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Dynamics and Self-Organization

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ABSTRACT

Modern views of qualitative dynamics seem to promise simple geometric models of complex behavior and a mathematical rationale for constraint of evolutionary processes to particular paths. Dynamical bifurcation theory should not be confused with elementary catastrophe theory or with classical bifurcation theory. A short history of dynamics is given. A new category of dynamical models for complex systems is proposed, based on networks of serially coupled dynamical systems. These models may potentially be extended to account for irreversibility, fluctuation, coherence, symmetry-breaking, complementarity, and other phenomena of self-organization. Finally, ten outstanding problems for dynamics that are central to the development of self-organization theory are described. —THE EDITOR

There are various, distinct mathematical viewpoints applicable to the description of self-organizing systems. Here my concern will be the assessment of (and speculation on) just one of these: the viewpoint of *dynamics*. This subject lies between mathematics and the sciences, and has been central to increasing hopes for a rigorous mathematical theory of morphogenesis. Optimism was generated by the visionaries—Turing and Thom—and justified by successful and spectacular applications to physics, especially to hydrodynamics. Extrapolating from these applications, dynamics seems to promise:

- Simple geometric models for complex and chaotic behavior
- A complete taxonomy of dynamic states or their attractors, and of developmental events (bifurcations) for morphogenetic sequences
- A mathematical rationale for the constraint of complex self-organizing sys-

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tems to a simple "homeorhesis" (in Waddington's terminology), or evolutionary process

The overpowering allure of dynamics for self-organization theory is the promise of a mechanical explanation for morphogenesis, a kind of rebirth of Greek rationalism, or scientific materialism. In fact, simultaneous with the emergence of dynamics as a field, dynamicism is emerging as a prospective scientific cosmology, a philosophical new wave.

We must be careful not to confuse *dynamical bifurcation theory*, which is the basis of this essay, with the rather similar subjects of *elementary catastrophe theory* (see, e.g., Zeeman, 1976, or Poston and Stewart, 1978), or *classical bifurcation theory* (surveyed in Marsden, 1978). The bifurcations of gradient dynamical systems, including elementary catastrophes, are less general than the dynamical bifurcations of dynamical systems, and these are in turn less general than the classical bifurcations of nonlinear operators and partial differential equations. All three fields use the same vocabulary in different ways—a circumstance that can lead to confusion. Furthermore, all three viewpoints may be applied to the same partial differential equations!

Dynamics has three aspects: mathematical; experimental; and applied. The history of experimental dynamics divides naturally into the subcategories of

- Real machines (since Galileo, 1600)
- Analog machines (since Bush, 1931)
- Digital machines (since Lorenz, 1963)

These are described in Abraham and Marsden (1978) and shown in Table 1. Before the widespread use of analog and digital computing machines in dynamics (i.e., before 1960), it was difficult to distinguish between experimental and applied dynamics, because real machines usually are not sufficiently tractable to permit serious exploration of their detailed dynamics. In fact, the capability of a machine to admit preparation in an arbitrary initial state effectively *defines* it as an analog computer. The evolution of this capability for nonlinear electrical oscillation ushered in the purely experimental period (Hayashi, 1953, 1964).

By a *real machine* we mean any system of the phenomenal universe that behaves sufficiently like a dynamical system, with states that may be related to an idealized state space, evolution from an initial state along a trajectory to a final motion, dynamics that can be changed by control "knobs," and so on. The real machines that have engaged dynamicists most seriously involve fluids, gases, elastic solids, chemical reactions, neurons, and so on.

In this chapter, I discuss the potential of applied dynamics for modeling self-organizing systems. First, I propose some dynamical models that in the future might be useful to describe self-organizing systems. Then I present some unsolved problems of dynamical systems theory that are important for the program suggested in the first part. [All of the concepts of dynamics essential for this discussion are presented in an elementary, intuitive way in Chapter 29, and at greater length in Abraham and Shaw (1983, 1984, 1985), in a similar style.]

