The Dynamics of Synchronization and Phase Regulation

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Abstract

This is a joint project begun in 1983 and first reported in (Abraham, 1989). We were inspired by Arthur Winfree and have in mind a number of applications to medical physiology and mathematical biology. The main theme is the role of the geometry of periodic attractors – the shape of an attractive limit cycle and its isochrons – in determining the phase synchrony of coupled oscillators. We present a profusely illustrated review of the geometric theory of the synchronization of periodic attractors, such as biological oscillators, by periodic forcing.

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INTRODUCTION

Some years ago we worked together on the microstructure of forced oscillators, while studying various biorhythms of mammalian physiology. While the global analysis behind our ideas was eventually published, the basic intuitions of our theory have been buried in our files until now. In this minimally technical tutorial, we present the essential geometry of synchronization and phase regulation of coupled oscillators.

We begin with a detailed description of the general setup in four sections. Our setting is the global analysis of differential manifolds. A basic reference is FM2.

1 Riemannian metrics (with differential geometry)

Recall from linear algebra, given a real vector space, E, an *inner product* on E is a 2-form, g, which is symmetric and positive definite. That is, given two vectors in E, e_1 and e_2, then $g(e_1, e_2)$ is a bilinear real-valued function (thus linear in each argument) that is symmetric, that is, $g(e_1, e_2) = g(e_2, e_1)$, and positive definite, that is, $g(e_1, e_1) >= 0$ and is 0 only for $e_1 = 0$. The *norm*, or length, of $\|e\|$, is the square root of $g(e, e)$.

Given a differentiable manifold, M, the metric and norm are structures that may be defined on the tangent bundle, $TM \to M$. That is, for each point $p \in M$ a metric $g(p)$ is defined on the tangent space T_pM, which depends smoothly on p. This is a tensorfield of covariant order 2. [FM2, sec. 1.7] From the metric $g(p)$, a norm may be defined as above.

The pair (M, g) is a *Riemannian manifold* by definition. In this article, the metric is used only to define the norm.

Example

In case M is Cartesian space R^N, the usual Euclidean scalar product provides a convenient Riemannian structure. This simple case is adequate for most of this article.

1.1 The norm

Returning to our basic context, a smooth vectorfield V on a smooth manifold M of dimension N, we now add a Riemannian metric, g. We obtain a norm from the metric by the formula,

$$\|e\| = \sqrt{g(e, e)} \tag{1}$$

1.2 The length of a curve

We may now consider the length of a C^1 curve $p(t)$ in M, from time t_0 to time t_1, to be the integral,

