COMPLEX DYNAMICAL SYSTEMS

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Abstract. Complex dynamical systems theory is a new development, in which concepts of nonlinear dynamical systems theory (static, periodic and chaotic attractors; basins and separators; structural stability; subtle and catastrophic bifurcations) are combined with concepts of system dynamics and control theory (input/output, feedback, networks) for the purpose of modeling complex systems. This paper presents an outline of the theory, simple applications, and simulation techniques.

Keywords. Dynamical systems, complex systems, nonlinear control systems, phase space methods, stability, bifurcation analysis, modeling, simulation

INTRODUCTION

Complex dynamics has evolved in attempts to model and simulate complex systems, especially in physiology. As the concepts are scattered throughout literature in diverse fields, we attempt here to collect and summarize them. More details and illustrations may be found in the original papers (Abraham, 1979, 1983; Abraham and Shaw, 1983b).

COMPLEX DYNAMICS

A complex dynamical system is a network, or directed graph, of nodes and directed edges. The nodes are *simple dynamical schemes* or dynamical systems depending on control parameters. The directed edges are *static schemes*, or output/input functions depending on control parameters. These provide the *serial coupling* from the instantaneous states at one node into the control parameters of another.

Simple Dynamical Schemes

These are variously known as control vectorfields, parameterized flows, and so on.

Definitions. Let C be a manifold modeling the control parameters of a system, and S another manifold, representing its instantaneous states. Then a *simple dynamical scheme* is a smooth function assigning a smooth vectorfield on S to every point of C. Alternatively, we may think of this function as a smooth vectorfield on the product manifold, C x S, which is tangent to the state fibers, {c} x S. For each control point, $c \in C$, let X(c) be the vectorfield assigned by the scheme. We think of this as a dynamical system on S, or system of first order ordinary differential equations.

Attractors and basins. In each vectorfield of a scheme, X(c), the main features are the *attractors*. These are asymptotic limit sets, under the flow, for a significant set of initial conditions in S. These initial states, tending to a given attractor asymptotically as time goes to plus infinity, comprise the *basins* of X(c). Every point of S which is not in a basin belongs to the *separator* of X(c). The decomposition of S into basins, each containing a single attractor, is the *portrait* of X(c). Attractors occur in three types - *static* (an attractor limit point), *periodic* (an attractive limit cycle, or oscillation), and chaotic (meaning any other attractive limit set).

Diagrams. For each point c of the control manifold, the portrait of X(c) may be visualized in the corresponding state fiber, {c} x S, of the product manifold, C x S. The union of the attractors of X(c), for all control points c of C, is the *attractrix*, or locus of attraction, of the scheme. The union of the separators of X(c), for all control points, is the *separatrix* of the scheme. These sets, visualized in the product manifold, comprise the *diagram* of the scheme. Many examples are fully illustrated in the literature.

Catastrophes and Subtle Bifurcations

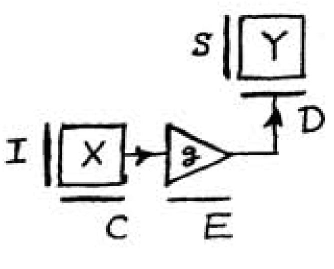
For most control points, $c \in C$, the portrait of X(c) is *structurally stable*. That is, perturbation of the control parameters from c to another nearby point cause a change in the portrait of X(c) which is small, and qualitatively insignificant. In exceptional cases, called *bifurcation control points*, the portrait of X(c) significantly changes as control parameters are passed through the exceptional point. Many cases, generic in a precise mathematical sense, are known, and the list is growing. These bifurcation events all fall into two categories. A bifurcation is *subtle* if only one attractor is involved, and its significant qualitative change is small in magnitude. For example, in a Hopf bifurcation, a static attractor becomes a very small periodic attractor, which then slowly grows in amplitude. All other bifurcations are *catastrophic*. In some of these, called *blue-sky catastrophes*, an attractor appears from, or disappears into, the blue (that is, from a separator). In others, called *omega explosions*, a small attractor suddenly explodes into a much larger one. All of these events are very common in the simplest dynamical schemes, such as forced oscillators. The bifurcations are clearly visualized in the diagram of a scheme, which is sometimes called the *bifurcation diagram*. The theory up to this point is adequately described in the literature (Arnold, Chow and Hale, Guckenheimer and Holmes, Hirsch and Smale).

