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Chaostrophes, Intermittency, and Noise

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Dedicated to René Thom

In 1972, we proposed the blue sky catastrophe for periodic limit sets. Here, we describe one for chaotic limit sets. This provides a pathway to chaos quite different from the usual ones, which are all sequences of subtle bifurcations. Further, models for intermittency and noise amplification are given, based on hysteresis loops in a serially coupled chain of dynamical schemes.

PART A. SUBTLE AND CATASTROPHIC BIFURCATIONS

The classification of bifurcations into these two types was suggested in 1966 by Thom(1972), and given explicit treatment (under the names leaps and wobbles) by the author (Abraham, 1976). In this part we have two goals: to define catastrophic borders in the control space of a complex dynamical scheme, and to discuss an example of a chaostrophe, that is, a catastrophic border for the domain of a chaotic attractor, in the context of a serial chain of three oscillators.

A1. PARTITIONS AND BORDERS

We consider a vector field depending upon a parameter, also known as a metabolic field, or dynamical scheme. Let $\underline{C}$ and $\underline{M}$ be manifolds of finite dimension, X(M) a space of vectorfields on M, and F: C → X(M) the dynamical scheme. If B(M) is the subset of X(M) consisting of structurally unstable vectorfields, then the bifurcation set of the scheme, B, is the inverse image of B(M) under F.

Imagining the phase portrait of F(c) in {c}xM for each c in C creates a control-phase portrait of F in CxM. We wish to concentrate on the attractors (in the sense of probability, for example) in this portrait, along with their basins and separators (the complements of the basins, elsewhere called separatrices). Let A denote the locus of attraction, the union of all the attractors of the scheme, and S denote the locus of separation, the union of all the separators of the scheme.

A relatively open subset of the locus of attraction will be called an attractrix. This is usally called a branch of the attractive surface in static catastrophe theory. A relatively open subset of the locus of separation, similarly, will be called a separatrix. This is also known as a branch of the repelling surface in static catastrophe theory.

We assume that the scheme, F, is generic in any reasonable sense. Specifically, it is as transversal to B(M) as possible, and over each point b in the bifurcation set, there is a single bifurcation event in the phase portrait of F(b). We see in examples that this bifurcation event normally involves a single attractrix and a single separatrix, or it involves no attractrix. Thus, B may be divided in two parts. Here, we will be interested in the attractrix bifurcations only, in which an attractrix and a separatrix are involved. Further, we will discuss only the hypersurfaces contained in this part of the bifurcation set, which we call the attractrix bifurcation hypersurfaces in the control manifold, C. And finally, these may be isolated hypersurfaces in the bifurcation set, or they may be hypersurfaces of accumulation, from one or both sides. We will refer to one of these isolated attractrix bifurcation hypersurfaces as a border, if an attractor appears or disappears during the bifurcation occuring across it. Otherwise, we call it a partition. The borders belong to the boundaries of the domains of attraction, the regions of control space in which certain attractors exist. These domains are the shadows (images in C under the projection from C x M onto the first factor) of attractrices, and the borders are shadows of boundaries of attractrices. Borders may always be oriented, by a normal vectorfield pointing toward the exterior of the region it bounds. Partitions belong to the interiors of the domains of attraction, and may radiate inward from a border. Precise definitions are given in Section A5.

A2. STANDARD EXAMPLES WITH ONE CONTROL

Specializing the preceding definitions to the case in which the control space, C, is a line or circle, yields the most important examples. Thus, F: C → X(M) is a generic arc or generic loop, the bifurcation set, B, is zero-dimensional, every point is a hypersurface, and we fasten attention upon the isolated points at which an attractrix appears or disappears. These are the borders in this context.

In case the dimension of the state space, M, is two, everything is known about the bifurcations of generic arcs. The attractrices correspond to static or periodic attractors. The separatrices are generated by the insets of limit points and cycles of saddle type. The isolated points of the attractrix bifurcation set belong to a known list of possible models, while the accumulation points (also called thick bifurcations, Abraham and Shaw, 1983) are due to a single phenomena: non-trivial recurrence on a torus.

The different types of known bifurcation obviously fall into two categories, subtle and catastrophic. The catastrophic ones are the borders, while the subtle ones are the partitions. This classification is given in Table 1. Drawings of the locus of attraction, in most cases, may be found elsewhere (Abraham and Marsden, 1978, Abraham and Shaw, 1983). An exception is solidification, a type of Hopf bifurcation, which is shown in Figure 1.

Another of these, the periodic blue sky event, will be described in detail in the next section.

If the dimension of the state space, M, is three or more, then chaotic attractorts may (and usually do) occur. The full list of these objects is not yet known, even in three dimensions. Their bifurcations, which include pathways to chaos, are just beginning to be discovered by

TABLE 1. ATTRACTRIX BIFURCATIONS OF GENERIC ARCS

CATASTROPHIC BORDERS	SUBTLE PARTITIONS
Static creation	
Static solidification	Hopf excitation
Periodic creation	
Periodic solidification	Neimark excitation
Murder	Subharmonic division
Periodic blue sky catastrophe	