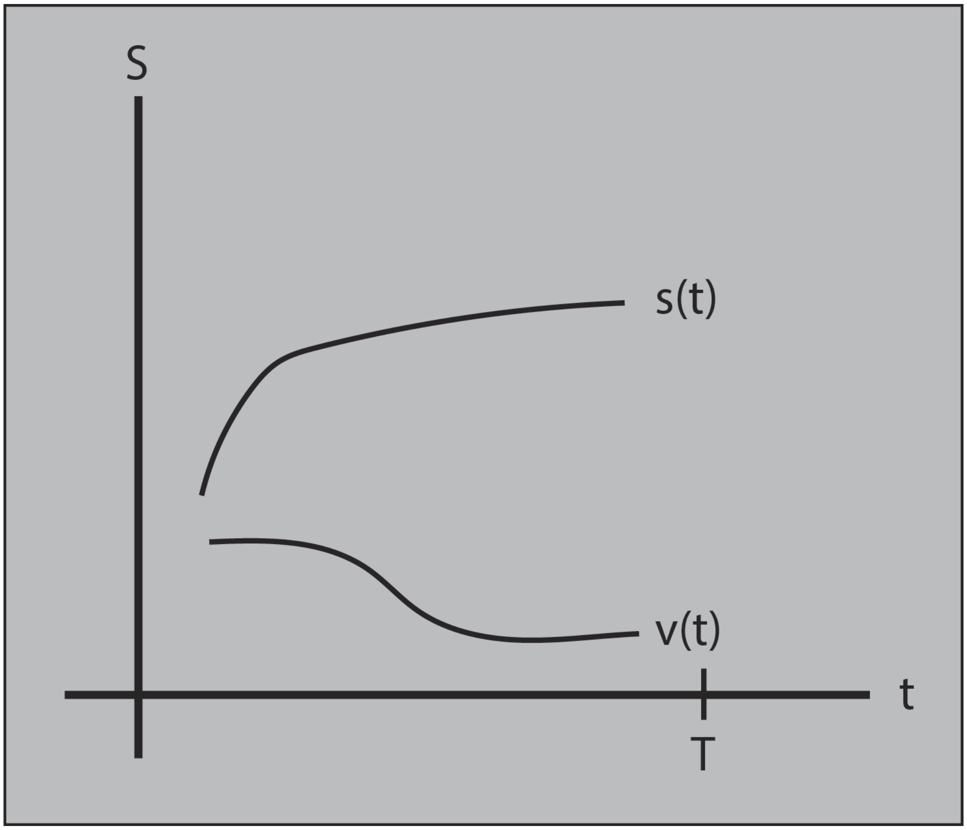

Figure 1: Arc length and velocity vs time showing the motion along Γ.

$$\int_{t_0}^{t_1} \|p'(t)\| dt \tag{2}$$

where $p'(t)$ is the tangent vector of p at t.

Choosing a point p_0 on the closed orbit Γ of V as a fiducial point, and assuming the time variable such that $t_{p_0} = 0$, we may now define a new parameter, arc-length, by:

$$s(t) = \int_0^t \|p'(t)\| dt \tag{3}$$

We define the length of Γ by $S = s(T)$, where T is the prime period of Γ. Plotting $s(t)$ vs t, we may view the motion along Γ, where velocity, $v(t)$ is the slope of the graph, as shown in Figure 1. This will be the context for our analysis of perturbations of V.