TABLE 1. The History of Dynamics

Date	Mathematical dynamics	Experimental dynamics			Applied dynamics
		Digital	Analog	Real	
1600					Galileo
					Kepler
	Newton				
	Leibnitz				
1700					
	Lagrange			Chladni	
1800					
				Faraday	
			Thompson		
	Lie			Rayleigh	
	Poincaré				
1900	Lyapunov				Ehrenfest
	Julia				Duffing
				Van der Pol	
	Birkhoff		Bush		
	Hopf		Philbrick		
	Peixoto		Hayashi		Turing
	Smale	Lorenz			Thom
			Rössler	Gollub/Swinney	Ruelle
2000					

Models for Self-Organization Suggested by Dynamics

I propose here a concordance between the concepts of dynamics and those of self-organization. I do not expect this to be a convincing case, because the worked examples necessary for proof would require an effort such as the 5 years Zeeman (1977) devoted to documentation of the Thom (1972) case for Elementary Catastrophe Theory models for morphogenesis.

Experimental Dynamics: A Canonical Example

Although hydrodynamics is a classical subject, its firm connection to dynamics began only recently, with Arnol'd (1966). Ruelle has traced the roots of the idea back to Ehrenfest's thesis, a century ago.

Three hydrodynamical machines have been very important in experimental dynamics: the Couette–Taylor stirring machine; the Rayleigh–Bénard simmering machine; and the Chladni–Faraday vibrating machine. Detailed descriptions and bibliographies may be found in the books cited in the References and in Abraham (1976b) and in Fenstermacher *et al.* (1979). The behavior of all three machines show great similarities, so here I shall consider explicitly only the Couette device, in which two concentric cylinders are mounted on a common vertical axis. Through the transparent outer cylinder, we observe a fluid contained between the two cylinders. The inner cylinder may be rotated by a motor, with the speed parameter controlled by the experimentalist. If we naively assume that there is an abstract dynamical system with a finite-dimensional state space that we can visualize as a plane, then exploration of the machine and direct observation of its attractors yields a partial map of the bifurcation diagram.

Initially the system has an equilibrium (a point attractor) corresponding to the rotation of concentric cylindrical lamellae that show a one-dimensional symmetry group of vertical translations, and another of cylindrical symmetry.

As the speed parameter is increased (visualized as moving to the right), the equilibrium ends, and the state leaps "catastrophically" to a new point attractor, corresponding to spiraling motion of discrete annular rings of fluid, the "Taylor cells." Taylor cells have a zero-dimensional, or discrete, symmetry group of vertical translations. This catastrophe is a prototypical symmetry-breaking bifurcation. Further extension of the control parameter (speed of rotation of the inner boundary cylinder) produces a new excitation in a sequence leading to turbulence.

Several specific sequences leading to turbulence have been suggested, and there are several proposals for a deterministic scheme for describing turbulence, based on finite-dimensional, dynamical bifurcation theory. This development, originating with Lorenz (1963) and Ruelle and Takens (1971), is of great interest to physics. The link between bifurcation diagrams and the classical partial differential equations of hydrodynamics (the Navier–Stokes equations) has been elaborated in considerable detail, especially in Marsden and McCracken (1976), Marsden (1977), Ratiu (1977), Smale (1977), Bowen (1977), Abraham and Marsden (1978), Ruelle (1980), Pugh and Shub (1980), and Rand (1980).

I shall describe briefly three problems of this treatment of the Couette machine, not only because of its role as an important example of morphogenesis itself, but because similar treatment of more general partial differential equations may be expected in the near future, especially equations of the "reaction–diffusion" type.

