1 CATEGORIES OF DYNAMICAL MODELS

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HISTORICAL INTRODUCTION

In 1966, my manuscript [10] was nearing completion. Its Conclusion [1], inspired by Duhem [14] and Velikovsky [22], was an essay on the stability of the solar system.[1] During the writing, a package arrived from Rene Thom. It was the first few chapters of his own manuscript.[2] Subtitled "An Outline of a General Theory of Models," it diverted me to this subject, and my Conclusion expanded accordingly. Despite these ambitious early efforts, little has passed to clarify the strategies of dynamical modeling, save the appearance of an increasing number of examples of the art.[3] In this paper, we will try again to move towards a general theory of dynamical models.

Since Newton created dynamical models, there have been two catastrophes in the basic paradigm. From Newton to Poincaré, a simple strategy guided all modeling. This strategy, which we shall call the classical quantitative scheme, evolved to suit the needs of the physical sciences. This classical scheme, clearly defined by Duhem [14], has been the subject of most of the literature on models in the philosophy of science.[4] It is described in detail in Section B1.

The crisis of celestial mechanics at the end of the last century (see [10] for the whole story) prompted Poincaré to create a new paradigm, which we shall call the modern qualitative scheme, in 1882. This was the subject of the Conclusion [1] and is also described in Sec. A5 below.

The needs of theoretical biology led Thom to create a new paradigm, which we shall call the dynamical bifurcation scheme, in 1966. His book [21], although now known primarily as the source of Elementary Catastrophe Theory, contains the more general General Catastrophe Theory (that is, the dynamical bifurcation scheme) and more general schemes as well. He related these schemes to morphogenesis in the abstract. Although there are exciting applications of the dynamical bifurcation scheme throughout the sciences, convincing models for biological morphogenesis have not yet appeared.[5] We review the dynamical bifurcation scheme in Section C3.

The complex bifurcation schemes introduced recently [8] are ramifications of Thom's simple bifurcation scheme, designed particularly for physiological modeling. Hopefully, they will be applicable to biological morphogenesis as well.

This paper is a tutorial on these schemes. Their applications in physiology will be discussed in a later paper. In Part A, we explain exactly what is meant by a scheme. In Parts B and C, we review the historical schemes. These are extended, in Part D, to the complex schemes which have been developed for physiological modeling. Our primary goal is to make these schemes, or dynamical modeling strategies, sufficiently

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1. This is not included in the second edition.
2. Now available in English [21].
3. The only mention of these events in the literature of the philosophy of science which we have come across is in Garfinkel ([16], p. 31).
4. See Hesse [18] and Garfinkel ([16], p. 170) for example.
5. For a summary, see Rosen [19].

clear that the reader may follow them in making new models. Secondarily, we hope to advance toward a general theory of models. Eventually, this would provide a synthesis of general systems theory, control theory, homeokinetics, dissipative structures, industrial and urban dynamics, and related concepts.

A. DYNAMICAL MODELS, SCHEMES, AND TYPES

A1. What Is a Model?

The sparse literature of the theory of models is troubled by this question. We wish to dispose of it at the outset by means of a naive map of the noosphere introduced earlier in this series [7].

What is the difference between a train and a model train? Between a solar system and an orrery? Between a Toyota and a toy auto? Besides having differing mechanisms, the two elements in any of these pairs of analogues differ in their degree of susceptibility to our influence. If we somehow made a mathematical model of the entire phenomenal universe, in which a particular phenomenon was represented by a point, we might imagine a real-valued function defined upon it, which is proportional to this susceptibility. This function would have a higher value at "model train" than at "train." We will call this a reality function. The level sets (contours) of this function would consist of equi-susceptible phenomena. We will call them levels of reality.

And we further may imagine our model to include the noosphere, or noumenal universe, with its own reality function. We imagine this as one model, and one reality function, with all noumena higher than any phenomena, and a gap in between. Thus, there is room in this vision for "train," "model train," and "mathematical model of a train," in an ascending sequence.

To simplify discussion, we may further imagine that the levels of reality have been discretized into a finite set. At the lowest level are the most intractable phenomena, such as "cosmos" or "solar system." These we call hardware. Up a level we find "orrery," "physics lab," "stirred chemical reacter," "meristem culture dish," and so on. These we call labware. All of these lower levels comprise phenomena. So a bit higher, we find software, or programs for labware (whether real, analog, or digital). Higher yet, there are "Schwartzschild universe," "Lorenz attractor," and other mathematical objects, or etherware. And at the top level, we may find "armchair experiment," "cognitive model," "scientific theory," and other flights of pure fantasy and science fiction, or knoware. See Figure 1. As we have noumena and phenomena in one bag here, we call the things in the bag objects, as in "mathematical objects" or "objects of thought." We are not trying to outrage Kant, but simply to create a practical framework for the working dynamicist.

Now should we observe objects on two different levels which are analogous, according to some pattern matching process among the cognitive skills of our collective nous, we say the higher one is a model of the lower one, the domain. This may occur by design or by chance. Accidents might occur because of some higher design, or concordance, as described earlier in the series [7]. In any case, we shall use the word concord for this perceived quality of analogy instead of analogy, metaphor, model, isomorph, etc. By modeling we shall mean the assumption of concord, especially between a well-known object and a poorly understood one, for the sake of study. In particular, dynamical modeling will denote the architecture of a mathematical model concordant with given dynamical phenomena.

The concord of two objects, as far as science knows at present, is observable solely through the medium of the human brain. The function of modeling (as defined above) is a cognitive one. It may be that we must be able to observe a concordance across several levels of reality in order to understand our surroundings in the phenomenal universe. This is the essence of theory. In any case, science is the pursuit of concordance across several levels, and mathematical modeling is part of this essence. Our goal here is to provide explicit schemes, or strategies, for dynamical modeling.