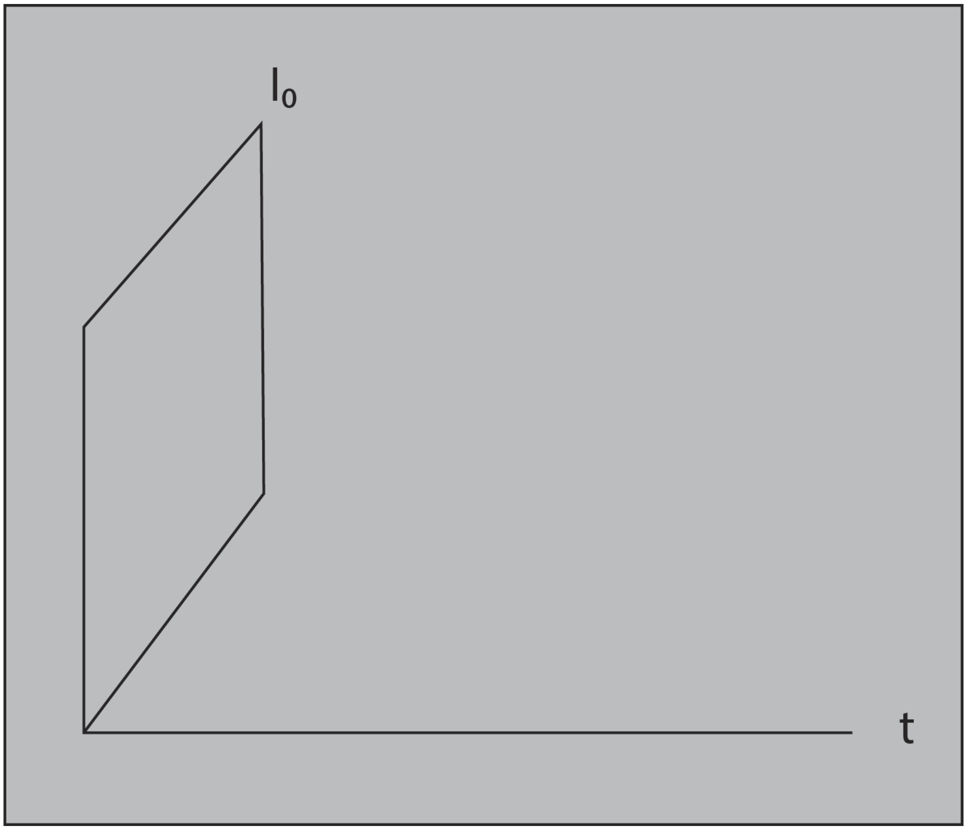

Figure 2: Normal coordinates for Γ, $s(t)$ vs t. The horizontal axis is time relative to t_0, and the complementary $N-1$ dimensions, shown as a plane, indicates the isochron, I_0, at time t_0.

We may use time and position in the isochron as a coordinate system for a neighborhood of Γ, as shown Figure 2.

2 The basic setup (with differential topology)

Let M be a differentiable manifold of dimension $N >= 2$ and differentiability class C^∞. We will work with vectorfields of class C^1, belonging to $\mathcal{V}$, the Banach space of all C^1 vectorfields on M, with the C^1 sup-norm topology. [MTA; p. 48] Henceforward *smooth* shall mean class C^1.

2.1 The periodic attractor

It is fine to think of M as a Euclidean space, or a hypersurface in a Euclidean space. Each vectorfield is a crosssection of the tangent bundle, $TM \to M$. And two vectorfields are close together in the C^1 norm topology if their values and derivatives are close at every point of M.

Now we consider a vectorfield V on M with a periodic attractor, Γ, of prime period T. Choose a base point, p_0, on Γ. Then we may parameterize the points, p in Γ, by either the time t that flows p_0 to p, or the phase, $\phi = (2\pi/T)t$ radians.

2.2 The isochrons

Let $\{F(t)\}$, with time t in R, be the flow of V, a curve of diffeomorphisms. Then $F(T)$, with T the prime period of Γ, is a diffeomorphism mapping p_0 to itself, and the derivative TF of $F(T)$ at p_0 is a linear isomorphism of the N-dimensional tangent space $T_{p_0}M$ to M at the point p_0. The vector $V(p_0)$ is tangent to Γ, and is mapped to itself by $TF(T)$. So it is an eigenvector of $TF(T)$ at p_0 with eigenvalue 1. The remaining $N-1$ eigenvalues of TF at p_0, the characteristic multipliers of Γ, have an $N-1$ dimensional eigenspace, $T_{p_0}I_0$, which is a linear compliment to the tangent space to Γ at p_0. It is tangent to the isochron of Γ, $I_0 = I_{p_0}$, an $N-1$ dimensional submanifold of M. The chief feature of I_0 is that it comprises all points that have asymptotic phase zero.

Finally, for any other point p_t of Γ, the isochron I_t is the image of I_0 under the diffeomorphism $F(t)$. The union of all isochrons of Γ, together with time t or phase ϕ, comprises a coordinate system for a neighborhood of Γ, called *normal coordinates.* [HPS]

By the way, the C^1 hypothesis and associated sup-norm is needed here, to guarantee that the characteristic multipliers of Γ (all within the unit circle in the plane of complex numbers, as Γ is an attractor) as well as Γ itself, vary continuously with perturbations of V.

3 Perturbations

In applications we frequently encounter a situation in which a dynamical system, a vectorfield V on manifold M, is perturbed, or weakly forced, by another vectorfield A, which may depend on a control parameter.

Thus, the perturbed vectorfield is,

$$\widehat{V(c)} = V + A(c) \tag{4}$$

where c is a point in another manifold C, the control manifold, and A is a smooth mapping of C into $\mathcal{V}$, the Banach space of C^1 vectorfields on M.

3.1 The periodic case

In a common case, C is a circle. In other words, $\{A(t)|t \in [0, T_2]\}$ defines a periodic cycle in $\mathcal{V}$, the Banach space of C^1 vectorfields on M. By small, we mean that this cycle is within an ϵ disk centered on V, for some small ϵ, defined by the C^1 norm.