The first problem is technically very severe: many kinds of partial differential equations on a finite-dimensional domain may be viewed easily as dynamical systems on an infinite-dimensional space. But they turn out to be *rough dynamical systems* (described later) instead of the smooth sort to which dynamical bifurcation theory easily applies. Specialists have taken two approaches to this obstacle: the easy way around consists of reducing the domain to finite dimensions by some special tricks (thereby losing credibility); the hard way consists of re-proving the necessary results of dynamical bifurcation theory in the rough context (losing readability). Both approaches have been pursued successfully (Bernard and Ratiu, 1977).

The second problem is equally severe: because the experimental equipment (e.g., a Couette–Taylor machine) is a real machine, *it cannot be prepared in an arbitrary initial state* (i.e., arbitrary instantaneous fluid motion). In fact, the fluid is always at rest between the cylinders (the zero point of the infinite-dimensional state space) when the experiment begins. Therefore, only a small section of the global bifurcation diagram may be discovered by experimental exploration. But the model, the global bifurcation diagram of the Navier–Stokes equations of fluid dynamics, could be mapped completely by digital simulation, if a large-enough computer were built. The techniques for this project comprise an entire field of numerical analysis.

The third problem seems potentially more tractable, although not yet solved. This is the *problem of observation.* Indeed, we may distinguish two problems of observation: *ignorance* and *error.* By "ignorance" I mean that we look at an infinite-dimensional state, but can record only a small number of parameters. Even if we measure without error, the data describe only a point in a finite-dimensional space. Thus, the observation procedure, at best, defines a projection map from the infinite-dimensional state space, *S*, to a record space, *R*, of finite dimension. I shall refer to this as the *output projection map.*

For example, in the recent work with the Couette machine described by Fenstermacher *et al.* (1979), observation is restricted to a single direction of fluid velocity at a single point in the fluid region, measured by laser–Doppler velocimetry. In other words, the dimension of *R* is one, an example of extreme ignorance. This difficulty is inescapable in any application of dynamical bifurcation theory to partial differential equations. Therefore, I have coined the word *macron* for the image of an attractor in *S*, projected into *R* (Abraham, 1976a). The question arises: can we recognize the attractor, having observed only its projected macron? We have not learned to recognize attractors yet, so this question is open.

By the problem of error I mean the problem of determining a point exactly, in a finite dimensional space. Errors arise either in the measurement of the coordinates, or in the storage of the data. This problem usually is handled by techniques of statistical physics or by information theory; Shaw (1981) presents a particularly interesting discussion of it.

In the future we hope for mathematical classifications of attractors and macrons that are: 1) experimentally identifiable, in spite of ignorance and error; 2) well-founded in the rough context of partial differential equations of evolution, viewed as dynamical systems on infinite-dimensional spaces; and 3) tractable in digital simulation. Substantial progress is being made, and I am assuming a satisfactory resolution of the above three problems in making the following prediction: *dynamical bifurcation theory will be useful for self-organization theory.*

Complex Dynamical Systems and Self-Organization

The Couette machine discussed above provides a canonical example of the role of dynamical bifurcation theory in modeling a simple self-organizing system. But to move to a more complicated example, such as slime mold aggregation (Chapter 10), would reveal quickly the inadequacy of our models based on a single dynamical system with controls. Rather, a continuum of dynamical systems with controls might be required, coupled together in a space-dependent way, with the continuum of controls manipulated by another such system! Therefore, I have proposed a richer class of models, called *complex dynamical systems* (Chapter 29). These new models combine dynamical systems that have controls into networks, by means of serial and parallel coupling. The full elaboration of the generic behavior and bifurcations of these complex systems—through theory and experiment—is still ahead. Yet I conjecture that their behavior will provide adequate metaphors and useful models for the homeorhesis of some self-organizing systems found in nature. For the present, I shall have to be satisfied with brief indications for the possible unification into this framework of some of the concepts described in the literature of self-organization theory.