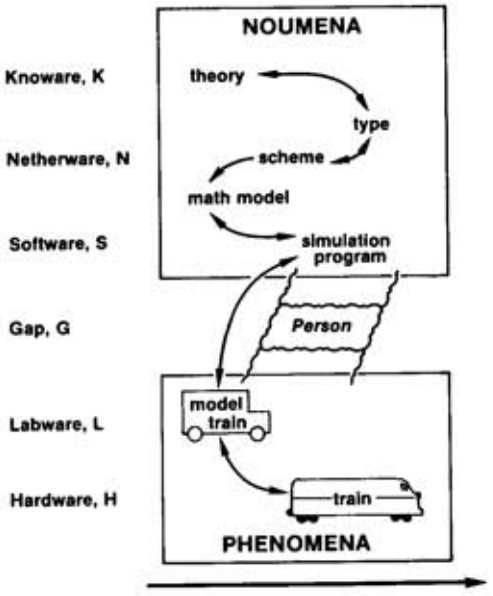

Fig. 1

So far, we have tried to give our naive answer to the question: what is a model? It is hard to be more precise without resorting to the concept of a scheme. We will outline it in the next Section.

A2. What Is a Scheme?

To Duhem,[6] a physical theory consisted of an experimental domain, D, together with a mathematical model, M, connected by a conventional interpretation, A. This connection is to be regarded as a contract, agreement, or accord, between discussers.[7] This accord identifies terms of D (especially observable physical parameters like voltage) with functions or variables of M. This picture of a theory is symbolized in Fig. 2.

In this context, Duhem identified two useful properties of a theory: adequacy, and stability. Adequacy refers to the quality of concordance between the predictions of M and the observations of D, as identified by A. Stability means the insensitivity of the predictions to changes in the model. In 1966, we tried to make precise these ideas of Duhem, and particularly these two useful properties, in a mathematical context appropriate to the physical sciences.[8] In this formalization, a problem develops.

First of all, as we have already described at length what we mean by a model, we shall use the word scheme for a mathematical object intended for modeling an unspecified phenomenon, or a whole family of virtual phenomena, of which an actual phenomenon may correspond to a single member. Schemes will occur in different types, which

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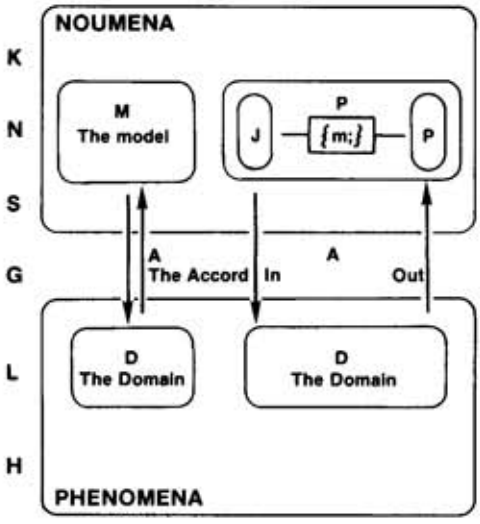
6. This idea must go back centuries, but we find a particularly clear discussion in Duhem [14]. As far as we know, this is the first use of a scheme in this context. This is summarized in the Conclusion [1] in mathematical terms, with explicit examples from celestial mechanics.
7. It is tempting to describe this in the language of categories and functors. In this account, we will be satisfied with a naive description.
8. See the Conclusion [1] for precise statements, and the illustrative models for the solar system.



(a) Singleton (b) Family-of-Singleton Fig. 2

we will not try to enumerate at present.[9] For example, Duhem's picture of a scientific theory, shown in Fig. 2(a), is a scheme of the singleton (S) type. It consists of two objects (a noumenon modeling a phenomenon) and a conventional interpretation.

But stability is not a property of a model, but of a scheme. It has nothing to do with the experimental domain. Further, it is not a property of an isolated scheme of singleton type, but only of a scheme belonging to a family of schemes. We wish to regard such a family as another type of scheme. We propose to call such a family a scheme of the family-of-singleton (F/S) type.[10]

Thus, Duhem's idea of a scientific theory belongs in the context of a particular scheme-type, which we shall now describe in more detail.

In a scheme of the family-of-singleton (F/S) type, there must be (among other things) two topological spaces. One of these, J, is a geometric model of the observed states of the experimental domain or device,. The other, P, corresponds to the predictions of the models. For each point j of J there is a model M(j), and a prediction of M(j), which is a point p(j) of P. Thus the predictions of the family of models $\{M(j)\}$ comprise a function $p: J \to P$. A model M(j) of this family is stable in the sense of Duhem if p is continuous at j. This scheme is shown in Fig. 2. Note that the accord, A, is represented here in two parts: input,[11] In(A), and output or observation, Out(A). The adequacy in the sense of Duhem of a model in this scheme becomes a mathematical question. For the observations and the predictions belong to the same topological space.[12]

This scheme-type of Duhem is sufficient for celestial mechanics, Thom's theory of abstract morphogenesis, and many important applications. But it is too limited for

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9. The types we know so far may all be described as categories.
10. Readers familiar with category theory will recognize here a functor: any type may be extended into a larger type, through the functor family-of, which is a universal construction.
11. This corresponds to the preparation procedure of Thom ([21], p. 15).
12. The two functions p and Out • D • In end up in the space P (D denotes the experimental process). Thus adequacy is defined by a neighborhood of the diagonal (uniformity), in the product space $P \times P$.

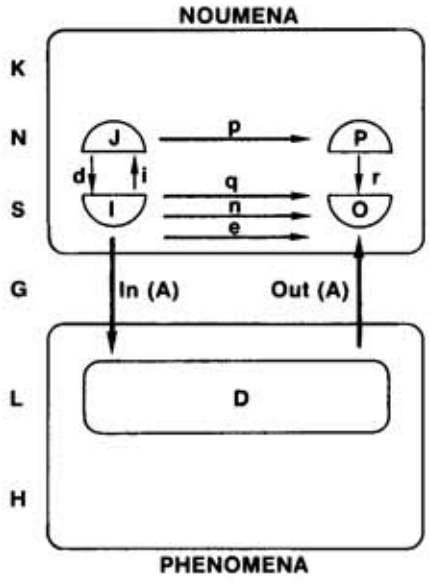

Fig. 3

others (especially self-organization[13]). Hence we proposed the complex scheme-types [8], which are developed further in Section D of this paper.

A3. The Essentials of a Dynamical Scheme

A dynamical model is a very special type of mathematical object. It may be in concord with dynamical phenomena in a physical, biological, or social realm. For the sake of definiteness, we will consider the target (domain) of the model to be phenomena on the labware level of reality. Thus, we will speak of the lab system, or the experimental domain, and use metaphors such as initial preparation, control parameters, observation of the device, etc. This should not be construed as the entire context of this theory, however. So when we say device, we may mean equally the social system, biological organism, or whatever is the target of the modeling, even if it belongs to the lowest (least susceptible) of the hardware levels of reality.