Thus, $A(t)$ is a periodic time-dependent vectorfield with prime period T_2. We might manage an arbitrary relationship of the two periods, that of A and that of Γ. But at this point, we will assume $T_2 \approx T$. Then a very small adjustment of the vectorfields V and A will make the two periods are identical, according to Peixoto's theorem. [DYN; ch. 12] Here, for simplicity, we will consider this case only.

3.2 The perturbation tangent to Γ

Now, for each t, we may extract from the perturbing vectorfield, $A(t)$, its component tangent to the periodic attractor, Γ, using the complementary eigenspace $T_{p_t}I_t$ tangent to the isochron I_t, along Γ. Thus, at the point p_t on γ,

$$A(t) = A_1(t) + A_2(t) \tag{5}$$

where $A_1(t)$ is tangent to Γ, and the complement $A_2(t)$ belongs to the isochron tangent space, $T_{p_t}I_t$. Similarly, we may decompose the original vectorfield,

$$V(t) = V_1(t) + V_2(t) \tag{6}$$

The decompositions may be extended to a neighborhood U of Γ using normal coordinates. In the following we shall assume this extension for $A(t)$ and $V(t)$. As $A_1(t)$ is parallel to $V_1(t)$, so there is a unique scalar $a(t)$ such that,

$$A_1(t) = a(t) * V_1(t) \tag{7}$$

This scalar function $a : U \to R$ is the sole component of A needed for our analysis. The intuitive understanding of this function is shown in Figure 3.

In the positive segment, the perturbed velocity vector $V(t) + a(t) * V(t)$ is faster than $V(t)$, so phase change is positive and in the negative domain, it is slower. Therefore, the downward crossing is an attractor of phase change and the upward crossing, a repellor. We call such a downward crossing a *favorite phase.*

4 The phase change

Given a small perturbation A to to V, which may depend on a parameter such as time, the periodic attractor Γ of V, with prime period T and arc-length S, is perturbed to a nearby periodic attractor $\widehat{\Gamma}$, with nearby prime period $\widehat{T}$ and arc-length $\widehat{S}$. We now investigate the phase relationship in the case of a syncronous periodic perturbation.

