Singularities & Dynamical Systems
S.N. Pnevmatikos (editor)
© Elsevier Science Publishers B.V. (North-Holland), 1985

#35

IN PURSUIT OF BIRKHOFF'S CHAOTIC ATTRACTOR

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A history of the chaotic bagel attractor, in theory and in experiments with forced oscillators, from 1916 to the present, including an account of its occurrence in catastrophic bifurcations.

1. HISTORICAL INTRODUCTION

There is a growing awareness of the gap between theoretical and experimental concepts in chaotic dynamics. As the bagel is unique among chaotic attractors in having a long history in theoretical as well as experimental dynamics, we have chosen to emphasize it here, in hopes of closing this gap.

In 1932, Birkhoff published a remarkable paper on *remarkable curves* [1]. These are curves only in the sense that they are closed subsets of the plane of measure zero, dividing the plane into two components. They originally arose as attractors in twist mappings of a plane annulus, in 1916. Birkhoff showed they are not Jordan curves, so he called them remarkable curves. In fact, they are fractals. The suspension of a twist map provides a flow on a thickened torus, with a remarkable surface, or fractal torus, as attractor. We call this a *Birkhoff bagel* .

Shortly after the appearance of Birkhoff's paper, Levinson conjectured that this bagel might occur in a forced dynamical system of second order [2]. At about the same time, Cartwright and Littlewood [3] guessed that the bagel had already been observed in this context, unknowingly, by Van der Pol and Van der Mark [4] in 1927. In fact, they reported the occurrence of an "irregular noise" in the earphone of a forced neon glow tube device. This device may be regarded as an analog simulator for some forced dynamical system of second order, but probably not the well known Van der Pol system.

In recent years, experimentalists searched in vain for a chaotic attractor in the forced Van der Pol system until 1980, when Shaw announced a sighting of the bagel, at last [5]. Very elusive, this *Van der Pol bagel* is very hard to find, but Shaw discovered a variant forcing of the Van der Pol system in which chaotic bagel attractors abound, which we call *Shaw bagels*. More recently, we have found abundant chaos in a forced Van der Pol system [6].

In this paper, we briefly describe these results, and present some conjectures on catastrophic bifurcations in which a bagel suddenly appears or disappears.

2. LIENARD'S DIFFEOMORPHISM

The Van der Pol system in Cartwright normal form,

$$P \begin{cases} \dot{x} = y \\ \dot{y} = -x + k(1-x^2)y, \ k > 0 \end{cases}$$

is obtained from Rayleigh's model for the clarinet reed,

$$R \begin{cases} \dot{u} = v \\ \dot{v} = -u + k(v-v^3/3)y, \ k > 0 \end{cases}$$

by differentiation. The equivalence of these two systems is conveniently seen by applying Lienard's map,

$$L: \mathbb{R}^2 \to \mathbb{R}^2; \ (x,y) \to (u,v)$$

defined by

$$L \begin{cases} u = -y + k(x - x^3/3), \ k > 0 \\ v = x \end{cases}$$

which is an area-preserving diffeomorphism. That is, $L_*P = R$. This equivalence is useful to obtain the phase portrait of the Van der Pol system, as the Rayleigh system is easier to analyze directly.

For example, the Rayleigh system is analyzed by means of the two characteristic curves,

$$C_1: \dot{u} = 0 \quad (u - \text{axis})$$

$$C_2: \dot{v} = 0 \quad (\text{graph of } u = k(v-v^3/3))$$

from which the limit cycle is found, as shown in Figure 1(a). The inverse image of these two curves, under L, consists of the two axes. However, the analysis of the Van der Pol system directly involves the curves,

$$D_1: \dot{x} = 0 \quad (x - \text{axis})$$

$$D_2: \dot{y} = 0 \quad (\text{graph of } y = x/k(1 - x^2))$$

as shown in Figure 1(b). Obviously it is easier to argue geometrically with the y-axis than with the three pieces of the curve, D_2.

