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Chaostrophes of Forced Van der Pol Systems

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Dedicated to Chihiro Hayashi.

In response to a recent conjecture, we explored two systems in analog simulation, in search of the blue bagel chaostrophe--in which a chaotic bagel attractor disappears into the blue. We found an abundance of these bifurcations in the forced Van der Pol systems.

1. INTRODUCTION

In a recent conjecture (Abraham, 1983a, Section A4 and Fig. 3) a blue bagel chaostrophe is proposed to exist in the forced Van der Pol system. This conjecture was based on the discovery of the chaotic bagel attractor in systems of this type by R. Shaw (1981). Our idea was to study these same systems in analog simulation, turning the knobs until the bagel attractor collided with a homoclinic tangle, its separatrix, in a mutual annihilation. We are grateful to Rob Shaw for sharing his lab with us for these experiments. What we found, more complicated than expected due to period doubling bifurcations, is presented here. The background of this entire cycle of ideas is a drawing by Hayashi, Ueda and Kawakami (1970) gracing the cover of Hayashi's volume of selected papers (1975, p. 186).

This drawing clearly shows an AIT (attractive, invariant torus) within a homoclinic tangle, occurring in a forced, conservative, Duffing system.

2. ACCELERATION FORCING

First we explored the conventionally forced Van der Pol system, in the form:

$$\dot{x} = ky + \mu x(a - y^2) + A\sin\theta$$

$$\dot{y} = -x$$

$$\dot{\theta} = 2\pi F$$

The fixed parameters, $a = k = 9$, $\mu = 32$, were chosen by twisting knobs and looking for likely attractors in the strobe plane. The analog setup in Rob Shaw's lab, used for this work, conveniently includes a strobe pulse and storage scope for direct observations of the of the Poincaré section. Amid a sea of complex bifurcations, we selected a simple arc, shown in Fig. 3. Here A = 0.420, fixed, while F varies from the frequency of the periodic attractor of the unforced system, 4.2 Hz, to double that, or 8.4 Hz. In Fig. 3 we see:

Two views of the fundamental oscillation--

A. Full periodic attractor, F = 4.20 Hz.

B. Strobed periodic attractor, F = 4.20.

Subtle bifurcation to chaotic torus--

C. Full baby torus, F = 6.60

D. Strobed baby torus, F = 6.60

E. Strobed large torus, F = 7.05

F. Two strobed phases, 0 and π, F = 7.05

G. Strobed torus, F = 8.00.

Catastrophic bifurcation back to the fundamental--

H. Strobed periodic attractor, F = 8.15.

These two bifurcations are apparently formed by a nearby homoclinic tangle. However, we did not observe it. The large attractor, an AIT, appears quasi-periodic because of noise modulation (see Abraham, 1983b).

3. VELOCITY FORCING

To observe analogous behavior in a chaotic system, we moved the driving oscillator form the acceleration equation to the velocity, following R. Shaw (1981). Thus, we explored the system

$$\dot{x} = ky + \mu x(a - y^2)$$

$$\dot{y} = -x + A\sin\theta$$

$$\dot{\theta} = 2\pi F$$

with the same parameters, $a = k = 9$, $\mu = 32$, $A = 0.420$, and F between 4.2 and 10.0 Hz. The results are shown in Fig. 4:

The fundamental oscillation--

A. Strobed periodic attractor, F = 6.15.

Chaostrophic bifurcation to a bagel--

B. Strobed bagel, F = 6.20.

This continues over a large range--

C. Strobed bagel, F = 9.20.

and chaostrophically collapses again to the fundamental,

D. Strobed periodic attractor, F = 9.25.

Again, these two catastrophic bifurcations are apparently related to nearby homoclinic saddles, but we did not observe them. In particular, the bagel in Fig. 4C clearly shows extensive dwell at the top and bottom, suggesting the locations of the nearby, invisible, saddle orbit of period two.