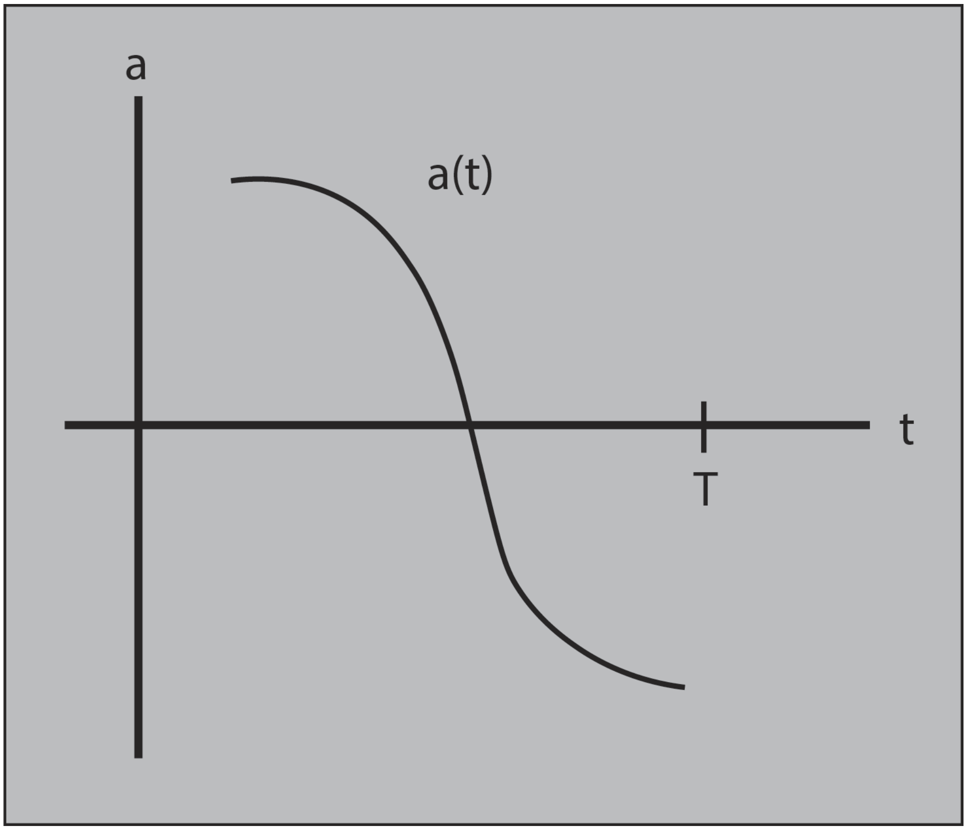

Figure 3: In this case, the function $a(t)$ is positive in the early part of the cycle, then crosses downward through the horizontal axis and stays negative until crossing again upwards.

4.1 The phase shift due to perturbation

The change in period, $\Delta T = \widehat{T} - T$, determines a change in phase relative to Γ, $\Delta\varphi = 2\pi * \Delta T/T$, the phase shift due to perturbation.

Choose a point p_1 on Γ, the periodic attractor of the vectorfield V. Let the time t_1, corresponding to the phase φ_1, be the time relative to the phase 0 at the fiduciary point, p_0 on Γ. After the small increment of time, Δt, the action of the flow of V moves p_1 to the point p_2 on Γ at time $t_2 = t_1 + \Delta t$, corresponding to phase $\varphi_2 = \varphi_1 + \Delta\varphi$.

Now consider the action of the perturbed vectorfield $\widehat{V}$ on its periodic attractor $\widehat{\Gamma}$ during Δt. Start again with the point p_1, which is near, but perhaps not on, $\widehat{\Gamma}$. The trajectory of $\widehat{V}$ will move p_1 closer to $\widehat{\Gamma}$, arriving at a point $\widehat{p_2}$ at time t_2.

Assuming the scalar $a(t)$ is positive during Δt, the point $\widehat{p_2}$ will be on the isochron I_{t_3}

