7

Is There Chaos Without Noise?

Ralph H. Abraham

Mathematics Board
University of California
Santa Cruz, California

Dedicated to Mauricio Peixoto.

The chaotic attractor of dynamical systems theory has been widely heralded as a new paradigm for chaotic and turbulent motions in nature. This idea has been strongly supported by the experimental discovery of the chaotic attractor of simulation machines. Is this experimental object an instance of the mathematical model, or an artifact of noise amplification? Here, we establish the existence of this artifact in the forced Van der Pol system, explain how it could account for the experimental chaos of the Lorenz, Rössler, and Shaw systems, describe a critical experiment to distinguish between the noise-amplification and the chaotic attractor models, and propose a new concept of dynamical stability.

1. INTRODUCTION

The chaotic attractor of mathematical theory began with Birkhoff in 1916. The chaotic attractor of simulation experiment arrived with Lorenz in 1962. (See Abraham and Shaw, 1983a, for historical details.) The identification of these two objects has not yet succeeded, despite many attempts during the past twenty years. Of course, everyone (including myself) expects this to happen soon (see Hirsch, 1983). Nevertheless, there is an important reason to consider the loathsome alternative: the quasi-periodic paradox.

2. THE QUASI-PERIODIC PARADOX

The attractive, invariant 2-torus is ubiquitous in dynamical systems. It occurs, for example, in the "main sequence" of bifurcations: static attractor to periodic attractor by a Hopf bifurcation, to an attractive invariant 2-torus (AIT) by a Neimark bifurcation. It is always found in forced oscillators,such as the Rayleigh or Van der Pol. According to Peixoto's Theorem on the open genericity of structural stability for flows on the 2-torus, the restricted flow on the AIT must almost always be a braid of periodic attractors. Thus, the power spectrum of one coordinate of a typical trajectory would reveal fundamental frequencies of two modes of oscillation, rationally related. However, most of the time, experimentalists observe not braids (rationally related frequencies) but quasi-periodic motions (apparently irrationally related frequencies). That is the quasi-periodic paradox. More than one scientist has lost faith in mathematics because of the ubiquity of this illegal motion in the natural world. In fact, in the forced Van der Pol system, quasi-periodic motion persists over most of the parameter space (see Abraham and Scott, 1983). We now present three competing explanations of this paradox.

3. THE THICK BIFURCATION MODEL

This scheme is due to Sotomayer (1974). We suppose that the dynamical system in question has one loose parameter. Thus, we are observing not a single generic dynamical system on a single AIT, but a generic arc of dynamical systems on a moving AIT. Therefore the braid bifurcations, at which one braid of periodic attractors changes to another (the ratio of frequences changes from one rational to another nearby rational), occur very frequently along this arc. In fact, Herman (1979) showed that the probability of bifurcation may be close to one. Thus, most of the time, quasi-periodic motions will be observed. This explanation of the quasi-periodic paradox is favored by mathematicians.

4. THE NOISE MODULATION MODEL

On the other hand, this one is favored by physicists (Ueda and Akamatsu, 1981; Shaw, 1981). Here we assume that the mathematical model is a single generic dynamical system, with a braid of periodic attractors on an AIT. But the experimental system simulating it has noisy imperfections. For example, the function generator providing the forcing voltage to an analog computer has low-level noise in its power spectrum. This noise, if its amplitude is sufficient, will cause the trajectory to leap from one basin of attraction, on the AIT, to another. The smallest distance form a periodic attractor to its separatrix (a periodic saddle) determines the critical amplitude of noise sufficient for quasi-periodic motion.

5. THE CHAOTIC ATTRACTOR MODEL

This rationale is a compromise of the two preceding ones. While admitting that noise modulation occurs in the simulation device, we suppose that there is a dynamical system model for the device, including its noise. At this point, we resort to the fact that chaotic attractors are known to exist in mathematical theory, even if we do not yet know whether the Lorenz system (for example) has one, or not. Thus, a mathematical model for the simulation device may be regarded as a coupling (generic perturbation) of the Cartesian product of an AIT (in three dimensions) and an unknown system with a chaotic structure (in three dimensions or more). Thus, our observation of the attractor in three dimensions is a projection of the actual chaotic attractor in six dimensions or more. Hence, there is no conflict with Peixoto's theorem.

This obviously models the noise modulation scheme. It may be applied to the thick bifurcation scheme as follows. Make a dynamical model for the ambient noise in the single loose parameter. Serially couple this model to the generic one, by selecting one coordinate (or some other real-valued function of the state space of the chaotic model) to control the loose parameter. The resulting coupled system will be quasi-periodic most of the time.