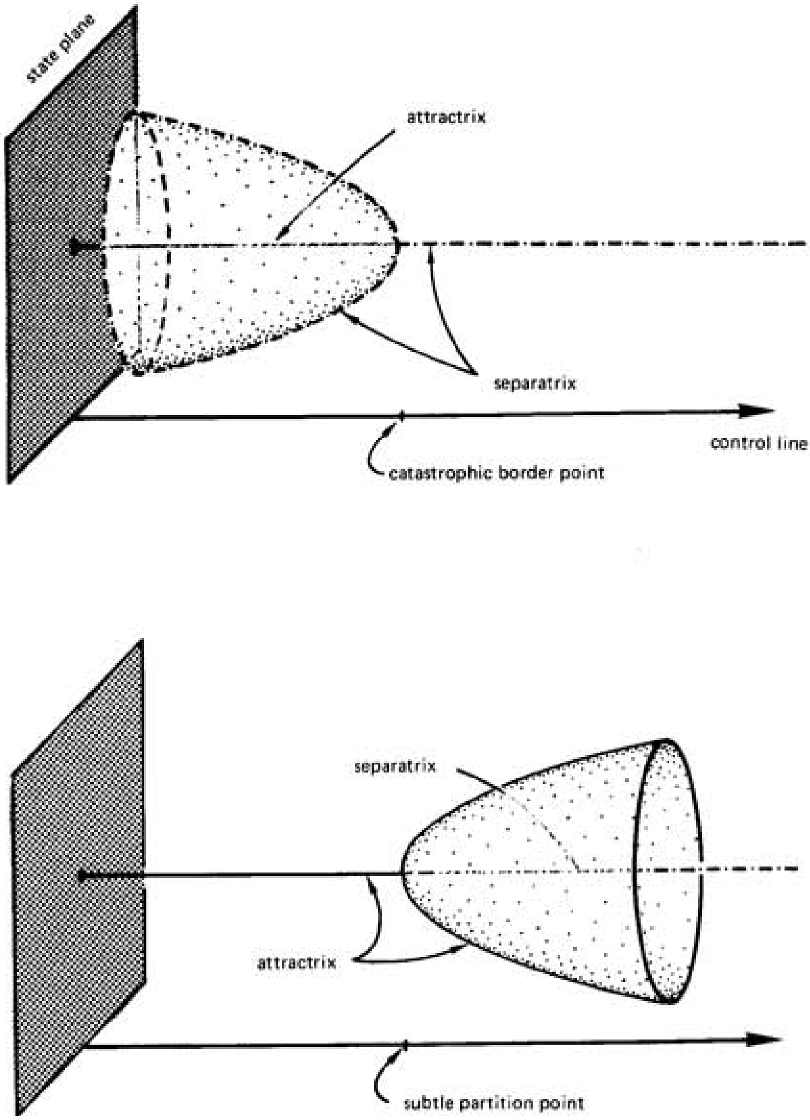

FIGURE 1. SUBTLE AND CATASTROPHIC HOPF BIFURCATIONS. The dashed curves repel, solid attracts.

experimental dynamicists. Yet we presume that these also will fall into the two categories, subtle and catastrophic. So far, the examples known are primarily of the subtle sort. (For some exceptions, see Arneodo, Coullet and Tresser, 1980; and Grebogi, Ott, and Yorke, 1982.)

A3. THE BLUE CYCLE PERIOSTROPHE.

Recall that in the dynamic annihilation catastrophe, a periodic attractor (attractive limit cycle, oscillation) vanishes. It collides with a limit cycle contained in its separator. Its attractrix meets its separatrix. In 1972, we conjectured the existence of a blue sky catastrophe, in the context of a dynamical scheme (Abraham, 1972). As in dynamic annihilation, a limit cycle would disappear into the blue sky. But in this case, it would not be cancelled through collision with another limit cycle. Instead, its period (length of its time cycle) would become infinite. It would just slow down, and cease to oscillate.

In the course of time, this conjecture was confirmed (Takens, 1974, Devaney, 1977). In the blue sky event now well known, expressed in the case in which the disappearing limit cycle is an attractor, a periodic attractor just slows down and stops. But in fact, at the moment of disappearing into the blue, it does indeed collide with another trajectory, also an oscillation of infinite period. This is a homoclinic trajectory, or saddle self-connection, associated with its separatrix. This also provides an illustration of basin catastrophe, as the basin of the periodic attractor vanishes at the moment of periostrophe, along with its (possibly unbounded) tail. The event is shown in Figure 2.

A4. THE BLUE BAGEL CHAOSTROPHE

This event will be constructed from the blue cycle periostrophe by Cartesian product with a circle plus a perturbation, to obtain a generic arc with known behavior. In other words, we perturb a blue cycle scheme with a forcing oscillation. Supposing the state space of the original scheme to be a plane, as shown in Figure 3, the forced scheme will have a solid ring for its state space, as shown in Figure 3. The plane within this ring corresponding to phase zero of the driving oscillation will be useful in our discussion. We call it the strobe plane (Abraham and Shaw, 1982).

Before the bifurcation of the forced scheme, near the border point of the original blue cycle scheme, we have an attractive torus. The torus