The device and its mathematical model are to be discussed in the context of a given theory: vocabulary, cognitive strategy, rules of inference, and so on. This idea of a theory is more general than Duhem's, and is conventional nowadays. What Duhem called a scientific theory may be closer to what we call here a scheme.

Among all schemes, there is a class of particular recipes which are most suitable for modeling dynamical phenomena on the lab level (or any other more-or-less susceptible level) or reality, which we call dynamical schemes. These might occur in any scheme-type. We now enumerate the essential ingredients of a dynamical scheme.

In a dynamical scheme, we must always have four geometric models: I, J, O, P, for the virtual states of the lab system or device. These will usually be topological spaces or differentiable manifolds of finite or infinite dimension. Beyond these, there must be four functions: p, q, r, s, relating the models in the commutative diagram shown in Fig. 3, in the box labeled "noumena." Finally, there must be the device, D, or lab system, shown in the box labeled "phenomena" in Fig. 3, and two interfaces: In(A), Out(A), across the noumenal/phenomenal gap.

There may be other ingredients, as we shall see in the examples in Part B. In this section, we discuss these essentials, one at a time.

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13. For a thorough discussion of the problems, see [9].

The space J is a geometric model for the virtual states of the system, in mathematical idealization. This means, primarily, as a continuum. In practice, the idealized states are subdivided in classes, such as S, phase space, a manifold of internal parameters (which evolve during a process according to the dynamics of the scheme), and C, control space, a space of control parameters which stay constant during the processes of the device. In this case, the topological space J is the Cartesian product of the differentiable manifold S and the topological space C.

The space P is a geometric model for the predictions of the scheme, and the function p from J to P is the prediction algorithm. In a dynamical scheme, this algorithm will typically involve the integration of a vectorfield.

These complete the essentials on the etherware level. We will see subsequently that the components of a scheme on this level comprise a subscheme called the N-scheme.

In 1966, this satisfied us as a precise description of the family of models. But in the meanwhile, the vagueness of the conventional interpretation has emerged as a practical problem in the workshops of applied dynamics. So we now add to this picture a parallel scheme on the software level, called the S-scheme.

Thus we have the space I of practical inputs to the device. This may be, for example, a finite set. The function d from J to I is the discretization algorithm, a decision procedure for sorting idealized, continuous, virtual states into bins. These bins, the elements of the space I, correspond more directly to actual states of the device, which can be prepared by the experimentalist as initial states for the dynamical process of the device.

The function i is the (partial) inverse of d. It is an injection algorithm, placing the bin labels within the cellular structure of J determined by d. While the elements of I are still noumenal (that is, belonging to mathematical reality, or to the world of thought) they are closer to the reality of the lab system than are the points of J.

Likewise we have the space O of practical outputs from the device, defined with due respect to the limitations of the observational system. The function r is the reduction algorithm relating idealized predictions and virtual observations.

The function q from I to O is simply the composite of i, p, and r. It is the numerical version of the prediction function. The function n from I to O is an (optional) numerical procedure, emulating the mathematical model, and executable by a machine (or a room full of people). In any case, q and n should be very similar.

We note here that the parallel subschemes (N and S) comprise a simple hierarchical scheme, and further levels could be added when necessary. For example, Thom's idea of a reduction from a metabolic model to a static one ([21], p. 52) could be expressed in this way.

The function s from I to O is not defined intrinsically to the mathematical scheme, but simply represents on the software level the result of three other processes: In(A), D, and Out(A). The first of these, In(A), is the actual input to the device of the initial state specified by a point of I. This process, which carries information across the gap, must be accomplished (barring psychic phenomena) by a person. Thus we shall call it a personal process. The second, D, is the dynamical process of the device (organism, etc.), in lab reality. The third, Out(A), is the transit across the gap of the actual output of the process D, observed and interpreted as a point of O, by another person of the experimental team. The two personal processes, In(A) and Out(A), comprise the conventional interpretation in this scheme. The composition of the three processes defines the mathematical function s, which we call the experimental system, or sometimes, the experimental team. Its precise specification requires the accord of the experimentalists, and their perfect performance according to the agreement. Thus, it may be a little optimistic to describe the experimental system as a mathematical function. So while we informally think of these as mathematical functions, we show them on Fig. 4 as half-arrows, to distinguish them from functions.

Finally, we may remark the similarity of this picture, as shown in Fig. 3, with the schemes of Duhem, as shown in Fig. 2.

We have emphasized the bilevel structure of the mathematical scheme, etherware and software, out of a practical necessity: an object is not a model, without a functional

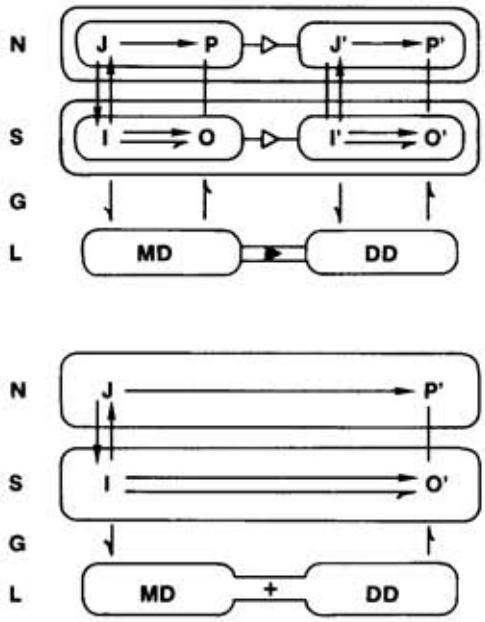

Fig. 4

prediction algorithm. At the present state of the art, this means the simulation of the etherware on an analog, digital, or hybrid computing machine.[14] For if the mathematical prediction, p, is not constructively defined, or effectively determinable, then the functional prediction algorithm will be the numerical procedure, n, which emulates it. Thus, the prediction algorithm on the software level, n, may involve a process in lab reality, and two personal transits across the gap, like the experimental system, s. And therefore n, like s, may be somewhat fuzzier than a mathematical function. So far, the theory (ambient cognitive scheme on the knoware level) has not taken this too seriously, although we have made some proposals along this line [12].

