers, 1972-1977* (Addison-Wesley, Reading, Massachusetts, 1977).
[30]S.-S. Chern and S. Smale, eds., *Global analysis*, Proceedings of Symposia in Pure Mathematics No. 14 (American Mathematical Society, Providence, RI, 1970) (Berkeley, CA, 1–26 July 1968).
[31]D. Ruelle, "Strange attractors," The Mathematical Intelligencer **2**, 126–137 (1980).
[32]C. Mira, "étude de la frontière de stabilité d'un point double stable d'une récurrence non linéaire autonome du deuxième ordre," in *Proceedings of the IFAC Symposium on Pulse-rate and Pulse-number Signals in Automatic Control* (1968) pp. D 43–7II.
[33]I. Gumowski and C. Mira, "Sensitivity problems related to certain bifurcations in non-linear recurrence relations," Automatica **5**, 303–317 (1969).
[34]D. Chillingworth, ed., *Proceedings of the symposium on differential equations and dynamical systems*, Vol. Lecture Notes in Mathematics 206, University of Warwick (Springer-Verlag, The address of the publisher, 1971) summer school, July 15-25, 1969.
[35]R. H. Abraham and Y. Ueda, *The chaos avant-garde: Memories of the early days of chaos theory* (World Scientific Publishing, Singapore, 2000).
[36]Www.visual-chaos.org.
[37]O. Gurel and O. E. Rössler, *Bifurcation theory and appliations in scientific disciplines*, Annals of the New York Academy of Sciences, Vol. 316 (New York Academy of Sciences, 1979) papers from the conference held on October 31-November 4, 1977.
[38]R. H. Abraham and C. D. Shaw, *Dynamics the geometry of behavior* (Aerial Press, Santa Cruz, CA, 1981-1988).
[39]J. Gleick, *Chaos: Making a new science* (Viking Press, New York, 1987).
[40]J. P. Crutchfield, J. D. Farmer, N. H. Packard, and R. S. Shaw, "Chaos," Scientific American **255**, 46–57 (1986).
[41]R. Shaw, *The dripping faucet as a model chaotic system* (Aerial Press, Santa Cruz, CA, 1984).
[42]R. May, "Biological populations with nonoverlapping generations; stable points, limit cycles, and chaos," Science **186**, 645–647 (1974).
[43]C. Letellier and V. Messager, "Influences on otto e. rössler's earliest paper on chaos," International Journal of Bifurcation & Chaos **20**, 3585–3616 (2010).
[44]Https://www.youtube.com/watch?v=Tmmdg2P1RIM.