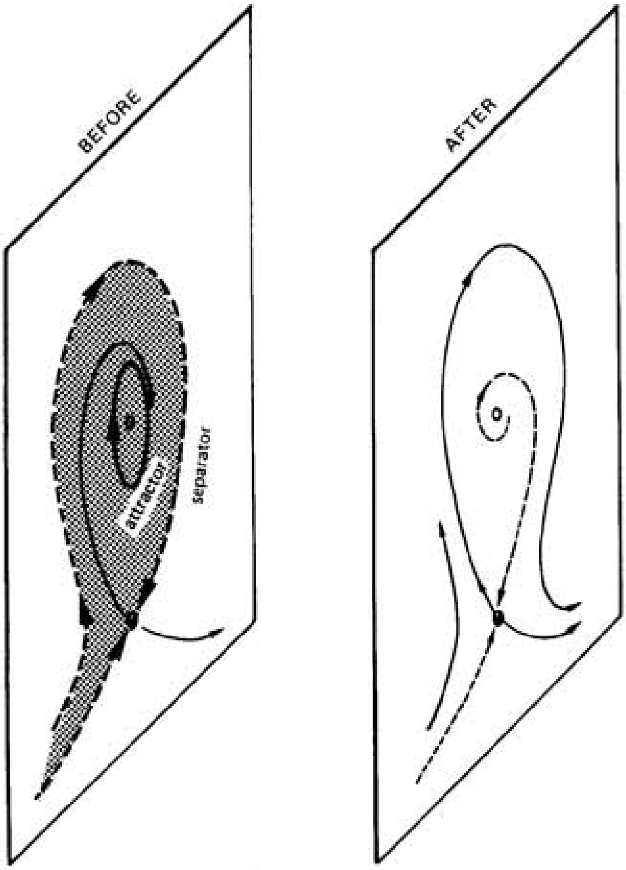

FIGURE 2. THE BLUE CYCLE PERIOSTROPHE.

meets the strobe plane in the periodic attractor of the original blue cycle scheme, the one which vanishes into the blue. This torus contains a braid of periodic attractors. Their basins, within the invariant torus, are separated by a complementary braid of periodic trajectories which are repelling, within the torus. The saddle point of the original scheme becomes a limit cycle of saddle type in the ring model (state space) of the combined scheme. The inset of this limit cycle is a scrolled cylinder, generated by the inset curve of the original scheme, visualized on the strobe plane. Likewise, the outset of the limit cycle is another, complementary, scrolled cylinder.

After the bifurcation event is over, well beyond the border point of the original scheme, the attractive torus is gone, braids and all,

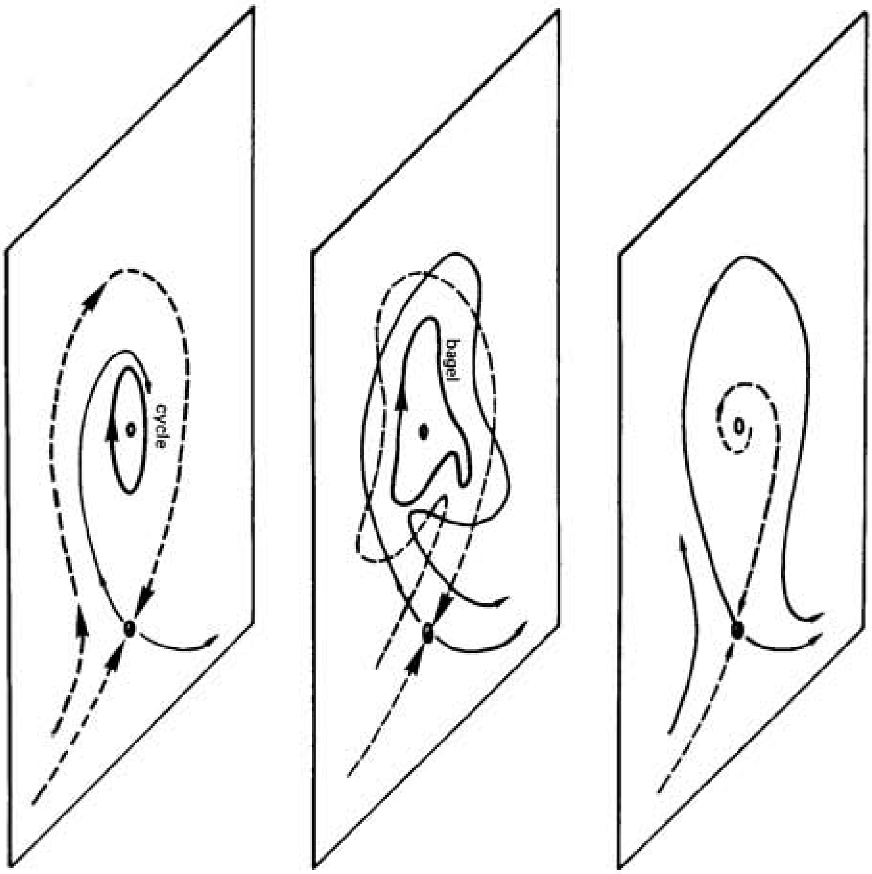

FIGURE 3. THE BLUE BAGEL CHAOSTROPHE, IN STROBE PLANE SECTIONS.

vanished into the blue yonder. The periodic trajectory of saddle type, along with its complementary scrolled cylinders, remains. But the relationship between them is reversed.

During the bifurcation event, they have crossed. The crossing does not occur at a single bifurcation value of the control variable, for we have perturbed the forced oscillation into a generic arc. Thus the inset and outset cylinders, visualized as curves within the strobe plane, must pass through each other within an interval of homoclinic transversal intersection. We call this the homoclinic interval. There are many possiblities for the prolonged passage, because of the likelihood of Birkhoff rechambering bifucations (Abraham and Marsden, 1978). One of the simplest is indicated in Figure 3. At some point in the homoclinic interval, the attractive torus disappears, braids and all.

So far, we have described aspects of this generic arc which are mathematically known. But now, we venture into conjecture:

Within the homoclinic interval, the tangled inset of the periodic saddle behaves as a repellor, and as the separator for the basin of the blue-bound attractive torus. Eventually, the torus will collide with this tangled inset. But before this happens, the convolutions of the tangled cylinder mold the torus into a form much like the chaotic attractor found by Rob Shaw. Discovered in experiments with the forced Van der Pol scheme, this object looks like a very dog-eared bagel (Abraham and Shaw, 1982). Thus, at some point within the homoclinic orbit, there is a subtle bifurcation, where the attractive torus becomes an attractive, chaotic bagel. At the final endpoint of the homoclinic interval, the chaotic bagel collides with the tangled inset, now tangent to the outset, of the periodic saddle, and vanishes into the blue.