Static Coupling Schemes

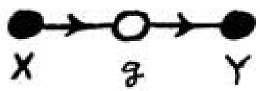
Consider two simple dynamical schemes, X on C x I and Y on D x S. The two schemes may be *serially coupled* by a function which, depending on the instantaneous state of the first (a point in I), sets the controls of the second (a point in D). A *static coupling scheme* is just such a function, but may also depend on control parameters of its own. Thus, let E be another control manifold, and $g:E \times I \to D$. Then the serial coupling of X and Y by the static coupling scheme g is a dynamical scheme with control manifold C x E, and state space I x S, defined by

$$Z(c,e)(i,s) = [X(c)(i), Y(g(e,i)(s)]$$

This is the simplest example of a *complex dynamical scheme*, symbolized by



or equivalently by



in the literature (Abraham, 1983a; Abraham and Shaw, 1983b).

Serial Networks

A large number of simple dynamical schemes may be coupled, pairwise, with appropriate static coupling schemes. The result, a *serial network*, may be symbolized by a directed graph. This is the full scale *complex dynamical system*. Its purpose, as a mathematical construction, is to model complex dynamical systems in nature. This strategy has been introduced in Abraham and Shaw (1983b).

SIMPLE NETWORKS

Several pedagogic examples have been presented (Abraham, 1983b, 1983c). We review them here.

Master-slave Systems

The simplest complex scheme consists of the serial coupling (as illustrated above) of two simple dynamical schemes. The behavior of these simple examples is notoriously complicated. Suppose that the control parameters of the first (or *master*) system are fixed. After startup, from an arbitrary initial state, the startup transient dies away, and the master system settles asymptotically into one of its attractors. We consider the three cases separately.

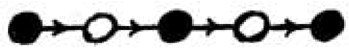
Static master. If the attractor of the master system is a static (point) attractor, and the control parameters of the coupling scheme are left fixed, then the control parameter of the second (*slave*) system are likewise fixed. Typically, this static control point of the slave system will be a typical (nonbifurcation) point, and the slave system will be observed in one of its attractors (static, periodic, or chaotic.

Periodic master. With fixed controls of the master and the coupling function, a periodic master attractor will drive the slave controls in a periodic cycle. This is the situation in the classical theory of *forced oscillation*. Experimental study of these systems began a century of so ago, and continues today.

Chaotic master. This situation, forced chaos, has received little attention so far. Preliminary discussion may be found in Abraham (1983c).

Chains of Dynamical Schemes

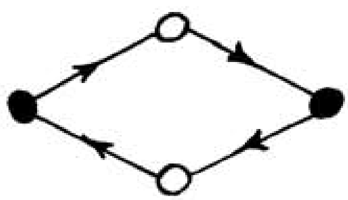
If three schemes are connected in a *serial chain* by two static coupling schemes:



a complex system with a very complicated bifurcation diagram may result. If the first pair comprise a periodic master forcing a simple pendulum, as described above, the terminal slave may be either a periodically or chaotically forced system. Of course, if all three systems are pendulum-like (one basin, static attractor) the serial chain is also pendulum-like. But a periodic attractor in either the first or second dynamical scheme is adequate for rich dynamics in the chain.

Cycles of Dynamical Schemes

If the directed graph of a complex scheme contains a cycle (closed loop) then complicated dynamics may occur, no matter how simple the component schemes. The minimal example is the *serially bicoupled pair*:



Even if the two dynamical schemes are pendulum-like, the complex system may have a periodic attractor. For example, Smale (1976) finds a periodic attractor in exactly this situation (and a Hopf bifurcation) in a discrete reaction-diffusion model for two biological cells. A cycle of three pendulum-like nodes is discussed below.

EXEMPLARY APPLICATIONS

We turn now to some simple examples of complex dynamical models.

Coupled oscillators

We consider now a master-slave system, in which the master system is following a periodic attractor. Controls of the master system and the coupling function determine the frequency and amplitude of the periodic cycle in the control manifold of the slave system.

Duffing system. If the slave system is a soft spring or pendulum, the coupled system is the classic introduced by Rayleigh, in which Duffing found hysteresis and catastrophes in 1918 (Abraham and Shaw, 1982). The bifircation diagram is very rich, full of harmonic periodic attractors and chaos.

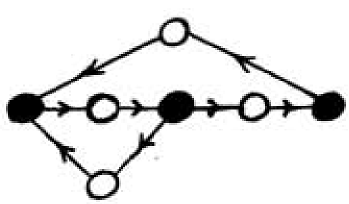
Van der Pol systems. If the slave system is a self-sustained oscillator, the coupled system is the classic introduced by Rayleigh, in which Van der Pol found subtle bifurcations of harmonics (Abraham and Shaw, 1982) and Cartwright and Littlewood apparently found chaos (Abraham and Shaw, 1983a). Both of these classical systems have been central to experimental dynamics, and research continues today.

