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# The Outstructure of the Lorenz Attractor

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**Dedicated to René Thom.**

In the chaotic attractors we have come to know experimentally, there frequently are distinguished critical points or closed orbits which organize the geometry. Here, we describe the geometry of the Lorenz attractor in terms of a yoke of outsets from three such distinguished organizers, and speculate on the generalization of this outstructure to other chaotic attractors.

1. Neat heteroclines. Consider two basic sets of a flow, Alpha and Omega. That is, each set is hyperbolic, invariant, and indecomposable, or Axiom A. These are heteroclinic if there is a trajectory from one to the other. Let us suppose there is a trajectory from Alpha to Omega. Thus, some trajectory has Alpha for its alpha limit set and Omega for its omega limit set. Then it follows that the outset (unstable manifold) of Alpha, Out(A), approaches arbitrarily close to the outset of Omega, Out(Y). Considering the implications of the hyperbolicity of Omega, and the invariant manifold theorem, there must be an intersection of the boundary of Out(A) with Out(Y) itself. We say that Alpha is neatly heteroclinic to Omega if the dimension of Out(A) is one more than the dimension of Out(Y),

and the boundary of Out(A) is identical to Out(Y). This is the case in the Lorenz attractor, as we shall see.

2. Yokes and Coboundaries. We now suppose, for simplicity, that the basic sets under discussion are hyperbolic critical elements, that is, critical points or closed orbits. Further, we consider three of these, A, B, and Y, where both A and B are heteroclinic to Y. We call this a heteroclinic yoke. We will see that these yokes can behave very much like homoclinic cycles in some flows: in the presence of reinsertion, they may make horseshoes, knots, and chaos. Now suppose the yoke is neat, that is, both of the heteroclinic links are neat. Then Out(A) and Out(B) are both bounded by Out(Y). Due to the hyperbolic structure of the three critical elements, the closure of the union of the three outsets is locally attractive. It is a candidate for an attractor, in fact.

3. Reinsertion. Note that the three outsets of a yoke must go somewhere. The omega limit sets in the boundary of these outsets are also yoked. But in the case of a neat yoke, if we suppose that the entire boundary of Out(A) and Out(B) is Out(Y), then Out(Y) has nowhere to go. So, this is only possible if Out(Y) either goes off to infinity, or it is reinserted, as Rössler would say. That is, the boundary of Out(Y) is found in the closure of the union of the three outsets, the candidate attractor. And both of these cases occur in the Lorenz attractor, as we show visually in the next section.

4. Example: the Lorenz attractor. Here is a neat yoke, expressed in a sequence of eight drawings which we made while trying to understand Perello (1980).