This is an example of a chaostrophe, as the chaotic bagel attractor has disappeared discontinuously. It is also an example, in the context of three-dimensional dynamical systems, of the fractal torus crisis described for three dimensional maps by Grebogi, Ott, and Yorke (1982).

We may end this fantasy with the further conjecture, that this event may be found experimentally in the forced Van der Pol scheme.

A5. CATASTROPHES WITH SEVERAL CONTROLS

Now, we approach our second goal in this part, the definition of subtle partitions and catastrophic borders, in the general context of a dynamical scheme with several control paramters. In review, the control space, $\underline{C}$, is a finite-dimensional manifold. The dynamical scheme, $\underline{F}: \underline{C} \to \underline{X}(\underline{M})$ is assumed to be generic in a sense we have not made precise. The bifurcation set, $\underline{B}$, is a subset of $\underline{C}$. We consider a subset $\underline{H}$ of $\underline{B}$ which is an oriented hypersurface of $\underline{C}$, and which is isolated in the sense that a neighborhood of $\underline{H}$ in $\underline{C}$ intersects $\underline{B}$ only in $\underline{H}$. (In what follows, it may only be essential that the hypersurfaces be isolated on one side.) Under these assumptions, the isolated hypersurface corresponds to a single bifurcation event in the portrait of the dynamical scheme.

Finally, the hypersurface is a border if this event involves either the appearance or the disappearance of an attractor, and the hypersurface is oriented toward the exterior of the domain of this attractrix. Otherwise, the hypersurface is a partition.

We now define a generic sub-arc of a dynamical scheme, which will facilitate a more precise distinction between subtle partitions and catastrophic border in this context.

As a hypersurface of $\underline{C}$, $\underline{H}$ is of codimension one. Any curve $\underline{d}: \underline{I} \to \underline{C}$ in $\underline{C}$ (where $\underline{I}$ is an open interval of real numbers) which is transversal to $\underline{H}$ may be composed with the scheme $\underline{F}$ to obtain an arc, $\underline{f} = \underline{F} \circ \underline{d}: \underline{I} \to \underline{X}(\underline{M})$. This is a one-parameter dynamical scheme, the sort discussed in Section A2. As $\underline{F}$ is generic, so is $\underline{f}$. We may refer to such a generic arc, derived from the dynamical scheme $\underline{F}$, by composition with a curve transversal to its bifurcation set, as a generic sub-arc.

Note that the bifurcation set of a generic sub-arc cutting $\underline{H}$ transversally at the point $\underline{h}$ (at least, if it is sufficiently short) consists of the single point, $\underline{h}$. Finally, we assume the short, generic sub-arc crosses the border in the direction of the outward normal, or orientation. We call this a transverse at $\underline{h}$.

This is the auxiliary notion we need for the definition of subtle partitions and catastrophic borders in this multi-dimensional context.

Here at last is the definition. Every point $\underline{h}$ in the hypersurface, $\underline{H}$, is either subtle or catastrophic. It is subtle if, roughly, the locus of attraction is continuous over it. More precisely, the point $\underline{h}$ is a subtle partition point if every sub-arc transverse at $\underline{h}$ has a subtle bifurcation at $\underline{h}$, in the sense of Section A2. Otherwise, the point $\underline{h}$ is a catastrophic border point. We use the word catastrophe at once for the point, $\underline{h}$, in the bifurcation set within the control space, and the discontinuity in the affected attractrix.

A subset of an isolated, oriented, attractrix bifurcation hypersurface consisting entirely of catastrophe points is called a catastrophic border, or just plain border. We define subtle partition, or partition, similarly. Every such hypersurface may be decomposed into a union of catastrophic borders and subtle partitions.

So much for the definition. A better idea of the distinction between a subtle bifurcation and a catastrophe may be gleaned from the example shown in Figure 4. Derived from the Andronov-Takens (2,-) model by symmetry-breaking, it is four-dimensional (Takens, 1974; Abraham and Marsden, 1978). Both $\underline{C}$ and $\underline{M}$ are planes. So we show the tableau of sample phase portraits within each region of the control plane. The borders, in this planar control space, are the four solid curves radiating from the central point. Omitting a neighborhood of this point, they are all isolated, and involve an attratrix catastrophe. Two of these are