The main ideas underlying this accommodation are that:

- Mathematical models for morphogenesis involve partial differential equations of the "reaction–diffusion" type (see the original pioneering papers of Rashevsky, 1940a,b,c, and Turing, 1952)
- The model equations may be viewed as "rough" dynamical systems on an infinite-dimensional state space (see Guckenheimer, 1980)
- These dynamical systems have controls, and may be combined into complex systems to model a given self-organizing system

For specific examples, see the articles by Carpenter (1976), Conley and Smoller (1976), Guckenheimer (1976), and Rinzel (1976); also Part III of Gurel and Rössler (1979); the master equation (Gardiner *et al.*, 1979) and the Fokker-Planck equation (Haken, 1977). The equations of elastodynamics have been treated in this way by Marsden and Hughes (1978) and Holmes and Marsden (1979). Applications of Elementary Catastrophe Theory in this area by Thompson and Hunt (1973), Zeeman (1976), and Poston and Stewart (1978) are relevant.

I now complete my case for the complex dynamical system scheme, by relating it to specific models emerging in self-organization theory.

My scheme is an extension of Thom's, and so his morphogenetic field is automatically accommodated. The proposal of Turing (1952), the dissipative structures of Prigogine (1978), and the synergetics of Haken (1977) all are based upon partial differential equations of the reaction–diffusion type; they all fit directly into this scheme. I suspect that the homeokinetics of Iberall and Soodak (Chapter 27) and the models of Winfree (1980) also fit.

The ideas of *irreversibility* and *fluctuation* emphasized by Prigogine are included in my scheme: irreversibility in the error problem (Shaw, 1981); and fluctuation in the fluctuating braid bifurcation, either in serial or parallel coupling (Chapter 29).

The ideas of *coherence* and *order parameter* emphasized by Haken (Chapter 21) are also natural in my complex scheme; coherence is an aspect of entrainment, and a special case of order parameter is known in dynamics under the name *slow manifold of an attracting point*. In fact, slow manifolds exist for a large class of attractors—those satisfying Axiom A (see Smale, 1967, 1971, or Irwin, 1980), and not just for rest points. Complementary *fast foliations* define a projection of the state space onto the slow manifold, so order parameters are included in this scheme as an output projection map. An outstanding pedagogic example of the slow manifold and fast foliation concepts is found in the heart and neuron model of Zeeman (1977, p. 81).

The concept of *symmetry-breaking bifurcation* fits the complex dynamical scheme well. The *hypercycle* idea of Eigen (Schuster and Sigmund, Chapter 5) has a dynamic interpretation as a limit cycle, and this has been generalized by Zeeman. But I would suggest a more extensive model for it, as a closed cycle of a complex dynamical system (Chapter 29, Figure 84).

Microcosmic/macrocosmic complementarity (see Prigogine, 1980, 1981) fits into my scheme as part of the problem of observation, as the output projection map of a model translates the microcosmic dynamics into macrocosmic observables. *But here I see the need for the further development of an extensive statistical and information theory of dynamical systems, along with near-Hamiltonian dynamics, to place thermodynamical laws on a firmer foundation* (see Shaw, 1981; Smale, 1980; Ruelle, 1980).

The *symbol/material complementarity* (Pattee, 1981; also Chapter 17) also may be discussed in the context of complex dynamical systems. An attractor functions as a symbol when it is viewed through an output projection map by a *slow observer*. If the dynamic along the attractor is too fast to be recorded by the slow-reading observer, he then may recognize the attractor only by its averaged attributes, fractal dimension, power spectrum, and so on, but fail to recognize the trajectory along the attractor as a deterministic system. See Chapter 29, Figure 69, for an example of such a material model for a cyclic symbol sequence with random replacements in a self-regulating dynamical system.

Tomović's idea (Chapter 20) of nonparametric control is accommodated naturally in my scheme. I rest my case here, for the present.