In Fig. 3, we conjecture a rough idea of the nearby homoclinic tangles, following the inspiration of Hayashi, Ueda and Kawakami (1969, p. 251): For the conventionally forced system of the preceding section, forced at about the fundamental frequency--

A. Corresponding to Fig. 3D, the baby torus in a homoclinic nest.

And at the end of its regime, at about double the fundamental frequency--

B. Corresponding to Fig. 3G, the great torus in a homoclinic box.

And for the velocity forced system of this section, at about the fundamental frequency--

C. Corresponding to Fig. 4B, the chaotic bagel in a nest-- and again at about double the fundamental frequency--

D. Corresponding to Fig. 4C, the chaotic bagel in a homoclinic box.

4. BIFURCATION CONJECTURES

Without further experimental work, we may only guess at the bifurcation sequences behind our observations, shown in Figs. 1 and 2. Here, we record a few guesses suggested by the observations. Consider first the bifurcation, shown in Poincaré section from Fig. 1B to 1D. As the

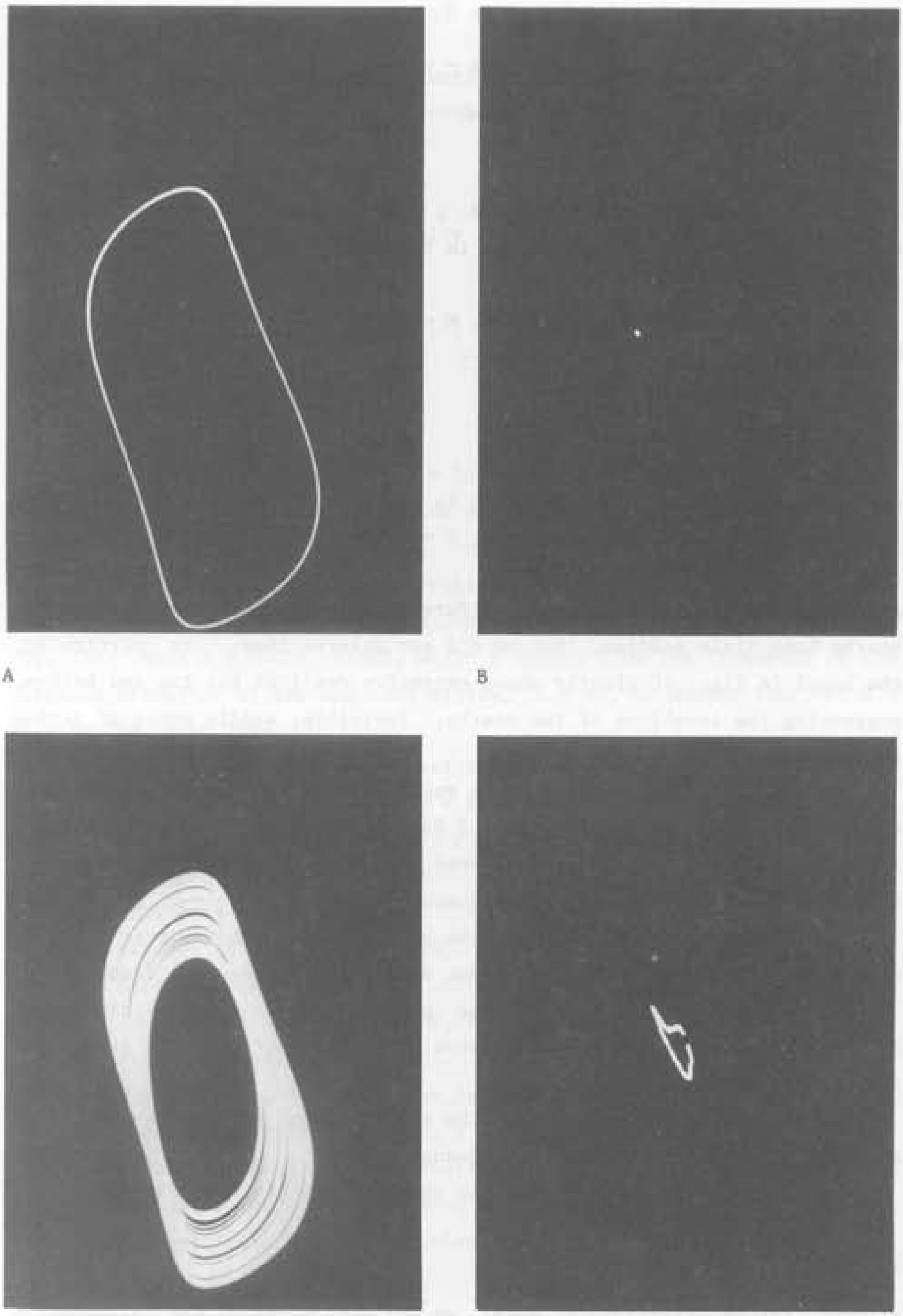


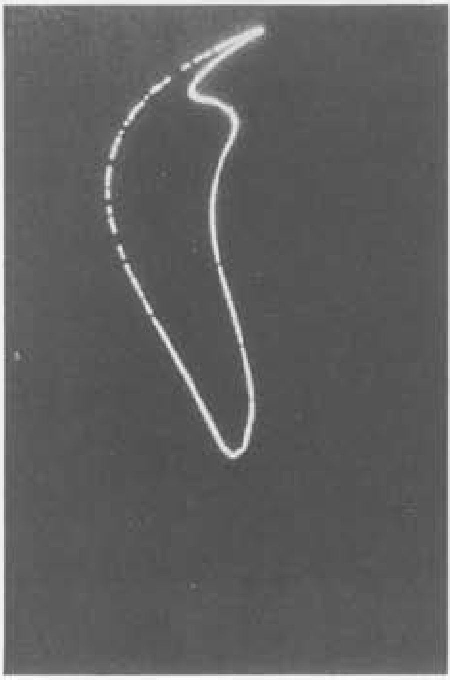
FIG. 1. AIT bifurcations observed in Van der Pol´s system.

A. F = 4.20 Hz., fundamental.

B. F = 4.20 Hz., strobed.

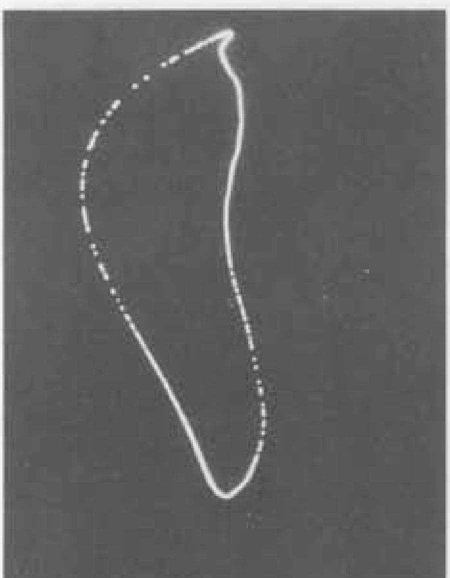
C. F = 6.60 Hz., baby AIT.

D. F = 6.60 Hz., strobed.



E

F



G

H

FIG. 1 (cont.)