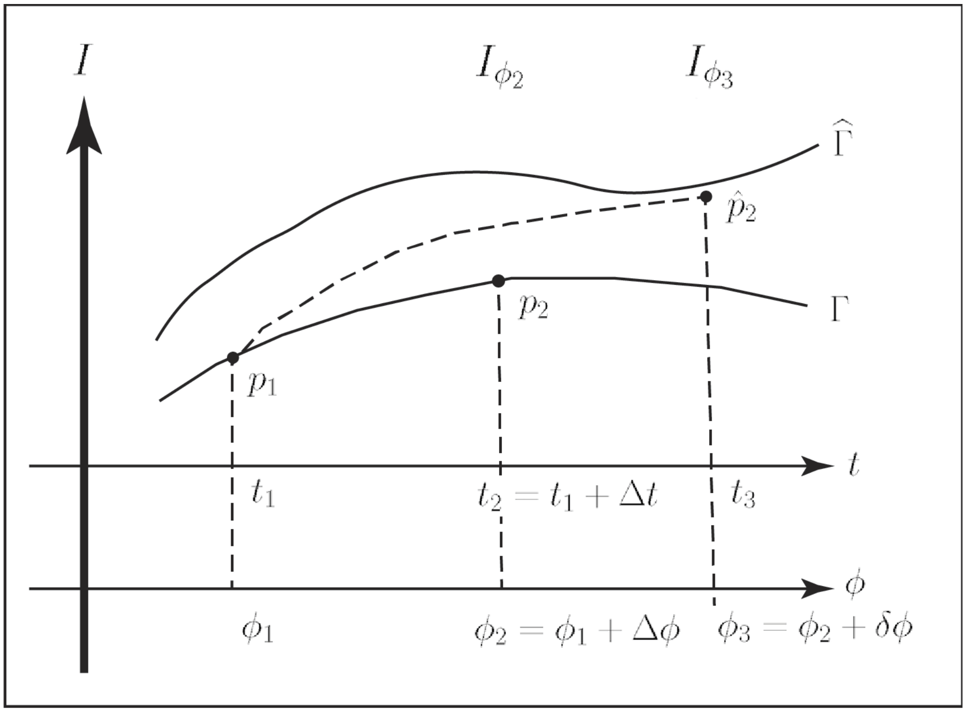

Figure 4: The trajectories and important points, in the normal coordinates of Figure 2, with the isochron dimensions indicated by a line, zooming in on a small interval of time.

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of V for some $t_3 > t_2$ as shown in Figure 4. The corresponding phase is $\varphi_3 = \varphi_2 + \delta\varphi$, and $\delta\varphi$ is the incremental phase shift caused by the perturbation A. Our task now is the determination of $\delta\varphi$ in terms of V and A. This configuration is shown in Figure 4.

Consider a small interval of time, Δt, and assume the perturbation A is constant in this interval. We seek an expression for the change in phase caused by A in this interval in terms of V and A. Our analysis follows from Figures 4 and 5.

The action of V in interval Δt moves the point p_1 to p_2 along Γ, as shown in Figure 4. This belongs to the isochron at time t_2 or equivalently φ_2.

Meanwhile, the perturbed action of $V+A$ moves p_1 to $\widehat{p_2}$ along the perturbed trajectory shown dashed in Figure 4. The position of this point is shown forward of p_2, which would be the case if a were mostly positive during the interval Δt.

This determines a time t_3 and equivalent phase φ_3 such that $\widehat{p_2}$ lies on the isochron I_{t_3}, and thus a small interval of phase, $\delta\varphi$ (note lower case δ), such that $\varphi_3 = \varphi_2 + \delta\varphi$, and similarly, $t_3 = t_2 + \delta t$.

The motion of p_2 under V along Γ from φ_1 to φ_2 determines an arc-length, ΔS, and