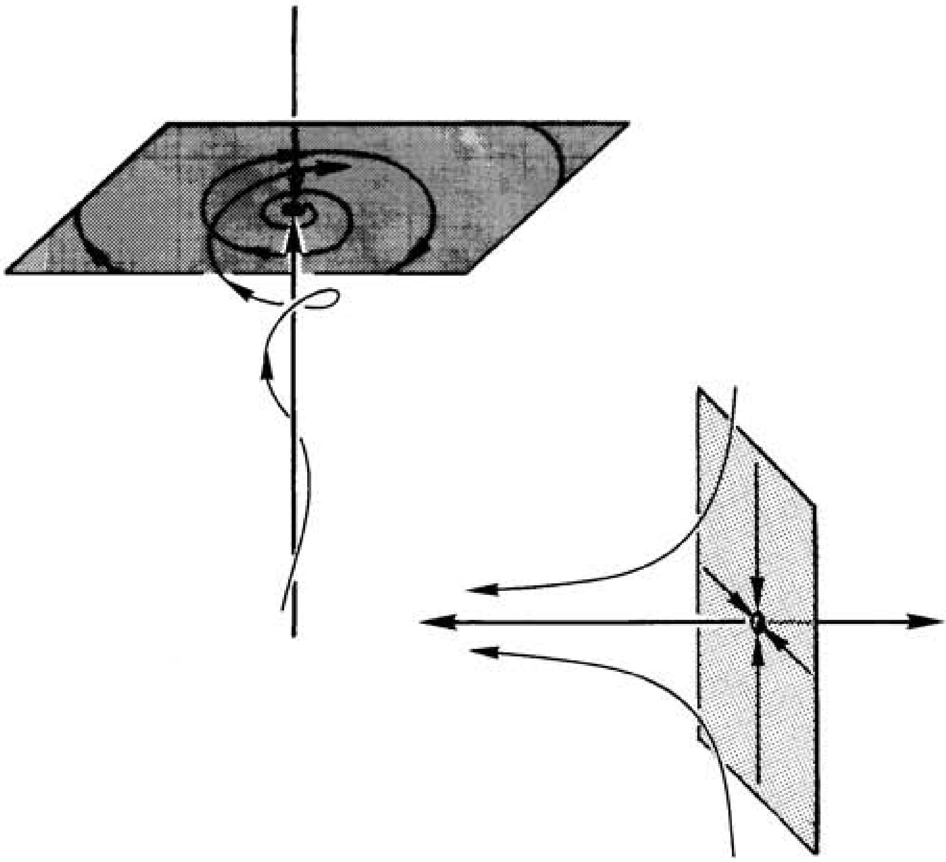

FIGURE 1. Here are two saddle points, A and Y. They are hyperbolic, in three dimensions. One, A, has index 2, with spiral dynamics on its planar outset (shaded). The other, Y, has index 1, with nodal dynamics on its planar inset (dotted), In(Y). The two outsets are attractive, as shown by the neighboring trajectories. As Out(A) and In(Y) are both two-dimensional, they could intersect transversely in three space. If they did, the transversal intersection would have to be a trajectory, the <u>heteroclinic trajectory</u>.

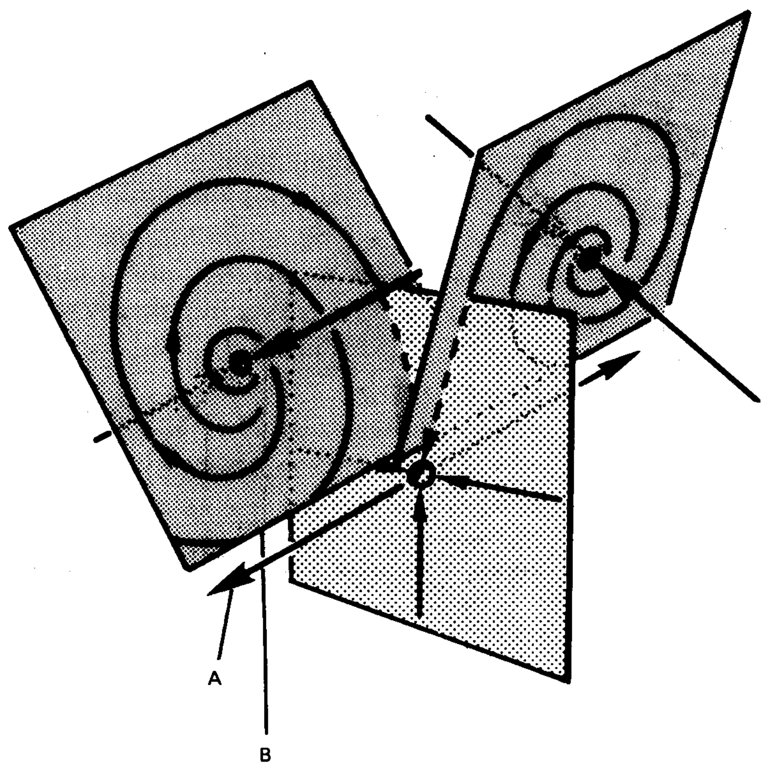


FIGURE 2. Adding another saddle point, B, essentially identical to A, we make a yoke like this. Both A and B are heteroclinic to Y. They are transversally heteroclinic, as the two planar outsets (shaded) intersect the planar inset (dotted) transversally. There are two heteroclinic trajectories in this yoke. Note that the arriving outsets are incident upon the departing outset, at Y. Thus, it is possible that this is a neat yoke. Next, we will see where these outsets end up.