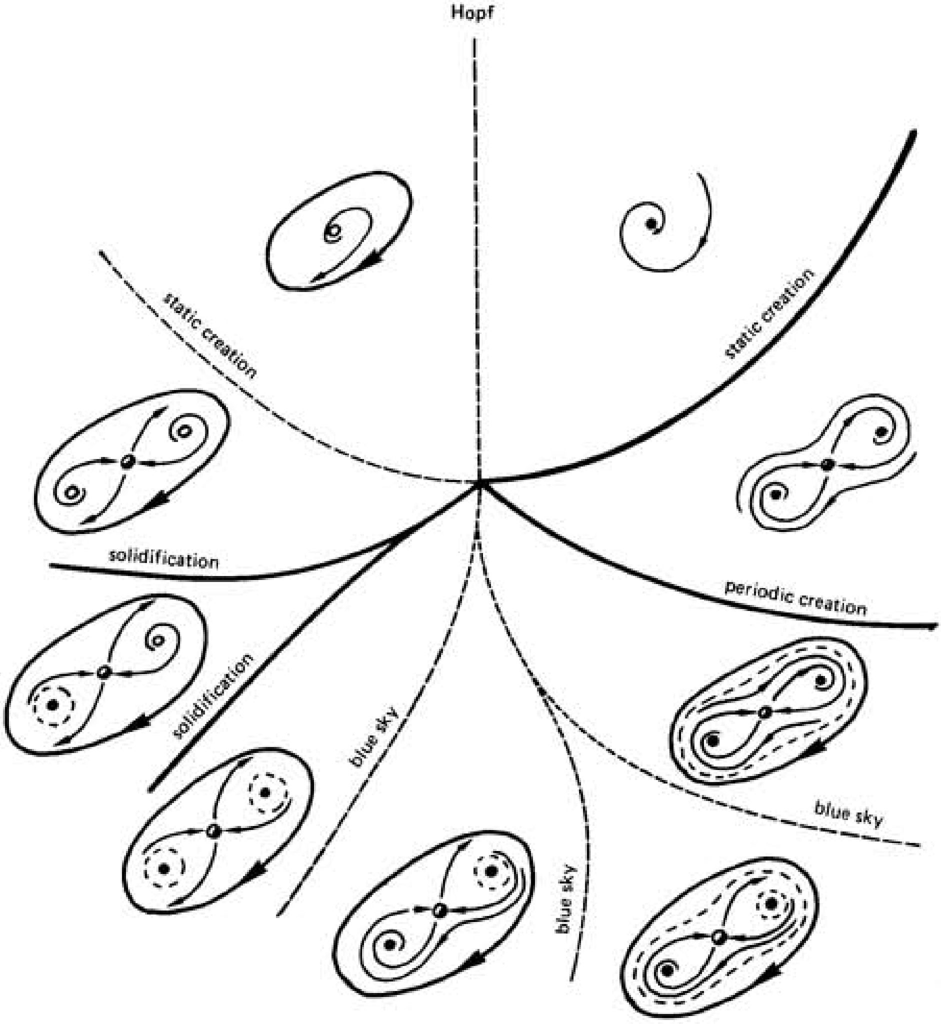

FIGURE 4. SUBTLE PARTITIONS AND CATASTROPHIC BORDERS.
Subtle partitions are represented by broken curves, catastrophic borders by solid curves.

solidifications. The other two are a static creation, and a periodic creation. The remaining five curves radiating from the central bifurcation point are subtle partitions.

PART B. HYSTERESIS, HOLONOMY AND NOISE.

Thom has written that the primary events in morphogenesis are the catastrophic bifurcations, and we share this view. In this part, we indicate some subtleties of catastrophic borders in the context of serially coupled chains of dynamical schemes. In particular, a chain of three oscillators serves as our standard example.

B1. HYSTERESIS WITH ONE CONTROL

Hysteresis refers to the failure of a system to return to its original state, after a temporary change of its controls. We may interpret this in the context of complex dynamical system theory (Abraham, 1983a) as follows.

We consider three simple dynamical schemes, the output of one determining the control parameters of another, in a chain. This is an example of a serial chain, and is shown in Fig. 5(a), with the standard convention of complex dynamics: the solid dots represent the component schemes, while the hollow dots denote the serial coupling functions. We will call the first scheme the master controller, and the whole serial chain driven by it the slave chain. This is exemplified by the classical model of Lord Rayleigh for forced oscillation, in which both systems of the slave chain are running in periodic attractors. We assume now that the master controller is an oscillator, relatively slow with respect to the dynamics slave chain.

This forced oscillation is an hysterical, (or, each cycle is an hysteresis loop) if the slave scheme (as a coupled dynamical system) does not return to the same attractor after each period of the forcing oscillation. Hysteresis is characteristic of serial chains.

We now illustrate this phenomenon in a system with a single master control. An early example, found by Duffing in Lord Rayleigh's model for forced oscillations of the damped harmonic oscillator, is shown in Figure 5 (Abraham and Shaw, 1982). This is a periodic version of a configuration common in elementary catastrophe theory, which may be called the periodic double fold.

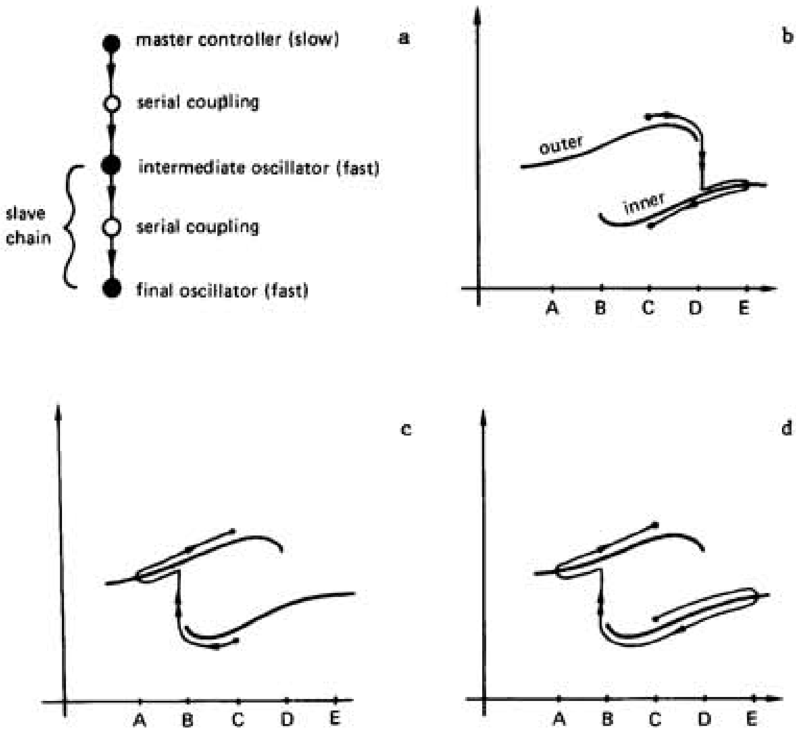

FIGURE 5. HYSTERESIS LOOPS OF DUFFING IN A THREE COMPONENT CHAIN.