E. F = 7.05 Hz., torus.

F. F = 7.05 Hz., opposite phases.

G. F = 8.00 Hz., blue torus.

H. F = 8.15 Hz., fundamental.

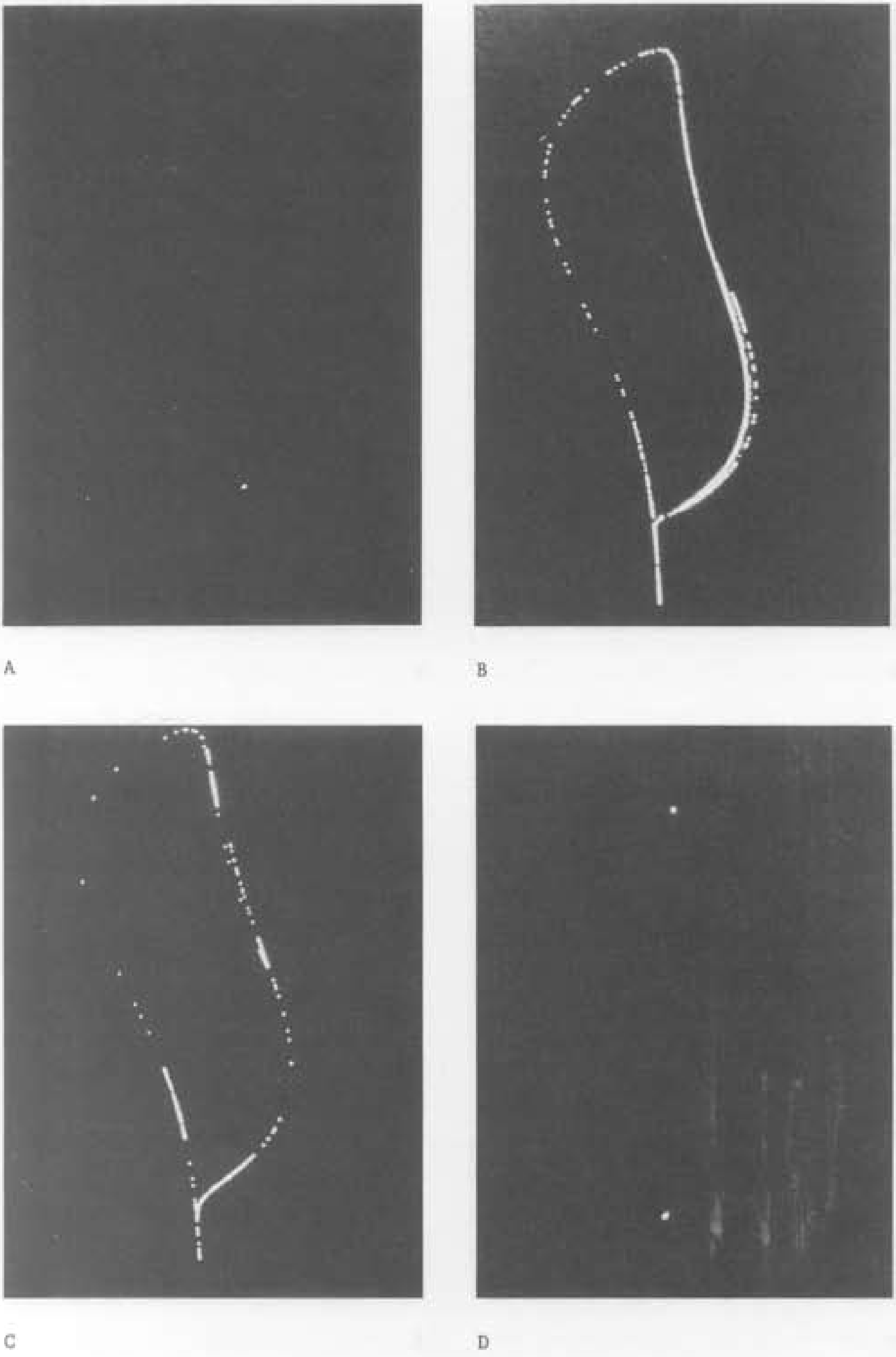


FIG. 2. Bagel bifurcations observed in Shaw´s system.