A4. A Simple Static Scheme

Besides dynamical schemes, there are many other strategies for modeling phenomenal systems. Thom [21] has described some very elaborate ones, called static models, based on singularities of mappings. Here we describe the simplest of these, which will be useful in constructing networks.

Suppose several systems, having total state spaces,

$$J_1, J_2, \cdots, J_k$$

jointly influence another system, through its control space, C. It may miraculously happen that this control space is segmented as a Cartesian product,

$$C = C_1 \times C_2 \times \cdots \times C_k$$

with one factor space for each of the controlling systems. Then the joint influence may be expressed by a set of functions,

$$f_1: J_1 \to C_1, \quad f_2: J_2 \to C_2, \quad \ldots, \quad f_k: J_k \to C_k$$

These functions may be combined in a single function,

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14. As recently as 1945, this functional prediction was carried out "by hand" by the experimental team, running around on a giant piece of paper in their socks. Thus, although personal processes were involved, no transits across the gap were necessary.

$$g\colon J_1 \times \cdots \times J_k \to C = C_1 \times \cdots \times C_k$$

defined by

$$g(j_1, \ldots, j_k) = (f_1(j_1), \ldots, f_k(j_k))$$

But more generally, the inputs may be combined by a function

$$g\colon J_1 \times J_2 \times \cdots \times J_k \to C$$

without any segmentation of C as a Cartesian product. This is what we call a static scheme. The inputs are instantaneously translated into an output by a (nonlinear) function of several variables. No dynamical evolution is required within the algorithm of the scheme.

A static scheme may also depend on control parameters, which change the algorithm of the scheme (that is, the function). This is simply a function of more variables,

$$g\colon J_1 \times \cdots \times J_k \times K \to C$$

This is a static scheme of family-of-singleton type, which will find application in Part D.

B. EXEMPLARY SCHEMES, SINGLETON TYPE

B1. The Classical Quantitative Scheme

This is historically the first dynamical scheme, created by Newton for the apple. It belongs to the singleton type. Its essential components are the following.

First, we have a manifold representing the virtual states of the device, S. The basic datum of the scheme is a smooth vectorfield, v, on S. To simplify the description, we will assume it is complete, so every integral curve may be prolonged indefinitely into the past and the future.[15] Then $J = S \times R$, and the prediction space, P, is just S. The prediction map, $p\colon S \times R \to S$, is defined by the motion of an initial state along the trajectory of the vectorfield for the prescribed time.[16] Thus, $p(s,t) = c(t)$, where c is the unique integral curve of v with $c(0) = s$. This function is continuous everywhere.

This completes the components on the etherware level, E. We may regard this as a subscheme, the N-scheme, and record its data as the object $(J, v, P, p\colon J \to P)$.[17]

The balance of the scheme, on the software level, S, involves the integration of the vectorfield by numerical methods, whether by power series or polygonal approximation, by hand or by machine. Thus, as described at the end of Section A3, I is a finite set, corresponding to cells of a partition of J. The discretization map, $d\colon J \to I$, assigns to a virtual state the label of its cell. The injection map, $i\colon I \to J$, is a right-inverse of d, defined by the specification of a preferred point, or nucleus, in each cell. Likewise, the space O is a finite set associated to a partition of P, and the reduction map, $r\colon P \to O$, is a discretization like d. As used in modern times, this classical scheme

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15. Otherwise we would take J to be the domain of the complete solution of v ([11], p. 68).
16. Alternatively, we could take, for the prediction space, P, a function space of curves in S, fibered over J by the map $b\colon P \to J$ which evaluates a curve at zero. Thus, the fiber P(j) is a function space of curves beginning at j. If H is the open half-plane of points (x,y) such that $x < y$, then P is also fibered over $H \times J$ by the map $a\colon P \to H \times J$ which assigns to a curve the endpoints of its domain, and its beginning point. We need not specify which function space, as we describe a scheme, not a model, but there are standard choices. Finally, the prediction function, $p\colon J \to P$ is defined by the maximal integral curves of the vectorfield v. This function will be continuous except at exceptional points.
17. These describe the objects of a category, for which the morphisms may be prescribed variously.

will have the optional numerical algorithm, $n: I \to O$, emulating the prediction function, $p: J \to P$, and approximating its parallel numerical version, $q = r \circ p \circ i: I \to O$.

The data of the full scheme described here comprise the object $(S, V, J = S \times R, P = S, p: J \to P, I, d: J \to I, i: I \to J, O, r: P \to O, n: I \to O)$. The first five components comprise the E-scheme. The components $(I, O, n: I \to O)$ comprise the optional S-scheme, a subscheme which could serve as a model without any E-scheme, as is frequently done in the computer simulation field.

B2. The Modern Qualitative Scheme

This is the revolutionary paradigm introduced by Poincaré in 1882. As in the singleton classical scheme, $(J = S \times R, v, P = S, p: J \to P)$, the basic datum is a single vector-field, v, on the virtual state space, S. This is called the dynamic of the scheme. We drop the time factor R in J, as the prediction will be not for a particular time in the future, but forever. Thus, the prediction is not a point of S, but a closed subset, the omega-limit set.[18] We imagine the phase portrait of v, behind the scenes, as a guide to the prediction algorithm. This contains an oriented, nonparameterized curve connecting each point s of S to its omega-limit set, p(s). We choose for P an appropriate space of subsets of S.[19] This completes the E-scheme, $(J = S, v, P, p: S \to P)$. Note that p is constant on the Insets of the various limit sets of the dynamic, and discontinuous between them. The extension of this subscheme to the S-level is routine and we omit the description of the discretization processes. The numerical procedure, n, is particularly problematic; see [12].

B3. The Attractor/Basin Scheme, Singleton Type

This is a slightly simplified variant of the Poincaré scheme of the preceding section, introduced recently to represent the minimum scheme in some sense [12]. Having fixed the basic datum, v in V(S), let A be the union of the attractors (suitably defined by one of the various definitions in vogue), B be the union of their basins, and X be the complement of B in S. We call A the locus of attraction, and X the locus of separation of v. The idea of this scheme is to ignore points in X, and then to proceed as in the preceding scheme. Thus we let J = B, P and $p: J \to P$ as before. Behind the scenes, we envision the attractor/basin portrait of v, instead of the phase portrait. This is particularly reasonable if X is rare, that is, of probability measure zero. For then, after discretization, S cannot be distinguished from B, as X is experimentally invisible.