This is the portrait of the slave chain, a forced oscillator. Here there is one control, corresponding to the frequency of the intermediate forcing oscillation. This control is to be determined by the master oscillator. The attractors are periodic, the separator is periodic, the bifurcation set in control space consists of two points (B and D). Both borders, the two attractrices (solid surfaces) overlap in the interval between these points, and in this interval they are separated by the separatrix (shaded surface).

Here are some exemplary hysteresis loops. (1) If the coupled (slave) oscillator is on the outer attractrix (oscillation) over control C, the cycle CEC in control space will leave it on the inner surface, as shown in Fig. 5(b). (2) Starting again on the inner attractrix over point C, the cycle CAC will return the slave system to the outer oscillation, as shown in Fig. 5(c). (3) The compound control cycle CECAC will not change the outer oscillation, but will change the inner oscillation to the outer, as shown in Fig. 5(d). We consider this compound cycle to be a hysterersis loop also.

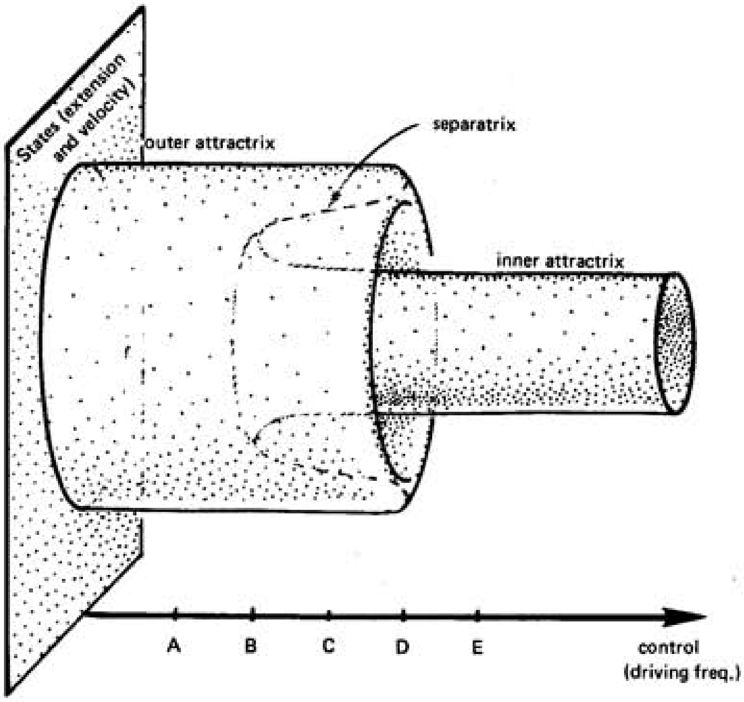

FIGURE 5 (cont.)

The maps from the set of attractors over C into itself, determined by all hysteresis loops at C, comprise what we have called (Abraham, 1983b) the holonomy monoid of C. It is clear in this example that the parameterizations of the hysteresis loops do not affect the holonomy maps. But in most cases, parameterization does affect the holonomy, as we shall see in Section B3.

B2. HYSTERESIS WITH TWO CONTROLS

The simplest example of hysteresis with two controls (two-dimensional control space of the driven system) is provided by the static cusp catastrophe of elementary catastrophe theory. This is essentially three-dimensional, and thus is easily visualized. A periodic version, the periodic cusp catastrophe, is essentially four-dimensional. One three-dimensional section is identical to Figure 5. But viewing the strobe-zero plane of one of the periodic attractors in place of the planar state space, we obtain another three-dimensional section of the four-dimensional diagram, as shown in Figure 6. Here, the borders are the two curves of

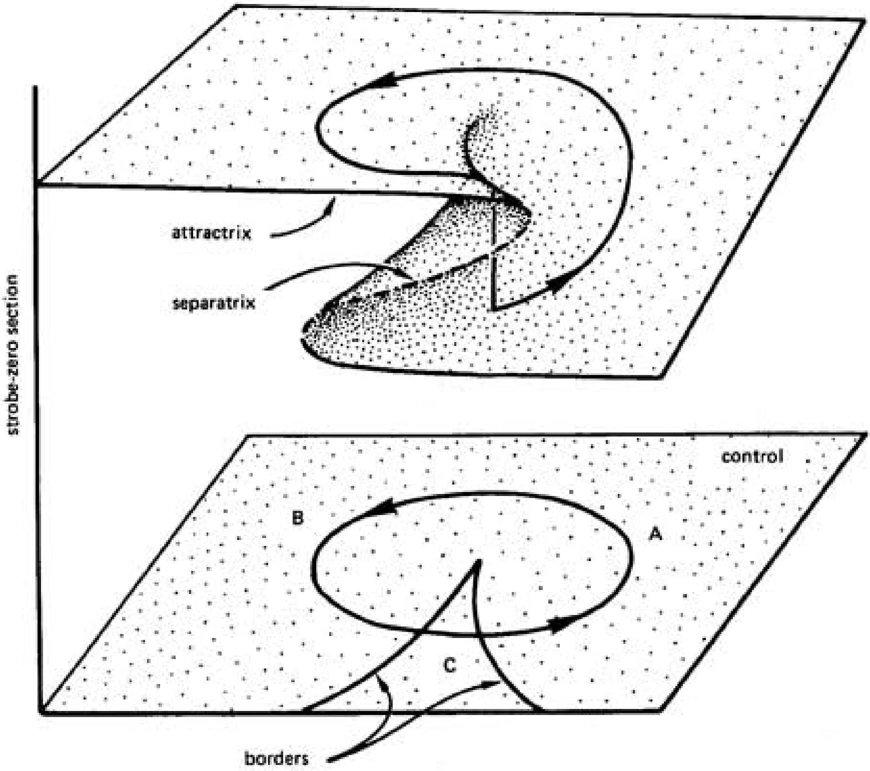

FIGURE 6. PERIODIC CUSP CATASTROPHE.

the cusp, in the control plane. The attractrices, actually three-dimensional hypersurfaces in four-dimensional space, appear in the section as surfaces.