Critique of Dynamical Models

As explained above, dynamicism is encroaching on our scientific cosmology. Let us suppose that the problems mentioned were solved, the properties of complex dynamical systems worked out, and specific models for self-organizing systems at hand, satisfactory from the point of view of explanation and prediction. Even then we would know precious little about the mechanisms of self-organization, without a parallel development of the qualitative, geometric theory of partial differential equations. But this effort has hardly begun, although classical bifurcation theory is a beginning (see, e.g., Haken, 1977). I shall refer to this emerging branch of mathematics as *morphodynamics*.

Compare my model for the Couette machine (which claims that patterns will change only in certain ways) with the morphodynamic analysis of Haken (1977, and Chapter 21) that actually discovers the patterns! We may regard dynamics (i.e., dynamic bifurcation theory, together with all its extensions described above) as an intermediary step. The development of this theory has its own importance in mathematics. Numerous applications in the physical, biological, social, and information sciences await its maturity. Yet for evolution, self-organization theory, morphogenesis, and related scientific subjects, it seems to me that dynamics will play a preparatory role, paving the way for a fuller morphodynamics in the future.

The benefits of using dynamical concepts at the present stage of development of self-organization theory fall in two classes: permanent ones—the acquisition of concepts to be embedded in morphodynamics, guiding its development; and temporary ones—the practice of new patterns of thought. In the first category, I would place the attractors, the stable bifurcations, and their global bifurcation diagrams, as essential features of morphodynamics. These may be regarded as guidelines, exclusion rules, and topological restrictions on the full complexity of morphodynamic sequences. The temporary category would include the specific models in which dynamics is applied to rough systems on infinite-dimensional state spaces, in order to accommodate partial differential equations. This application is valuable, because the basic concepts of our scientific cosmology—noise, fluctuation, coherence, symmetry, explanation, mathematical models, determinism, causality—are challenged by dynamical models and essentially altered. Thus, the philosophical climate for the emergence of morphodynamics is created.

Elementary Catastrophe Theory has similarly provided some excellent examples of scientific explanation, permanent members in the applied mathematics Hall of Fame. Yet its most important function in the history of mathematics may turn out to be the *practice* it provides scientists in geometric and visual representation of dynamical concepts, paving the way for understanding of dynamics, bifurcations, and the rest.

In summary, dynamicism is without doubt an important intellectual trend, challenging the fundamental concepts of mathematics and the sciences. I see its importance for self-organizing system theory as temporary and preparatory for a more complete morphodynamics of the future. And yet dynamicism even now promises a permanent legacy of restrictions, a taxonomy of legal, universal restraints on morphogenetic processes—a Platonic idealism.

We must be careful not to cast aside dynamics, as Newton cast out wave theory, for not explaining forms. Whitehead (1925) wrote at the end of *Science and The Modern World*:

> . . . a general danger [is] inherent in modern science. Its methodological procedure is exclusive and intolerant, and rightly so. It fixes attention on a definite group of abstractions, neglects everything else, and elicits every scrap of information and theory which is relevant to what it has retained. This method is triumphant, provided that the abstractions are judicious. But, however triumphant, the triumph is within limits. The neglect of these limits leads to disastrous oversights . . . true rationalism must always transcend itself by recurrence to the concrete in search of inspiration.

Problems for Dynamics Suggested by Self-Organization

I now shall describe ten open problems for mathematical dynamics that are especially significant to the program proposed here for self-organization theory.

1. *Taxonomy of Attractors*

The observed "states" for a dynamical system are its attractors (Chapter 29). The need for a complete taxonomy of equivalence classes of generic chaotic attractors is well appreciated, and several attacks on this problem are under way. The meanings of "equivalent" and "generic" are still evolving. I think that a scheme based on the geometric features of experimentally discovered attractors will triumph eventually. These features include: the distribution of critical points and other particulars of shape (especially "ears"); the gross reinsertion maps (as described by Rössler, 1979); the fractal dimension; and the shuffling map of the Cantor section. Analytic features, such as the power spectra, entropy, and Lyapunov characteristic exponents determined by these attractors, may be important (but probably insufficient) for a complete classification.