C. EXEMPLARY SCHEMES, FAMILY-OF-SINGLETON TYPE

C1. The Modern Quantitative Scheme

Here we have a simple example of the functor family-of. We begin with a classical singleton scheme as in Section B1, $(J = S \times R, v \in V(S), P = S, p: J \to P)$, and a new topological space or manifold, C. This is to model the control (static) states of the target device or system. This is the classical system of first order differential equations, with coefficients depending on parameters.

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18. Actually, this is the coarse prediction. In many applications, one would want a fine prediction. This would be the omega-limit set (coarse prediction) together with some information about the dynamic on it. In the case of a limit point, there is none. For a limit cycle, one might want to predict some qualitative information, such as the period and amplitude. And for a chaotic limit set, one could ask for topological invariants (equivalence class of the limit dynamic, entropy, fractal dimension) and perhaps some qualitative information, such as the characteristic exponents, amplitude, etc. But we will not formalize this variation here.
19. For example, the set of all subsets of S, with the Hausdorff pseudometric topology induced by a metric on S, see ([11], p. 515).

Let V(S) denote the space of smooth vectorfields on S. The basic datum, replacing the vectorfield v of V(S) in the singleton scheme, is a function, $F: C \to V(S)$. Like the metabolic field of Thom ([21], pp. 40, 46, 52; [23], pp. 293, 616, 633), this assigns a dynamical system (vectorfield) on S to each point of C. We will prefer to call this the dynamical field, or just the dynamic, of the scheme.

The prediction space is again S, and the prediction p(c, s, t) is the position at time t along the trajectory of F(c) beginning at s. The complete E-scheme in this case is $(J = C \times S \times R,\ F: C \to V(S),\ P = S,\ p: J \to P)$. This is completed to the full scheme exactly as before, except that C must also be discretized. This is a scheme of family-of-singleton type because for each point c of C, we have a singleton classical scheme, $(S \times R,\ v = F(c),\ P = S,\ p: S \times R \to P)$. Each of these will be stable in the sense of Duhem, as the function F (and thus also p) is continuous everywhere.

C2. The Structural Stability Scheme

This idea, introduced by Andronov and co-workers in 1937, is a maximal family construction. Let S be the dynamical state space, or phase space, and V(S) a space of smooth vectorfields on S. Once again, the basic datum of the scheme is a chosen vectorfield, v, of V(S). But as one is never sure to choose the right one, we choose an open neighborhood, U, of v in V(S). Further, it is traditional to regard the phase portrait as the prediction, even though this is much richer in information than any experimental process. Thus we take J = U, P the space of phase portraits, and $p: U \to P$ assigns to each vectorfield its phase portrait. But what is P exactly?

Although we think of the points of P as phase portraits (a system of oriented, nonparameterized paths) in S, it is easiest to define P as the underlying set of V(S), with a new topology. This topology makes any two vectorfields close, which have close phase portraits in the following sense: there is a homeomorphism of S carrying one to the other, which is close to the identity (in an appropriate topology on the group of homeomorphisms of S).[20]

Thus the components of P are equivalence classes of phase portraits, and the interior of these components are the structurally stable vectorfields. The function p is continuous on these interiors. The scheme may be considered a reasonable one if p is continuous on J = U, so this space should be in the interior of one of the equivalence classes. Such an E-scheme, $(J, P, p: J \to P)$, is a structural stability scheme. This is a scheme of the family-of-singleton type, in that for each point of u of J = U, we have a singleton qualitative scheme (as in Section B2), $(S \times R, u, S, p(u): S \times R \to S)$. And each of these is stable in the sense of Duhem, as p is continuous by construction: J = U is contained in one component of P. If J is not restricted to one interior, it is a bifurcation scheme, to which we now turn.

C3. The Dynamical Bifurcation Scheme

This is essentially the scheme proposed by Thom in 1966, under the name metabolic model ([21], pp. 40, 46). We prefer to call this a dynamical model. In any case, this is in contrast to his static models, little used outside of catastrophe theory, but potentially very useful in a more general setting.

In a dynamical bifurcation scheme, we have, as in the family-type quantitative scheme (Section C1), two spaces of virtual state variables: the control space C of control (static) parameters, and the phase space S, a manifold of internal (dynamic) variables. The virtual state space, then, is $J = C \times S$. Originally, Thom chose the control space, C, belonging to space-time ([21], pp. 15, 40). Later he generalized this to an open set of Euclidean space ([21], p. 52). At some point, this was further generalized to a finite-dimensional manifold. The first explicit description of this case we have

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20. The uniform topology induced by a metric on S may be a good choice here. Thus for a positive real, r, the homeomorphisms within uniformity r of the identity define an r-disk in the phase portrait topology on V(S).

found is Zeeman ([23], p. 289). And finally, infinite-dimensional manifolds have been considered as virtual state spaces, for both static and dynamic states ([4], p. 142).

Next, we have for basic datum, the dynamical field (metabolic field, in the language of Thom), $F: C \to V(S)$, which replaces the inclusion $U \to V(S)$ of the open set U, in the preceding scheme. (Thus, the scheme of Andronov is a special case of the scheme of Thom, if we allow infinite-dimensional spaces for C.)

Finally, for the prediction, we could take the phase-portrait as in the preceding scheme (C2), or the omega-limit set, as in the singleton qualitative scheme (B2). The later is more practical, if the scheme is to be useful in the sense of Zeeman, that is, constructively predictive ([23], p. 293). Thus, let P be the appropriate space of closed subsets of S, as before (B2). The complete E-scheme is ($J = C \times S$, $F: C \to V(S)$, P, $p: C \times S \to P$). This scheme is of the family-of-singleton type, in that for each virtual control setting, $c \in C$, we have a singleton qualitative scheme (S, F(c), P, $p: S \to P$). One of these will be stable in the sense of Duhem if c is a regular point of the field: F(c) is structurally stable. Otherwise, c is a bifurcation point of F.

A useful conceptual model in this scheme is the geometric picture composed of all the phase portraits (or simpler yet, the attractor-basin portraits) stacked side-by-side over a picture of C. This is called the bifurcation diagram. The literature of dynamics (e.g., [11]) and catastrophe theory (e.g., [23]) abound in these diagrams, which contain all the predictions of the bifurcation scheme.

C4. The Stable Family Scheme

Just as the first family-type scheme was introduced to deal with the question of stability of a singleton model, the first family-of-families scheme was introduced, by Sotomayor [20], to deal with the question of stability of a family-type scheme, the dynamical bifurcation scheme.