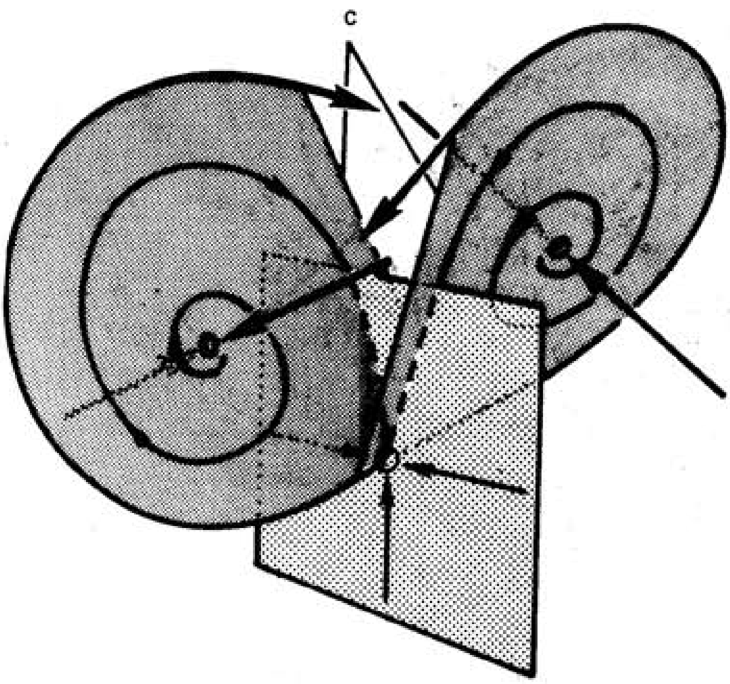


FIGURE 3. As the arriving outsets, Out(A) and Out(B), both have spiral dynamics, the departing outset which bounds them, Out(Y), swirls around and reinserts, as shown here. It can not go off to infinity.

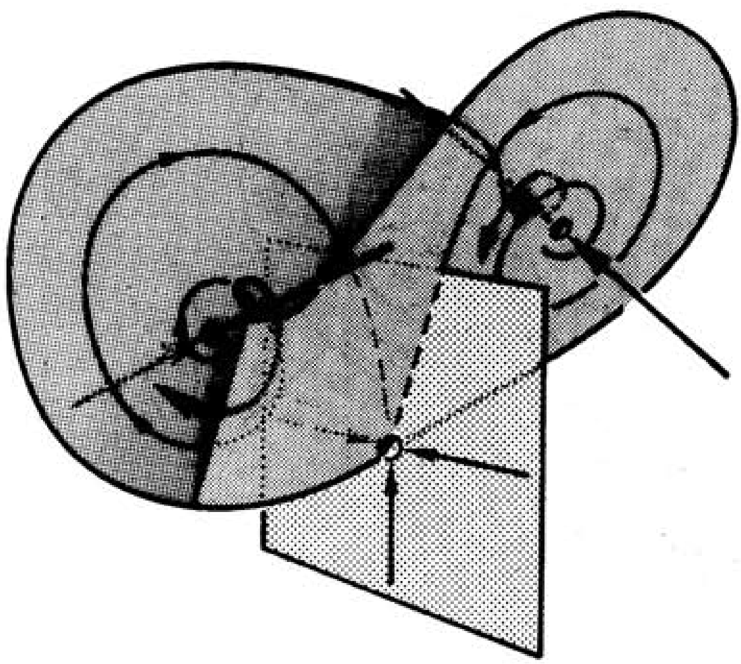

FIGURE 4. The result of reinserting is this: as each branch of Out(Y) swirls around one of the shaded outsets, it approaches near the other shaded outset. It gets attracted, as outsets are attractive. Thus, the omega limit set of Out(Y) is within the closure of the union of the three yoked outsets.