Periodic Hysteresis

If a periodic attractor in the master system is coupled to the planar control space of a slave system with a rich bifurcation diagram containing multiple hysteresis (blue-sky catastrophe) curves, the coupled system may exhibit erratic behavior, in a nearly periodic pattern. An example of this *periodic hysteresis*, based on the Andronov-Takens bifurcation diagram, has been fully described (Abraham and Shaw, 1983b; Abraham, 1983c).

Intermittency in an Endocrine Model

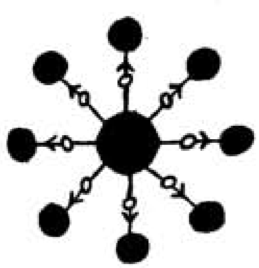
Models for physiological and biochemical systems have a natural complex structure. A recent model for the reproductive system of male mammals (hypothalamus, pituitary, testes) is a very simple network (Abraham, Kocak and Smith, 1983)



Although the simple dynamical scheme at each node is a point attractor in a one-dimensional state space, the complex system may have two periodic basins, each containing a periodic attractor. This phenomenon, sometimes called *birhythmicity*, has also been found in a biochemical model (Decroly and Goldbeter, 1982). Small changes in the control parameters of the coupling functions cause intermittent jumps between the two distinct oscillatory states.

Reaction-Diffusion Systems

An unusual example of serial coupling is provided by the reaction-diffusion model for biological morphogenesis, introduced by Kolmogorov, Piskunin, and Pontriagin, Rashevsky, and Turing (Abraham and Shaw, 1982; Abraham, 1983b). Given a spatial domain or substrate, D, and a biochemical state space, B, the state space is an infinite-dimensional manifold of functions from D to B, $\mathfrak{F}$. The reaction-diffusion equation may be regarded as a simple dynamical scheme of vectorfields on $\mathfrak{F}$, depending on a control space, C. Meanwhile, the spatial substrate is actually composed of biological cells, considered identical in structure. As the reaction-diffusion scheme, the master in this context, determines instantaneous states of biochemical (morphogen) concentrations in the substrate, $f:D \to B$, the cell at a fixed position in the domain will extract the values of this function at its location, $f(d)$. This is a point of B, which may be regarded as the control space for another simple dynamical scheme, modeling the dynamics within the standard cell. Let $g_d(f) = f(d)$. Then g_d is the static coupling function from master to slave. But there are many slaves, each distinguished by its own location, hence coupling function. The directed graph is thus a radial spray, or star, of slaves of a common master. If in addition each cell may be a source or sink of biochemical controls, then each connection is a serial bicoupling.



SIMULATION TECHNIQUES

After the strategies of complex dynamical systems have been used in an application, the resulting model is simply a large dynamical scheme. That is, a system of coupled ordinary differential equations, or partial differential equations of evolution (parabolic or hyperbolic) must be explored experimentally, to obtain the bifurcation diagram, which is the useful outcome of the modeling activity. As the exploration of the bifurcation diagram is an unfamiliar goal of simulation, we review here some of the strategies used.

Orbit Methods

When the dynamical scheme consists of a modest number of ordinary differential equations of first order, simulation by the standard digital algorithms (Euler, Runge-Kutta and so on) and analog techniques provide curve tracing in the bifurcation diagram. A large number of curves, for various values of control parameters and initial conditions, reveal the principal features of the diagram.

Relaxation Methods

When partial differential equations--reaction-diffusion, hydrodynamic, plasma, liquid crystal, solid state, elastodynamic, and so on--are part of the model, they may be treated most naturally as dynamical systems by discretization of the spatial variables. Thus, the infinite-dimensional state spaces are projected into finite-dimensional approximations. Finally, these may be treated by orbit methods, to obtain a bifurcation diagram with loci of attraction and separation. This is essentially the relaxation technique of Southwell.

Dynasim Methods

With small or large, ordinary or partial, the exploration of a bifurcation diagram by analog, digital, or hybrid simulation is extremely time intensive. A considerable gain in speed may be obtained with *dynasim methods* (Abraham, 1979). Here, special purpose hardware traces a large number of orbits in parallel. Having thus found all the most probable attractors at once, time is reversed and the basin of each is filled with its own color. This process is repeated (perhaps in parallel) for different values of the control parameters. When dimensions are large, new techniques of visualization must be developed.

CONCLUSION

Complex dynamical systems theory is a new development. An outgrowth of nonlinear dynamics and control theory, it aims to provide complete strategies for modeling and simulation of complex systems, whether in the physical, biological, or social sciences.

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