A. F = 6.15 Hz., fundamental.

B. F = 6.20 Hz., blue bagel.

C. F = 9.20 Hz., blue bagel.

D. F = 9.25 Hz., fundamental.

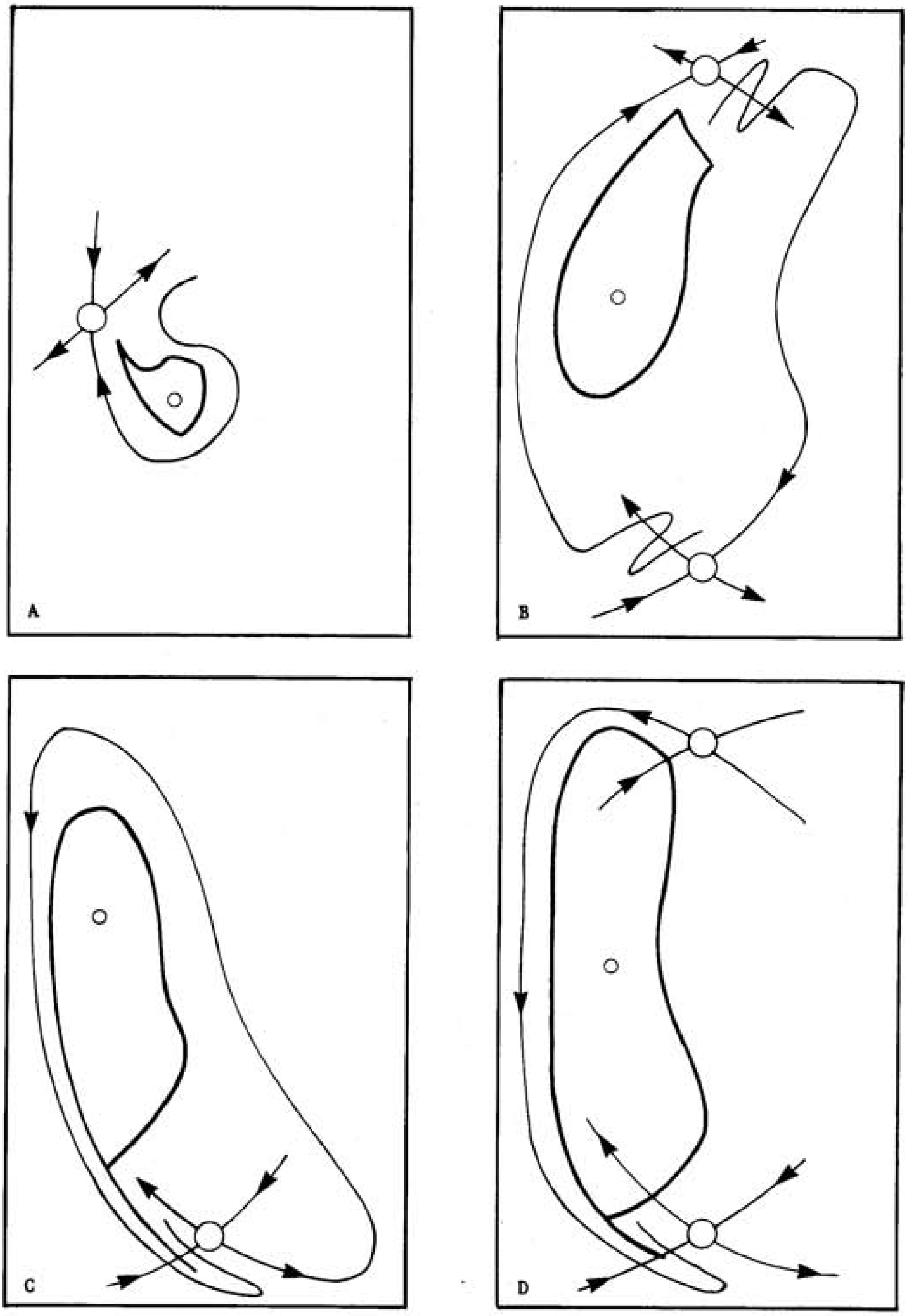


FIG. 3. Conjectured Homoclinic tangles.