Ignoring the technical details (which in any case are not yet firmly fixed) we have an additional control space, B, a function space of dynamical fields, $M = M(C, V(S))$, and the basic datum is a map, $G: B \to M$. Alternatively, this may be considered as a dynamical field, $F: B \times C \to V(S)$. Thus the scheme is the object ($J = B \times C \times S$, $F: B \times C \to V(S)$, P, $p: B \times C \times S \to P$). This is exactly the same as a dynamical bifurcation scheme, except for the interpretation: a derived family-of-singleton scheme, ($C \times S$, $G(b): c \to V(S)$, P, $p: C \times S \to P$) is a stable family if G is continuous at b. The tricky part, the definition of the topology of M in generalization of the phase-portrait topology of V(S) described previously (Section C2), is omitted. The general idea is due to Thom ([21], pp. 44, 320), and the details to Sotomayor [20].

D. EXEMPLARY SCHEMES, COMPLEX TYPE

D1. The Serial Coupling Scheme

Suppose we have two lab devices, and a satisfactory model for each. Then we couple them together in such a way that the first (master) is almost totally unaffected by the work it is doing in changing the control (static) state variables of the second (slave, or driven) device or system.[21] We call this a serial coupling. Now we want to construct a model for the system composed of the coupled devices. We may simply model the combined system, using a dynamical bifurcation scheme. But this ignores the important information concerning the coupling. Thus, we want a scheme with more structure for modeling the combined system, which allows the serial coupling to be modeled as well. Such is a serial coupling scheme, the first example of the complex scheme-type. In

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21. The outstanding example is the forced oscillator, discussed in the next section. More extensive discussions may be found in the literature of nonlinear dynamics [13].

this Section, we describe the serial coupling of two dynamical bifurcation schemes. Figure 4 shows both schemes for the combined device: with and without explicit coupling.

Now let MD = [C × S, F: C → V(S), P, p: C × S → P] be the E-scheme of the master device, and DD = [D × E × T, G: D × E → V(T), Q, q: D × E × T → Q] be the E-scheme of the driven device.

Here we have deliberately expressed the control space of the driven system as a Cartesian product of two spaces, D and E. This is to allow the enslavement of some controls of the driven system by the master device (those of D) while others (those of E) remain free. The coupling is now expressed by a function d: C × S → D from the total, virtual state space of the driving system to the slave variables of the driven system.[22] The effect of this coupling is to remove the control space, D, from the scheme. Thus our serial coupling scheme is the object,

$$[(C \times S,\ F,\ P,\ p\colon C \times S \to P);\ (D \times E \times T,\ G,\ Q,\ q\colon D \times E \times T \to Q);$$

$$d\colon C \times S \to D,\ H\colon C \times E \to V(S \times T),\ r\colon (C \times E) \times (S \times T) \to R]$$

or equally,

$$[MD;\ DD;\ d\colon C \times S \to D,\ H\colon C \times E \to V(S \times T),\ r\colon (C \times E) \times (S \times T) \to R]$$

where $H(c,e)(s,t) = (F(c)(s);\ G(d(c,s),\ e))(t)$, and r is the prediction algorithm of the coupled scheme: for given values of the free controls (C × E) and preparable initial states (S × T) it assigns the omega-limit set of the combined dynamic. Thus R is a space of closed subsets of S × T.[23]

Essentially, the serially coupled scheme is completely specified by the data [MD; DD; RC], where RC denotes the rigid coupling, d, because H and r are derivable from these.

On the other hand if the combined system is regarded as a simple device, the dynamical bifurcation scheme for it would be the object:

$$[C \times E \times S \times T,\ H\colon C \times E \to V(S \times T),\ Q,\ r\colon C \times E \times S \times T \to R]$$

We regard C × E as the control space in this case, and S × T as the dynamical (phase) space. Thus the prediction, $r\colon C \times E \times S \times T \to R$, is the same as in the coupled scheme above. Both are shown in Fig. 4.

This scheme forgets the full coupled structure of the combined system. It is obtained from the serial coupling scheme by a forgetful functor.[24]

So far, we have described the serial coupling scheme previously proposed for complex systems [8]. Finally, we must add to this scheme a new complication: the coupling function, $d\colon C \times S \to D$, may be changed by a parameter in an additional control space, B, belonging to the coupling itself. This is something like the static model of Thom ([21], p. 40), and is analogous to flexible coupling in the context of the conventional (parallel) coupling of dynamical systems [5]. We will call this one an adjustable coupling, as opposed to the rigid coupling of the preceding paragraph. Thus we replace the rigid coupling function, $RC = d\colon C \times S \to D$, in the scheme above, by an adjustable coupling function, $d\colon B \times C \times S \to D$. This produces the object:

$$[(C \times S,\ F,\ P,\ p\colon C \times S \to P);\ (D \times E \times T,\ G,\ Q,\ q\colon D \times E \times T \to Q);$$

$$B,\ d\colon B \times C \times S \to D,\ H\colon B \times C \times E \to V(S \times T),\ r\colon (B \times C \times E) \times (S \times T) \to R]$$

or equally,

$$[MD;\ DD;\ B,\ d\colon B \times C \times S \to D;\ H\colon B \times C \times E \to V(S \times T),\ r\colon (B \times C \times E) \times (S \times T) \to R]$$

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22. We have adopted this terminology from Haken [17].
23. Warning: this is larger than P × Q.
24. See [15] for an introduction to local category theory, which is the background for our treatment of schemes.

where $H(b,c,e)(s,t) = (F(c)(s); G(d(b,c,s), e)(t))$ and $r((b,c,e), (s,t))$ is the omega-limit set of (s,t) in the dynamical system $H(b,c,e)$. As in the rigid case, these are derivable from the essential data [MD; DD; AC], where $AC = [B, d: B \times C \times S \to D]$, the adjustable coupling.

Some symbols for these schemes are introduced in Figure 5 (see also [8]).

D2. The Canonical Example: Forced Oscillation

Here at last is an actual model. The lab devices are a robust oscillator (master device, MD) and a weighted spring or pendulum with damping (driven device, DD). These are illustrated in the figure on p. 133 of [13]. Nonlinear dynamics texts explain all one would want to know about these devices, and the result of coupling them [13]. Our formulation is a slight variation on the classical theme.