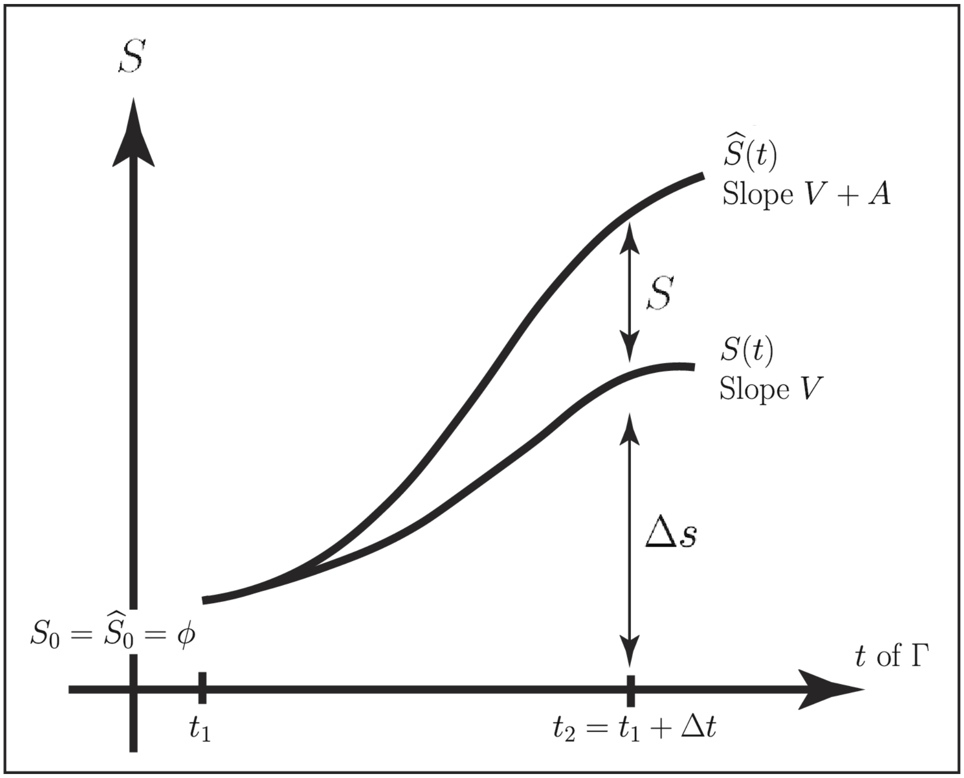

Figure 5: Enlargement of Figure 1, zooming in on a small interval of time, Δt.

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from phase φ_2 to phase φ_3 determines an arc-length δS.

Now we normalize the vectorfield V_1. Let $v = \|V_1\|$, and $W = V_1/v$, the unit vectorfield along Γ. Then we have $V_1 = vW$ and $A_1 = aV_1 = avU$.

Then we have $av = \delta S/\Delta t$, or

$$\delta S = av\Delta t \tag{8}$$

Similarly, $v = \delta S/\delta t$, or

$$\delta S = v\delta t \tag{9}$$

Eliminating δS between these two equations, we have, $\delta t = a\Delta t$, so,

$$\delta\varphi = a(2\pi/T)\Delta t \tag{10}$$

This is the expression we need going forward.

4.2 The phase shift integral formula

Let us assume the $t_0 = 0$. Let $Z(t)$ be the accumulated phase shift due to the perturbation A, from time 0 to t, so $Z(T)$ is the phase shift due to one turn around Γ, starting from p_0. The phase shift of $V + A$ at time t is thus $\varphi(t) + Z(t)$. where $\varphi(t) = (t/T) * 2\pi$.

Note the perturbation $A(t)$ can be arbitrary. For the analysis in an interval Δt, we approximate the time-dependent perturbing vectorfield, $A(t)$, by its average over the interval, or its value at the beginning of the interval.

Evidently, $Z(t)$ is the integral of $\delta\varphi$, which is given in equation 10. So we have:

Theorem

$$\int_0^t \delta\varphi = \int_0^t a(t) * (2\pi/T)dt \tag{11}$$

5 A simple example in the plane

Let V be a vectorfield on the Euclidean plane, R^2 with the standard Euclidean metric and norm. Suppose V has the unit circle as a periodic attractor, Γ, and that at every point p of Γ, the value of V is a unit vector directed clockwise, as shown in Figure 6. The arc-length of Γ is 2π, and the period is also 2π.

Let the perturbation, A, be the vectorfield with constant components $(0, 0.25)$. That is, vertical, directed upward, and one quarter of a unit vector in norm. The perturbation is constant, yet its component $a(t)$ tangent to Γ varies, as shown in Figure 7.