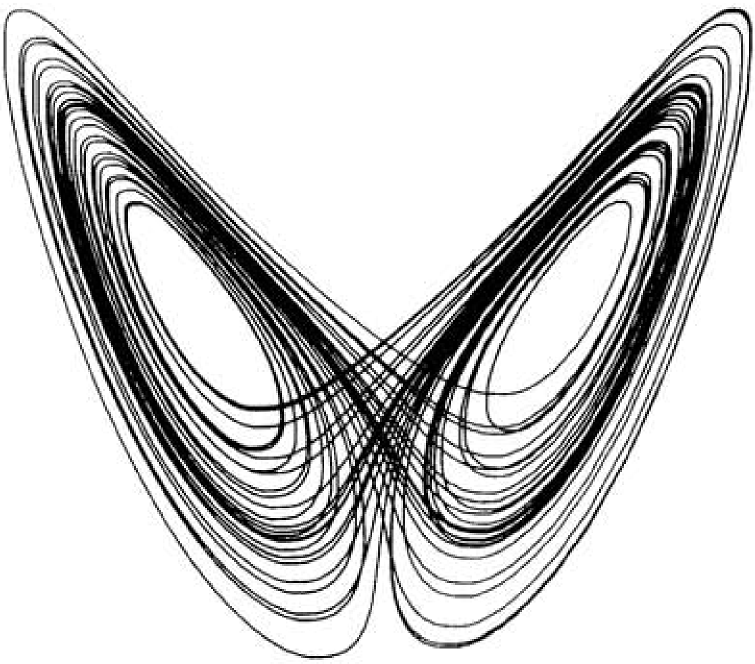

FIGURE 5. And here, for comparison, is a computer drawing of the Lorenz attractor. Inspection of the equations reveals the three distinguished saddle points, right where we want them. But the planar inset of the saddle point in the lower center is qualitatively invisible. It is a kind of separatrix. Now we will add it to the picture, with its full extension.

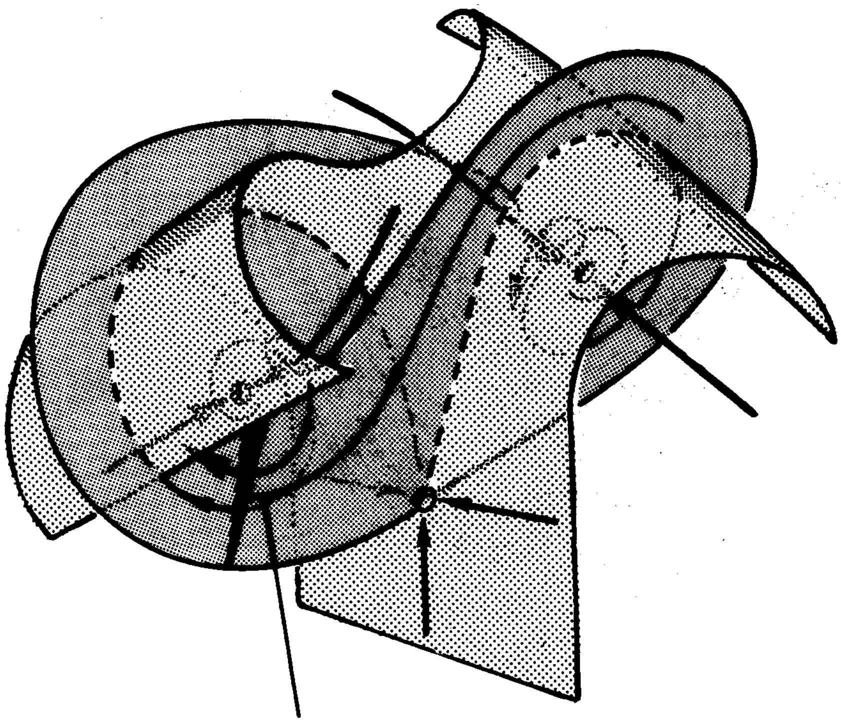


FIGURE 6. Referring to Figure 4, we run the flow backwards in time, to extend the planar (dotted) inset outwards from Y. It follows the heteroclinic trajectories (dashed) back to the yoked saddles, A and B, scrolling as it goes.