We consider a closed curve in the control space, as shown, oriented CABC. This may be the image of a periodic attractor of the master system, a forcing oscillator. As this path crossed borders, it is a hysteresis loop. Its holonomy map takes the lower oscillation attractrix to the upper one, which is left fixed by the map. As in the preceding section, the parameterization of this path does not affect its holonomy.

This diagram occurs in Lord Rayleigh's model for the forced oscillator, as Duffing discovered, if the amplitude of the intermediate oscillator is controlled by the master, as well as its frequency. Typical output, represented as a time series of a single state variable of the final, driven oscillator, is shown in Figure 7. An interesting application to memory has been made by Zeeman (1977).

FIGURE 7. TIME SERIES OF AN HYSTERICAL CYCLE.

B3. STOCHASTIC HOLONOMY

Here is another example of hysteresis with two controls. It is extracted from Figure 4. This is the portrait of another forced oscillator scheme, which will again play the role of the slave system. Adding a closed curve to the control space, enclosing the central point of bifurcation, we consider the holonomy of this curve. A master oscillator may be imagined, driving the controls of the slave system around this closed curve. This time, the parameterization of the curve will affect the holonomy map. For ease of visualization, we replace the control space by this curve, so that the restricted diagram becomes three-dimensional. That is, we visualize the Cartesian product of the planar state space and the control cycle. Ignoring subtle bifurcations, the result is shown in Figure 8. (See also Abraham and Shaw, 1983.) Beginning at point B on the control cycle, there is a single attractor, a point. Moving counterclockwise, this soon becomes periodic, by a Hopf bifurcation. But this is subtle, we pay no attention. Later, there is a static creation event, but our attractor is not affected. Two repellors and a saddle are created, all static.

At last, there is a border at point x. One of the point repellors has a Hopf bifurcation, creates a nearby periodic repellor, and itself becomes a point attractor. This is a solidification. So at point C, there are two attractors in competition, one static and one periodic. Soon, at point y, the other point repellor solidifies also. Now there are three attractors in competition, at point D.

There follow three blue sky bifurcations. But these involve the appearance out of the blue of periodic repellors. So, although they are catastrophic, no attractor is affected. We ignore these also.

At point z, there is a periodic annihilation, and the periodic attractor disappears. This is the crucial event in this example. At

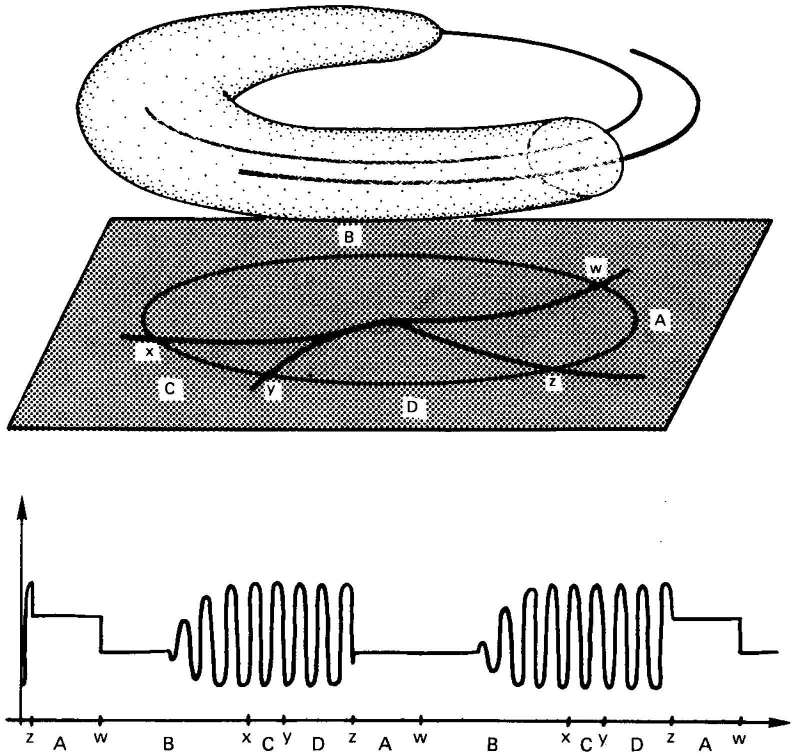

FIGURE 8. A STOCHASTIC HOLONOMY.

point A, there are only two attractors in competition, both static. The basins are intertwined teardrops, like yin and yang.

At point w on the control cycle, there is a static annihilation catastrophe. One of the static attractors collides with its separator and is cancelled. One point attractor is left, and we are back at the beginning of the cycle, at point B.