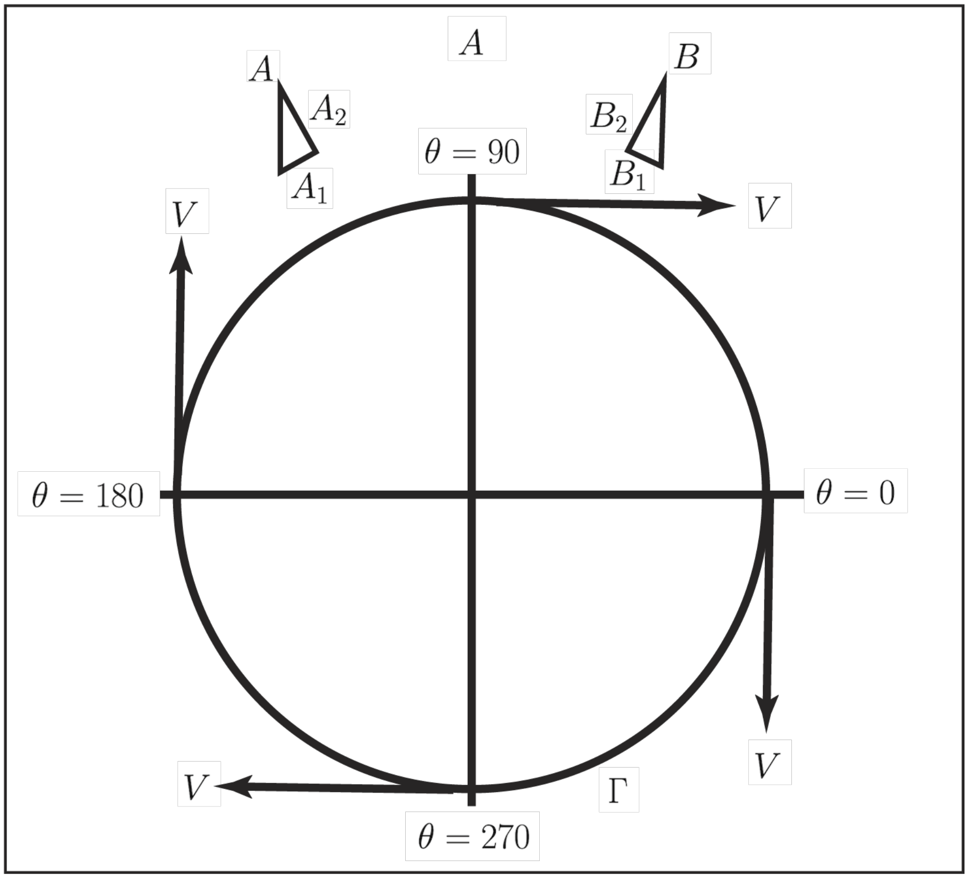

Figure 6: The state space of the example.

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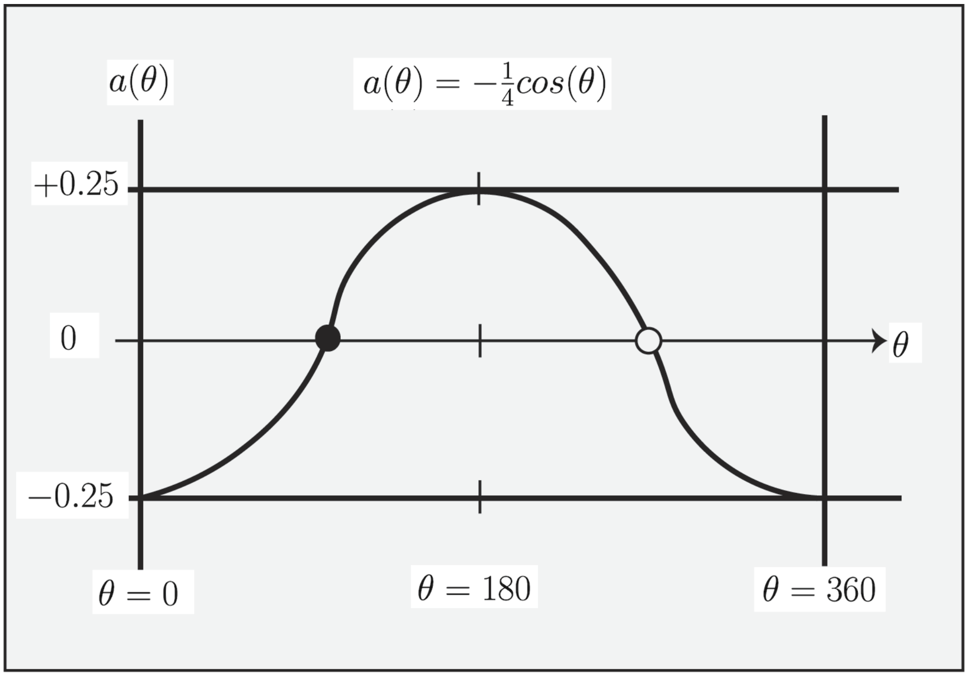

Figure 7: The tangential perturbation. The solid dot indicates the attractor, while the hollow dot shows the repellor.

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