2. *Separatrices and AB Portraits*

A dynamical system determines a *separatrix* in its domain that separates the domain into distinct open basins, each containing a unique attractor. The separatrix may contain vague attractors (Chapter 29, Figure 20) in vague basins (not open, yet of positive volume). These attractors and basins are of primary importance in applied dynamics. The decomposition of the domain, by the separatrix, into basins, and the location and identity of an attractor in each, comprises the *AB portrait* of the dynamical system. The *yin-yang problem* for AB portraits consists of finding a set of dynamical systems, the *good set*, which is large (yin) enough to be "generic" and small (yang) enough to be classifiable into "equivalence" classes of AB portraits.

As shown by the pioneering program of Smale (1967, 1970; see also Abraham and Marsden, 1978), an enormous simplification of this problem may be expected by taking into account topological restrictions on generic separatrices. Still, this problem is so difficult that it has been solved only for orientable surfaces (Peixoto, 1962). Even on the Möbius band it is still an open problem.

3. *Local Bifurcations*

In $D(S)$, the set of all smooth dynamical systems on a given state space S, the bad set B consists of the dynamical systems not structurally stable. The bad set is very bad (Chapter 29, Figure 37), and the *problem of local bifurcations* consists of discovering the full structure of B by the method of constructing local cross sections, called "generic bifurcations" or "full unfoldings." To Poincaré or Hopf, before the discovery of chaotic attractors (and exceptional limit sets), this problem looked mathematically tractable. Opinion changed after the discovery of the braid bifurcation by Sotomayor (1974). Now our hopes are based on experimental exploration. We can expect the discovery of most important bifurcations of equilibria and limit cycles, elucidation of new universal bifurcation sequences, and a growing insight into the pathways for the onset of chaos (Chapter 29).

4. Global Bifurcations

Dynamical bifurcation theory, up to the present, has been concerned mainly with the study of local bifurcations. When the control parameters are extended over large ranges of values, as they must be in most applications, we get *global bifurcation diagrams* (Chapter 29, Figure 56). The outstanding problem here is to find generic properties and topological restrictions for these diagrams. A historic first step was taken by Mallet-Paret and Yorke (1982). The invariants they discovered for "snakes" of limit cycles may be extended to families of tori and chaotic attractors. Even a partial resolution of this problem may have tremendous implications for future applications, such as sociodynamics and brain theory.

5. Symmetry-Breaking Bifurcations

The discovery of generic properties of dynamical systems with symmetry, and the classification of the symmetry-breaking bifurcations, has begun only recently. For the early results, see Schechter (1976), Golubitsky and Schaeffer (1978, 1979a,b,c, 1980), Buzano *et al.* (1982), and Fields (1980).

6. Near-Conservative Systems

We identify a *conservative* subset $C(S)$ in $D(S)$, the set of all smooth dynamical systems in state space S. This consists of the canonical Hamiltonian equations of classical mechanics, each determined by an energy function. Many famous equations of physics can be described as Hamiltonian systems on an infinite-dimensional state space (Abraham and Marsden, 1978): for example, the Schrödinger equation; Korteweg–de Vries equation; Euler equations of a perfect fluid; Lagrangian field theory; and Einstein equations of general relativity. (Precise details are given in texts on mechanics, e.g., Abraham and Marsden, 1978.) However, many important applications of dynamics address dynamical systems that are not in the set $C(S)$ of conservative Hamiltonian systems, but only near it. We call these the *near-Hamiltonian systems*. For a simple example, see Holmes (1980). The applications under consideration here, in fact, will involve near-Hamiltonian systems on an infinite-dimensional state space, S.