Finally, we compute the holonomy of this curve, but beginning at point A. We may start with either of the static attractors existing at this point, which we shall call yin and yang. At the end of one period of the driving cycle, the driven system will be in one or the other of these states. The same result will be obtained, no matter which attractor we start with, because at point B there is only one attractive state anyway. Thus the result of one control cycle, yin or yang, defined the holonomy of this closed curve, ABCDA. Which will it be?

All is simple until we reach point z. Just before this time, the system must be in the periodic attractor, even though two new attractors, the points yin and yang, have been born. At the moment border z is crossed, the periodic attractor disappears. At this moment, the driven system is in a well-defined phase of its oscillation, and thus in some computable point in its planar state space. This point is either in the basin of yin, or it is in the basin of yang. We discount the possibility that it is in the separator at that moment, as this has probability zero. If it is in the yin basin at border crossing time, it will stay in the yin basin until the next border crossing. Thus back at point A, after a full cycle of the driving oscillation, it is still yin, and the holonomy of the closed curve is yin: both yang and yin are mapped to yin.

We see that the holonomy is computable in this example, and that it depends critically on the phase of the driven oscillation at the z-border time. Thus, a slight change in the parameterization (not necessarily changing the period of the forcing oscillation) can switch the holonomy of the curve from yin to yang!

But what would be the holonomy in case the attractor vanishing into the blue at phase z was chaotic, instead of periodic? In this case, which must occur very frequently in applications, we may only hope to calculate the probability of the yin and yang results of a holonomy, based upon the measure of the intersection of the disappearing chaotic attractor with the basins of the non-disappearing attractors. These probabilities will obviously be independent of the parameterization of the closed curve. They depend only on the borders crossed. This is what we mean by stochastic holonomy. It applies equally to the preceding example, in which the attractor vanishing at z is periodic. The stochastic holonomy, averaged over the attractor, apparently depends continuously on the parameterization of the curve. This is more useful than the exact holonomy, which depends hypersensitively on the curve.

B4. MODELS FOR INTERMITTENCY AND NOISE

We have already seen, in Figure 7(b), a time series exhibiting a typical example of intermittency. This is characteristic of serial chains, as we have indicated. In this case, the model behind the time series is made of three oscillators, serially coupled in a chain. The slave chain in this case, a forced oscillator system, is characterized by the periodic cusp catastrophe of Duffing. A related phenomenon, shown in Figure 7(a), differs in that the master cycle has been deformed, through a nontransversal intersection with the borders, so as to significantly change the holonomy of the hysterical cycle. Through training, we may learn to recognize holonomy from the observed behavior of a system, and thus to create a suitable complex dynamical model for it. We will discuss a few variations on this scheme, to give four examples of this modeling strategy.

A. In the situation illustrated in Figure 7(a), let the periodic attractor of the master system be replaced by a chaotic attractor. Then the transitions between the two periodic states of the slave chain (forced oscillator) will become aperiodic in time. The time series will appear noisy, and its spectral analysis will reveal the spectrum of the master system, with the discrete spectra of the two periodic states of the slave system superimposed. The distribution of power between these two discrete spectral sequences will be indicative of the stochastic holonomy of the hysterical master macron: the master attractor, imaged in the slave control space by the serial coupling map. This is a strategy for modeling systems displaying intermittency.

B. Suppose that without changing its stochastic holonomy type, the macron is shrunk to a very small object near the cusp. At the same time, we will imagine the two attractrices of the slave portrait to be periodic motions of the slave system of very disparate amplitudes. In this situation, the noise of the master system is amplified greatly by the slave system. This is a strategy for modeling nonlinear noise amplifiers, which contribute their own periodic sequences to the output power spectrum, but do not otherwise change the noise charateristics of the input signal.

C. Suppose the slave system is made chaotic, so that while the cusp portrait still applies, each attractrix is the locus of a chaotic bagel. The borders, comprising the curves of the cusp in the control plane, are blue bagel chaostrophes, as described in Section B4. Then periodic input from the master system is amplified to periodically intermittent noise. A long time series might permit the recognition of the bagels from their characteristic power spectra, if the master oscillation is known. However, if the master system also becomes chaotic (through the action of a fourth system on its controls, a four-component serial chain), for example by a thick bifurcation resulting in a Rössler band, then the analysis of the component systems from the output time series could be hopeless. Still, exploration of serial chain behavior through fast simulations could provide enough experience to enable a strategy to evolve for modeling complex systems such as physiological or ecological networks.

D. Replace the portrait of Figure 7 with that of Figure 8. With a periodic master system, the macron (master cycle imaged in control space of the slave system by the serial coupling map) is hysterical, with a time series such as that shown in Figure 9. Here the static attractors of the slave system, yin and yang, occur periodically. But which one occurs depends on the exact state of the trajectory of the entire serial chain at the moment the master cycle crosses the critical border (blue cycle periostrophe) of the slave system. So if we now allow the master system to become chaotic through a subtle bifurcation (again, with a four-component chain), these occurrences of yin and yang will become stochastic. Still, their average frequencies of occurrence will reveal the stochastic holonomy of the master macron in the control plane of frequency and amplitude of the slave system. In this way, some information about the component systems of a serial chain may be gleaned from a time series output from the final system of the chain.

Finally, we note that the longer the serial chain, the more difficult the analysis of its output in terms of qualitative behavior of its component systems.

ACKNOWLEDGEMENTS

These ideas have been instigated by René Thom. In 1972, inspired by his book, I visited him at Bures-sur-Yvette. He showed me a book, Kymatiks, which led to my meeting with the author, Hans Jenny, to my eventual construction of the 4 inch Jenny Macroscope in Santa Cruz in 1974, and to the many experiments with three-components serial chains (using forced oscillation in fluids) which continue even now. These experiments are the source of the ideas presented here.

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