A. Like 1D.

B. Like 1G.

C. Like 2B.

D. Like 2C.

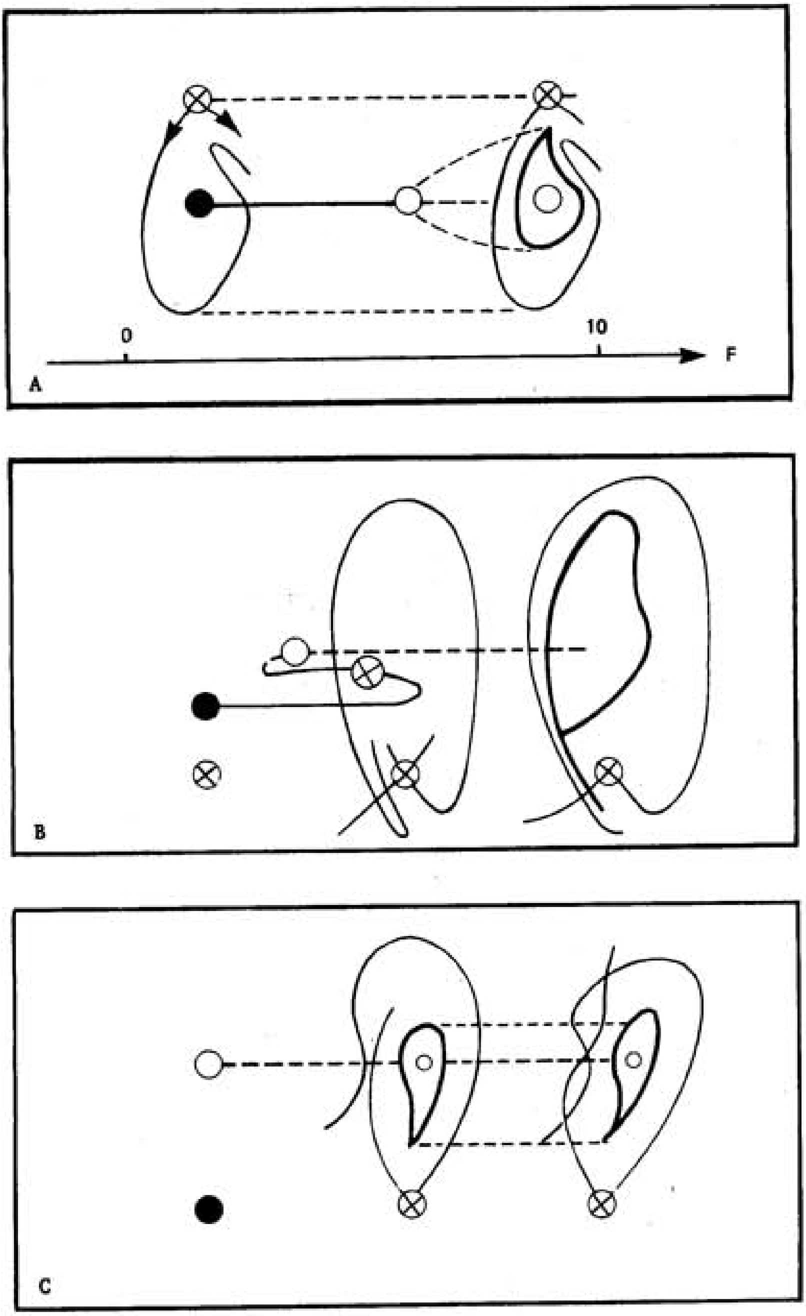


FIG. 4 Conjectured bifurcation diagrams.