The model for the master device (MD) is:

control space: $C = R$, the real numbers, representing the angular velocity,
phase space: $S = T^1$, a circle, representing the phases of the oscillator,
dynamical field: $F: C \to V(C)$, where $F(c)$ is the vectorfield (expressed as a first order ordinary differential equation for the sake of familiarity)

$$s' = c$$

The model for the damped oscillator (DD; we choose the spring, for the sake of definiteness) is:

control space: $D = R$, the real numbers, representing the acceleration of the support point of the spring,
phase space: $T = R^2$, the Euclidean plane, representing the extension and velocity of the spring, (x,y)

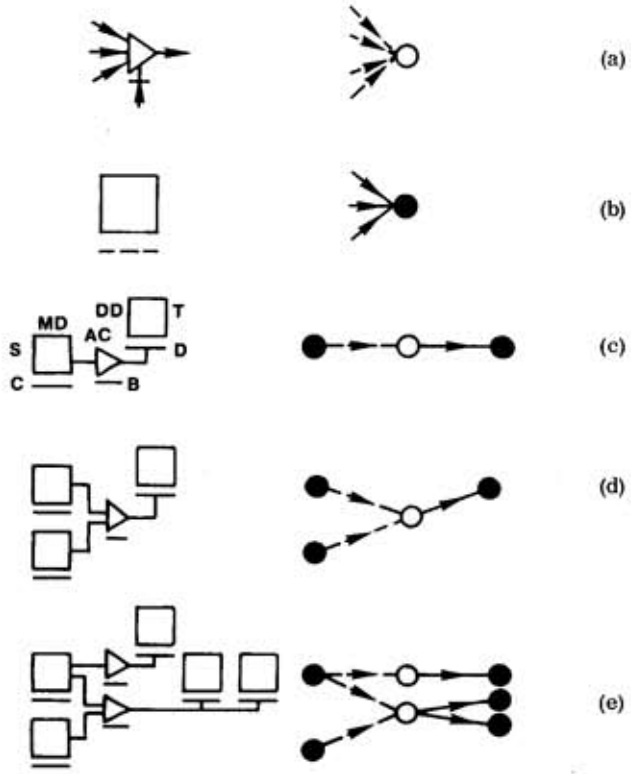

Fig. 5

dynamical field: G: D → V(T), where G(d) is the vectorfield (expressed as a system of two first order, ordinary differential equations, more or less as in the books [13], p. 212)

$$x' = y$$

$$y' = -(1/m)f(x) + ky + d$$

where f is the restoring force function of the spring, k the coefficient of friction or damping, and d the assumed uniform acceleration of the support point.

The adjustable coupling (AC) is given by:

control space: B = R, the real numbers, for amplitude,
coupling function: d: B × C × S → D, defined by

$$d(b,c,s) = b \sin(s)$$

which does not depend directly on c, the angular velocity of the oscillator. But integrating the oscillator equation at the initial condition, s = 0, we find s = ct (phase = angular velocity times time) so c controls the frequency of the forcing term,

$$d(b,c,s) = b \sin(ct)$$

so the flexibly coupled device (CD)—found by replacing d in (DD) by the function (AC)—is:

control space: B × C = R × R, representing amplitude and angular velocity of the driving oscillator (MD),
phase space: S × T = $T^1 \times R^2$, for phase of the driver, extension of the spring, and velocity of the weight,
dynamical field: H: B × C → V(S × T), expressed as a system of equations,

$$s' = c$$

$$x' = y$$

$$y' = -(1/m)f(x) + ky + b \sin(s)$$

which are the standard equations of this classical model ([13], p. 213). Other examples of serial coupling schemes, intermittency for example, have been proposed [8].

D3. Serial Network Schemes

It is now our goal to extend the considerations of the preceding section to complex systems. We might consider a fixed quiver (directed graph), with a dynamical bifurcation scheme at each vertex and a serial coupling on each directed edge. When several edges depart (spread) from one vertex, we have a plausible situation: the instantaneous state of the master device is coupled simultaneously to different driven systems, through different serial couplings, rigid or adjustable. But when several directed edges arrive (fan) to a single vertex, how can we represent the cooperation of several masters in driving a single scheme? We must have a multicoupling scheme to represent this cooperation. This is an example of a static scheme (see Sec. A4).

For example, if two masters, MD1 = (C × S, F, P, p) and MD2 = (D × T, G, Q, q), share control of a single slave, DD = (E × U, H, R, r), we must have a serial multicoupling function, e: C × S × D × T → E, to represent the determination of the driven control, e, by the instantaneous states (c,s) of MD1 and (d,t) of MD2, through the function, e(c,s,d,t). Or in the case of an adjustable multicoupling, we must have a scheme, AC = (B, e: B × C × S × D × T → E), where B is an additional control space for the multicoupling.

Then for a fan of directed edges to arrive at a common vertex (dynamical bifurcation scheme), they must converge first as inputs to a common multicoupler, from which a single output directs the target dynamical scheme. Thus, as a diagram of the network,

we must consider an odd mathematical object: a directed graph with two kinds of vertices—say green (for dynamical bifurcation schemes) and red (for static multicoupler schemes).

This diagram will be easiest to understand if we associate all directed edges arriving at a vertex with that vertex. Although not essential, it simplifies our discussion. Thus, edges arriving at a red vertex are red. This red fan represents a static multicoupling scheme, as shown in Fig. 5(a). The red edges represent the arguments of the multicoupling function, whether to be used as inputs from dynamical systems, or adjustable coupling controls. Similarly, edges arriving at a green vertex are green. A green fan represents a dynamical bifurcation scheme, as shown in Fig. 5(b). The green edges represent factors of the segmented control space (see Section A4). As segmentation of a control space is a special structure adapted to a particular multicoupling situation, it is more desirable to represent all multicouplings by a serial multicoupling function (red fan). Thus, we prefer to consider green fans having only one or two input segments. One of these is for the controlling input, arriving from the output of a multicoupling function (red vertex). The other (optional) represents free controls.

The fans are to be connected in a schematic diagram for a network. The rules are:

1. Red vertices may be joined only to the input ends of green edges.
2. Green vertices may be joined only to the input ends of red edges.

Some examples are shown in Fig. 5, both in pictographic and in schematic diagrams. Let us choose a specific schematic diagram, such as the one shown in Fig. 5(e).