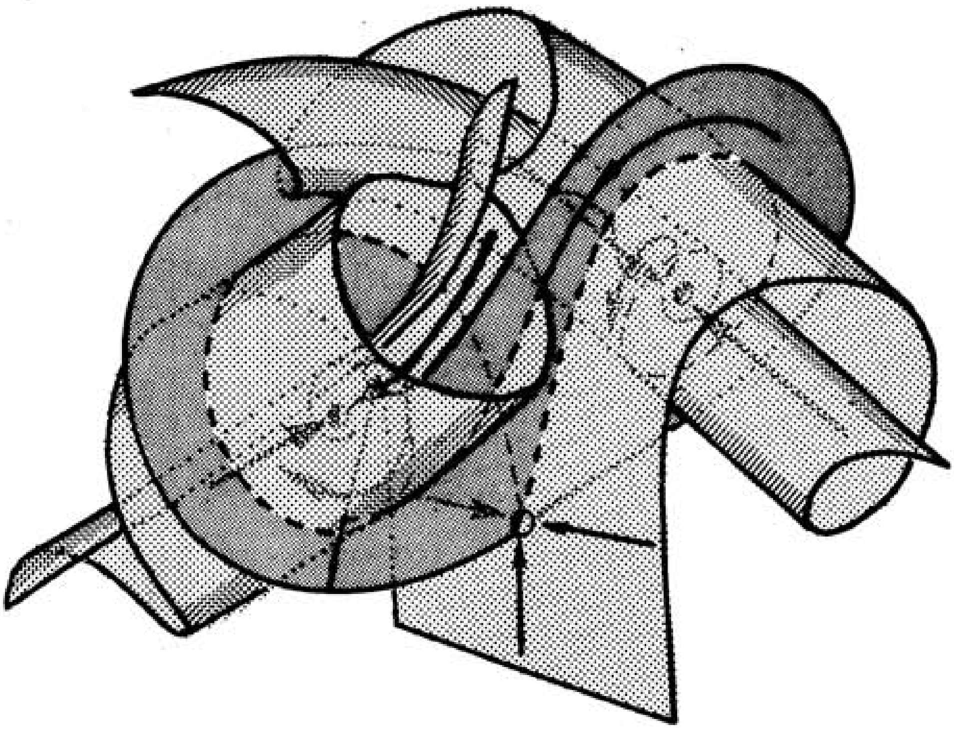

FIGURE 7. Extending the dotted inset farther backwards in time, it scrolls up tightly around the one-dimensional insets of A and B, In(A) and In(B). We have not said much about these curves so far. But if we could reverse the arrow of time for a moment, we would have a neat heteroclinic from Y to A (likewise, from Y to B) and thus In(A) and In(B) comprise the boundary of Out(Y). We may call this a neat reverse yoke. The boundary of Out(Y) also contains the repellor at infinity.

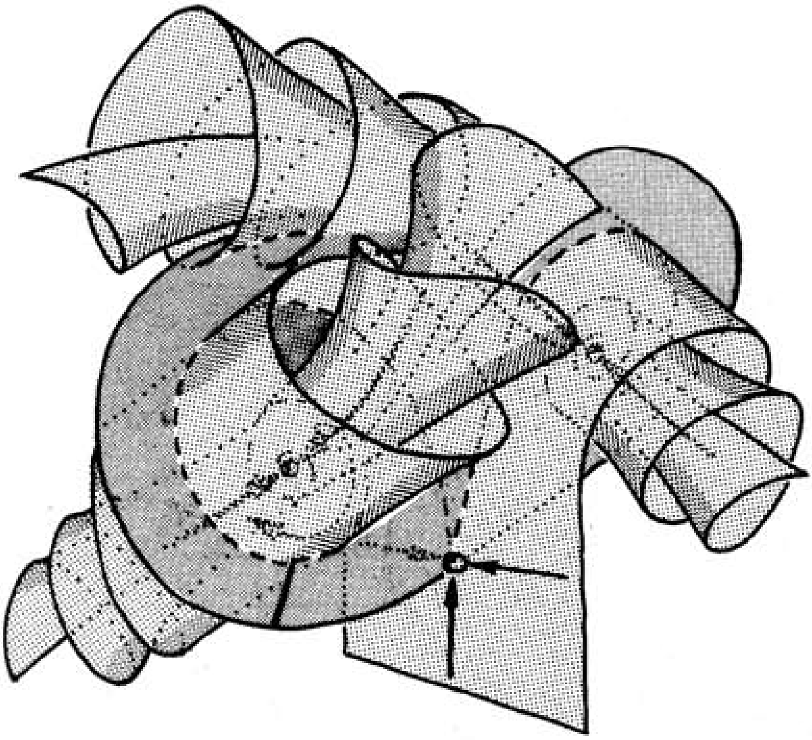

FIGURE 8. Extending the dotted inset farther backwards still, the four ends of the scrolls are pulled out along the curves, In(A) and In(B), toward their source at infinity.

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