A. Hopf in a nest.

B. Blue torus or bagel.

C. Captive balloon.

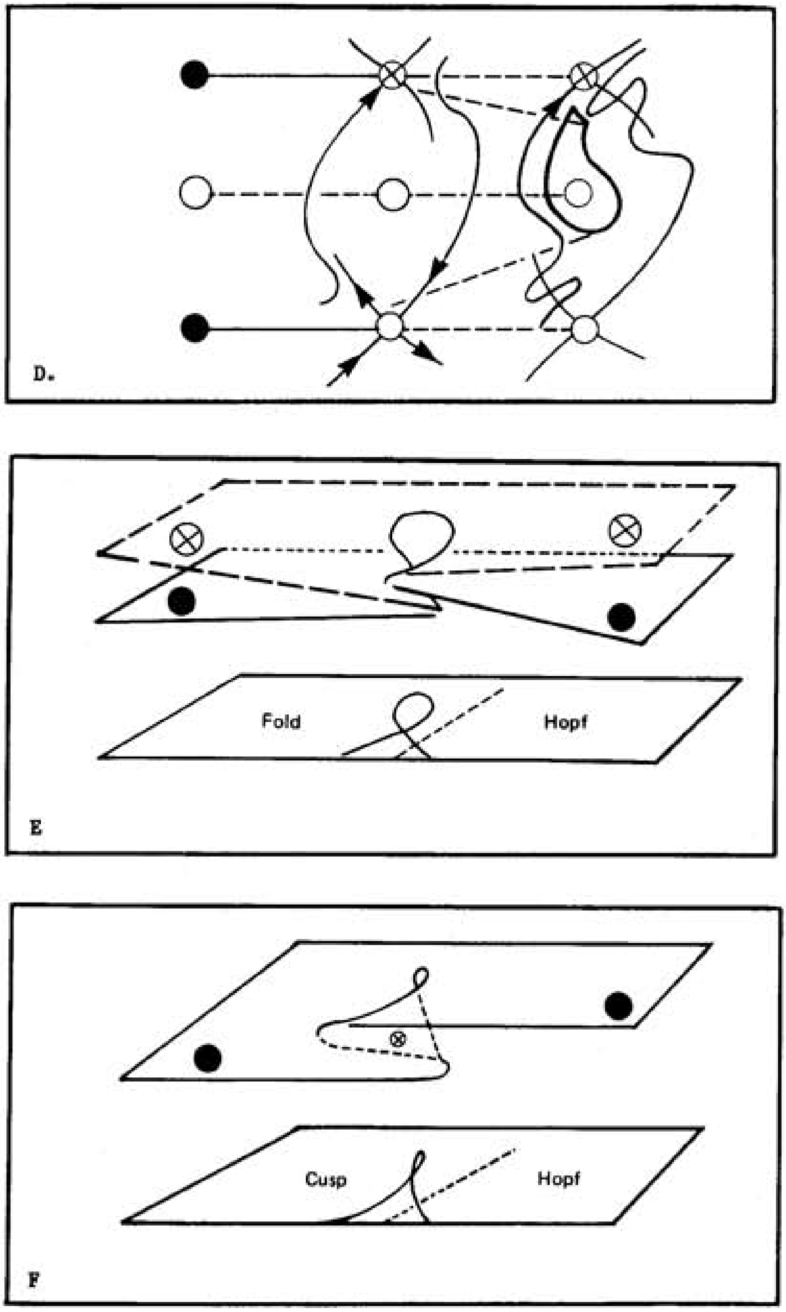


FIG.4 (cont.)

D. Subharmonic balloon.

E. Poppyseed.

F. Blueberry.

teardrop shaped section of an AIT was observed to grow from a point as the driving frequency, F, increased, a Neimark bifurcation within a homoclinic tangle is an obvious guess, as shown in Fig. 4A.

However, in the corresponding bifurcation in Shaw's system, from Fig. 2A to 2B, the chaotic bagel springs forth fully grown. So we conjecture here a blue bagel catastrophe, as shown in Fig. 4B. The disappearance of the fundamental periodic attractor, due (in this proposed model) to a static annihilation (saddle-node) catastrophe as the frequency is increased, drops the trajectory onto the bagel, recently appeared out of the blue, through the formation of a homoclinic tangle by the lower periodic saddle. We could apply this model to the preceding bifurcation as well.

These models are consistent with the well established theory of generic arcs of diffeomorphisms. But based on the observations, we would prefer a model in which the fundamental attractor destabilizes into a periodic saddle with a homoclinic tangle, within which an AIT or a bagel simultaneously forms out of the blue, as shown in Fig. 4C. This captive balloon catastrophe is theoretically unlikely, except at a bifurcation of codimension two. Nevertheless, we seem to observe this repeatedly in the analog simulations. This is an unsolved paradox at present, and deserves further study. Suitable bifurcations of codimension three, containing all three proposed arcs (4A, 4B, 4C) are shown in Figs. 4E and 4F. A third control parameter, not shown, creates a saddle connection. The bifurcations at the higher forcing frequencies, from Fig. 1H to 1G and from 2D to 2C, appear to be subharmonic (period two) versions of the captive balloon catastrophe, as shown in Fig. 4D.

5. CONCLUSION

The occurrence of toral and bagel chaostrophes in forced Van der Pol systems is established. It remains to draw the surrounding homoclinic tangles, in the wonderful style of Hayashi, by actual simulation instead of fantasy. But that is very difficult. The ubiquitous coincidence of two bifurcation events, called here the captive balloon catastrophe, also suggests further work, in search of a poppyseed bifurcation.

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