Corresponding to each such schematic, there is a category of complex schemes. Each of these schemes is determined by essential data: a dynamical bifurcation scheme for each dynamic vertex (green fan), and an adjustable static multicoupling scheme for each static vertex (red fan). Thus, each schematic diagram determines a category of schemes. And all these together comprise the complex type of schemes.

Surely this seems too arduous, yet we shall see in later papers that complex systems in nature, especially in ecology and physiology, present such schematic diagrams in a very natural way.

D4. Serial Cycles and Parallel Coupling

While the coupling concept in the context of dynamical systems theory is classical, it is commonly applied only to the parallel coupling of two systems. To further clarify the serial coupling concept we have emphasized here and elsewhere [8, 9] we contrast the two in this section.

Recall that for two vectorfields, F in V(M) and G in V(N), their direct product is a vectorfield (F, G) in V(M × N), defined by

$$(F,G)(m,n) = (F(m), G(n))$$

Let P be any very small vectorfield on M × N. Then the sum (F, G) + P is a weak parallel coupling of F and G. If further P depends on a parameter, P: C → V(M × N), the dynamical field (F, G) + P: C → V(M × N) is a flexible coupling of F and G [5]. Typically, C is a neighborhood of the origin in Euclidean space, and P(0) = 0. A strong parallel coupling refers to an arbitrary (possibly enormous) perturbation of the direct product, (F, G). In other words, this could mean any vectorfield in V(M × N).

Now let us express both of these coupling concepts in the specific case of the forced oscillation model of Sec. D2. In the notation of first order systems of ordinary differential equations, we have:

The uncoupled system:

$$a' = w$$

$$x' = y$$

$$y' = -(1/m)f(x) + ky + d$$

With serial coupling, $d(a,b) = b \sin(a)$:

$$a' = w$$

$$x' = y$$

$$y' = -(1/m)f(x) + ky + b \sin(a)$$

With parallel coupling, $P = (p,q,r)$:

$$a' = w + p(a,b,x,y)$$

$$x' = y + q(a,b,x,y)$$

$$y' = -(1/m)f(x) + ky + d + r(a,b,x,y)$$

So we see that adjustable serial coupling is a special case of flexible parallel coupling. But serial coupling provides a natural way to specify the very restricted class of perturbations which occur in the serial coupling situation so prevalent in the phenomenal universe, while weak parallel coupling provides a natural way to model the nonspecific perturbations of unknown coupling mechanisms.

We may end here with a serial coupling situation which comes closer to parallel coupling in its specific form. Consider a typical network of serially coupled schemes. Very frequently, this will contain a serial cycle. That is, the schematic diagram contains a directed subgraph which is a closed cycle, as shown in Fig. 6(a). These occur naturally, for example, in the ring of cells used by Rashevsky and Turing in modeling biological morphogenesis ([13], p. 109). The serial cycle implies a very intimate coupling. The shortest case is a serial coupling from the output of a dynamical bifurcation scheme to its own control input, as shown in Fig. 6(b). Expressed in equations, this is:

MD = DD: $x' = F(c,x)$

AC: $c = c(x)$

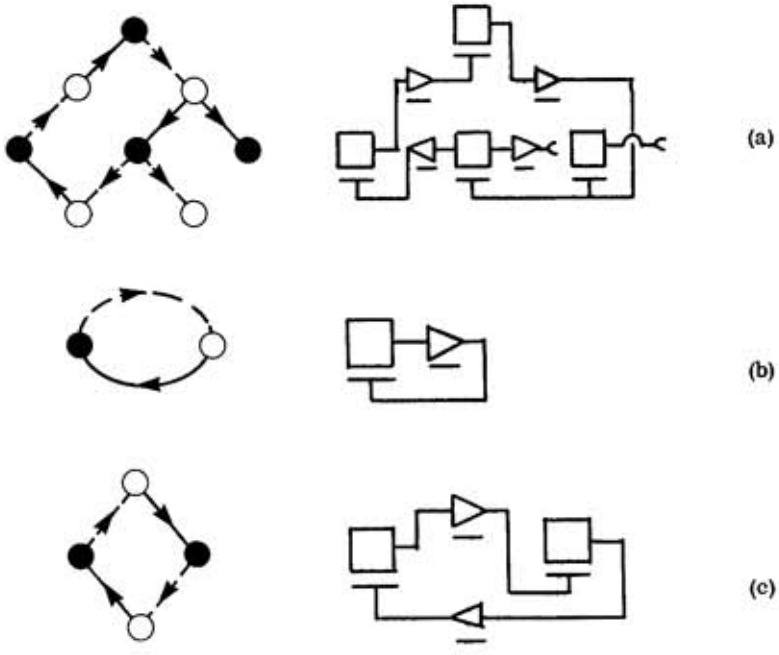

Fig. 6

Bicoupled system:

$$x' = F(c(x), x) = f(x)$$

The effect of the short feedback is to eliminate the control parameters.

Now suppose we consider the next shortest case, as shown in Fig. 6(c). In a simple version, d does not specifically depend on c nor vice versa, and both are rigid. Expressed in equations, this means:

$$D1: x' = F(c, x)$$

$$D2: y' = G(d, y)$$

$$C1: d = d(x)$$

$$C2: c = c(y)$$

Bicoupled system:

$$x' = F(c(y), x) = f(x, y)$$

$$y' = G(d(x), y) = g(x, y)$$

This is essentially a very strong parallel coupling, and the controls have been eliminated. In conclusion, serial coupling is a very useful concept, which may replace parallel coupling in many applications, giving additional structure and better modeling.

Previously, we have described a kind of multiplication table of attractors, based upon flexible parallel coupling ([5], Sec. 2; [8], Fig. O1; [9], Sec. 12). In the context of serial coupling networks, there is an analogue, the composition of bifurcations. For example, if two dynamical bifurcation schemes are serially coupled, and each exhibits a single canonical bifurcation, what are the possible bifurcation diagrams of the coupled system? As we have suggested earlier, this may be found to depend on generic properties of the adjustable coupling function, which maps the total state space of the master scheme (containing the locus of attraction) onto the control space of the driven scheme (containing the bifurcation set). The desired generic property is the transversal intersection of the locus of attraction of the master with the bifurcation set of the slave [9], Sec. 13). Here is a rich source of problems for dynamicists, fundamental for understanding the behavior of complex systems.

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