Dissipations (that appear at any small perturbation of the dynamic away from the energy-conserving form) validate the ergodic hypothesis. This postulate seems to be correct for near-Hamiltonian systems, but wrong for Hamiltonian systems, as explained by Smale (1980). Peixoto (1962) first publicized the problems of near-Hamiltonian systems. How can it be that near-Hamiltonian systems appear (approximately) to conserve energy? In other words, can the ergodic hypothesis and the law of conservation of energy both be derived from dynamics, or not? I am not going to hazard a prediction for the resolution of this problem, but I can imagine that it may be related to another hard one, called geometric quantization.

7. Infinite-Dimensional State Spaces

This problem consists of the extension of the preceding ideas to the context of partial differential equations of evolution, viewed as "rough" dynamical systems on infinite-dimensional state spaces. Two techniques have been described above: reduction to finite dimensions (to recover the topological exclusion principles of "snakes"); and direct mathematical assault on the properties of the rough system. It is at this point that we could imagine a renaissance of the art of applied mathematics.

8. Macroscopic Observation

The identification of a chaotic attractor in a real machine, in a simulation device, or anywhere else in the phenomenal universe—in the sense of matching an observation of "noise" to the taxonomy of attractors—is very restricted by the limitations of observation: ignorance (i.e., projection down to a few dimensions); and error, as described above. Thus, even if we someday recognize that an extraterrestrial radio source has generated a signal and not noise, will we be able to decode it as a sequence of symbols, i.e., as a bifurcation sequence of chaotic attractors? It seems likely that the classification of macrons (lower-dimensional projections of attractors) by dynamical topology will follow closely the classification of the attractors themselves. Some progress has been made (Froehling *et al.*, 1981; Frederickson *et al.*, 1983; Russell *et al.*, 1980; Takens, 1980).

The problem of errors seems unsolvable, limited as we are by the uncertainty principle. Thus, the theory behind the preceding problems needs to be recast in a *decidable scheme*, as described in Abraham (1979). An important step in this direction is found in Shaw (1981). What is needed here is a precise *theory of observation*: the relationship between the ideal mathematical model—the reversible, microscopic, dynamical system with its numerous dimensions and its allowable states, attractors—and the low-dimensional, irreversible, *macroscopic* world, with its observed macrons, finite-state memory media, and so on.

9. Parallel Coupling

The coupling of two dynamical systems is a perturbation of the Cartesian product of the two systems. Elsewhere (Abraham, 1976b) I have introduced a version of this, called *flexible coupling*, in which the perturbation depends upon control parameters. This version is obviously appropriate for many applications, and also for Thom's "big picture" concept of *unfolding* an unstable system into a stable family. We call the full unfolding of a coupled system by flexible coupling a *parallel coupling*, in a generalization of the entrainment of coupled oscillators (Chapter 29). To understand fully the possible results of such a coupling process, it seems sufficient to couple the attractors of one to the attractors of the other, in pairs. The bifurcation portraits obtained by coupling any two attractors may be viewed as a sort of *multiplication table* for attractors. The full determination of this table is the *parallel coupling problem.* For example, a point "times" a point is a point (no bifurcations); a point times a cycle is a cycle; but a cycle times a cycle is a *braid bifurcation,* as the unfolding of the coupled oscillators is complicated by entrainment (Chapter 29). This result is the first nontrivial example of the parallel coupling problem. Circle "times" chaos and chaos "times" chaos are open problems, perhaps important in forced oscillations, in the behavior of Langevin's equation (Haken, 1977), and in future applications in which noise becomes recognizable as signal. These instances may provide an understanding of the massive coupling and entrainment (coherence) phenomena of biological and social interaction.

10. Serial Coupling

Charting the behavior of complex dynamical systems is the newest problem of this list. Probably the first step will be experimental work, primarily to discover the basic properties of serially coupled hierarchical systems. The simplest case, two dynamical systems with controls, with a linkage map from the output projection of one to the controls of the other (Chapter 29, Figure 68), involves a "generic coupling" hypothesis for the linkage map, problems of observation on both levels, and new questions of entrainment between levels.

There are miles to go before we sleep. . . .

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