6. APPLICATION TO EXPERIMENTAL CHAOS

We may take the best known chaotic flows of experiment (Lorenz, 1962; Rössler, 1971; Shaw, 1981) as examples. For the sake of discussion, we suppose (banish the thought) that these systems do not contain a mathematical chaotic attractor. Then their apparently chaotic behavior may, like the quasi-periodic paradox, be explained by the noise modulation (and related) schemes. For in each of these cases, the subject dynamical system is known to have an attractive invariant set (AIS) which behaves like the AIT of the quasi-periodic paradox. For example, the AIS of the Lorenz system is the outset structure of its three saddle points (Lorenz, 1963; Abraham and Shaw, 1983b). The AIS of the Rössler system is the outset, a Möbius band, of its fundamental limit cycle (Rössler, 1979; Abraham and Shaw, 1983a). Finally, the AIS of the Shaw system is the usual AIT of a forced oscillator (Shaw, 1981; Abraham and Shaw, 1983a; Abraham and Scott, 1983).

And thus our question: is there chaos without noise? That is: is there a chaotic attractor in these particular three-dimensional systems (Lorenz, Rössler, Shaw) without modulating noise?

7. A CRITICAL EXPERIMENT

We do not know the answer to this question. It could be yes for the Shaw system, for example, and no for the Lorenz and Rössler systems. We do know that it is no for the quasi-periodic motion on an AIT. So we imagine there could be an experimental test for noise modulation; that is, a procedure to exclude noise modulation as a model for chaotic behavior of experimental systems. Here is a rough sketch of one such procedure.

Suppose a system is given with a single control parameter, which has a bifurcation to chaos at one bifurcation value of the control. Then, if the chaotic behavior is due to noise modulation, the bifurcation to chaos will depend on the amplitude of the noise. For example if the noise in the forced oscillator system is reduced, the appearance of quasi-periodic motion on the AIT will occur for a higher value of the amplitude of the forcing oscillation. Let noise ratio denote the ratio of power in the continuous part of the power spectrum of the simulated system, to that of simulation device at rest. Then the noise ratio, as a function of the bifurcation parameter, could discriminate between noise-modulated, versus truly chaotic, behavior.

8. DYNAMICAL STABILITY

The ubiquity of structurally unstable motions, in conflict with Peixoto's Theorem in the AIT context, suggests that structural stability is not an appropriate concept for experimental systems. Here, we suggest an alternative, very much in the spirit of Ueda and Akamatsu (1981). Suppose our dynamical system, for simplicity, has a single attractor, and upon perturbation , it still has a single attractor. In fact, let us suppose it is structurally stable, so that the attractor is not significantly changed by any small, static perturbation. Supposing the system depends on control parameters, let us not serially couple the output of another (possibly chaotic) system to these controls. Then the original system is dynamically stable if its attractor is not significantly changed by this dynamical perturbation, provided it is sufficiently small.

For example, the braided attractors on an AIT are structurally stable, in the static, classical sense, according to Peixoto's Theorem. But they are not dynamically stable, because any amount of dynamical perturbation (even periodic perturbation) may produce a chaotic attractor.

9. CONCLUSION

The formulation of the chaotic attractor model for noise modulation solves the quasi-periodic paradox. It may solve a chaotic attractor paradox, if there is one, in specific systems such as the Lorenz system. We still do not know if there is chaos without noise in these systems, or not. But noise ratio experiments may shed some light on this question. This is a special case of a more general question: are these systems dynamically stable? Here we may hazard a conjecture: all natural systems are dynamically stable. In fact, we will probably evolve the definition of stability until this conjecture becomes true.

REFERENCES

1. Abraham, R. H. and K. A. Scott, 1983. Chaostrophes of forced Van der Pol systems. This volume.
2. Abraham, R. H. and C. D. Shaw, 1983a. Dynamics, the Geometry of Behavior. Part Two: Chaotic Behavior. Aerial, Santa Cruz, CA.
3. Abraham, R. H. and C. D. Shaw, 1983b. The outstructure of the Lorenz attractor. This volume.
4. Herman, M., 1979. Sur la conjugaison différentiable des difféomorphismes du circle à des rotations. Publ. Math. I.H.E.S., 49.
5. Hirsch, M. W., 1983. The chaos of dynamical systems. This volume.
6. Lorenz, E., 1963. Deterministic nonperiodic flow. J. Atmos. Sci. 20: 130-141.
7. Rössler, O. E., 1976. An equation for continuous chaos. Phys. Lett. 57A: 397-398.
8. Shaw, R. S., 1981. Strange attractors, chaotic behavior, and information flows. J. Naturforsch. 36a: 80.
9. Sotomayer, J., 1974. Generic one-parameter families of vector fields in two-dimensional manifolds. Publ. Math. I.H.E.S. 43: 5-46.
10. Ueda, Y. and N. Akamatsu, 1981. Chaotically transitional phenomena in the forced negative-resistance oscillator. IEE Trans. CAS-20: 217-224.