3. ACCELERATION BIAS

The forced Van der Pol system is conveniently regarded as a serially coupled scheme of two dynamical systems, one of which is the Van der Pol system with an acceleration bias, or constant forcing term,

$$\text{PA} \begin{cases} \dot{x} = y \\ \dot{y} = -x + k(1-x^2)y + b \end{cases}$$

while the forcing system is

$$\dot{\theta} = 2\pi f \quad (\text{constant})$$

and the coupling function is

$$b = A \sin \theta$$

where $A > 0$, or sometimes, to break the symmetry,

$$b = A \sin \theta + a$$

for some constant, $a > 0$. The factorization of the forced Van der Pol system into these two subsystems allows a geometric intuition on the behavior of the forced system, at least, for slow forcing. For the response diagram of the driven system, PA, may be obtained from the Lienard diffeomorphism. In the Rayleigh coordinates, the pushforward, L_*PA, is the velocity-biased Rayleigh system,

$$\text{RV} \begin{cases} \dot{u} = x - a \\ \dot{x} = -u + k(v \quad - v^3/3) \end{cases}$$

for which the two characteristic curves are as shown in Figure 1(a), except for C_1, which is raised to the horizontal line, v = a. Obviously, the limit cycle will disappear for $|a| > 1$, leaving an attractive point. Evidently, there are two Hopf bifurcations in the response diagram of this system, which we call the *red cigar* , shown in Figure 2(a). Verification of this response has been obtained by simulation [6].

Returning to the problem of the elusive bagel, we see that forcing the Van der Pol system will not be likely to produce a chaotic attractor unless the force is large enough to periodically pass at least one of the Hopf bifurcations. Also, it will help if the forcing function is asymmetric, and rapid. In fact, these intuitions have succeeded in producing abundant chaos [6].

4. VELOCITY BIAS

The Shaw variant of the forced Van der Pol system consists of applying the force to the velocity rather than the acceleration. Factoring into a serially coupled scheme of two subsystems, we obtain

$$\text{PV} \begin{cases} \dot{x} = y - c \\ \dot{y} = -x + k(1 - x^2)y \end{cases}$$

As the Lienard diffeomorphism is no help in this case, we may study the direct characteristic curves, as in Figure 1(b), with the horizontal line, D1, raised to level, c. This shows that in general, there are two critical points, a saddle and a repellor. Thus, as the level, c, increases, the limit cycle may interact with the invariant curves of the saddle. In fact, for some critical level, c = $\pm c_0$, the limit cycle vanishes in a blue sky catastrophe, as simulation has shown [6].

Thus the response diagram of this system contains a bounded snake of periodic attractors, the *blue sleeve*, shown in Figure 2(b). This shows that the trajectory grows rapidly when the force exceeds the critical values, $\pm c_0$, and gives some intuitive explanation for the abundance of chaotic bagels found in this system by Shaw.

5. BAGEL CATASTROPHES

The *blue bagel catastrophe* is, roughly, the suspension of the blue sky catastrophe by a periodic forcing system. Previously, we had speculated on the occurrence of this bifurcation in the forced Van der Pol system [7]. In simulation, we did not find it. Instead, we found a *red bagel catastrophe*, in which a periodic orbit catastrophically explodes into a bagel [8]. We end this bagel review with yet another catastrophic scenario, related to the bifurcation of codimension two studied by Chenciner [9].

We consider a flow in $S^1 \times \mathbb{R}^2$, obtained from a forced oscillator with a single control parameter. Before the catastrophic bifurcation, we have an attractive invariant 2-torus (AIT) with a rational flow, and braided periodic attractors and saddles. As the control is increased, the toral flow twists faster, and the inset cylinders of the braided periodic saddles become tangent to the outsets (first on one side, later on the other) and transversely homoclinic. With only one side homoclinic, we have a chaotic limit set of saddle type, as in Smale's original horseshoe. But with both sides homoclinic, the tangle is attractive, and comprises a bagel attractor. Thus, in a bifurcation sequence of codimension one, an AIT *explodes* into an attractive bagel. The rotation number is replaced by a rotation interval, probably containing the original rational in its interior. Many related events, such as the annihilation of two bagels, suggest themselves. It seems increasingly likely that chaotic attractors abound in typical forced oscillators. Some further evidence for this is given in a companion paper [6].

6. ACKNOWLEDGEMENTS

Many valuable discussions took place at the conference. In particular, it is a pleasure to thank Marc Diener for his explanations on the red cigar and the canard which he had independently discovered [10], Alain Chenciner for clarifying the technical details of Birkhoff's attractor, and Jean-Claude Roux for pointing out its occurrence in his experiments [11]. The drawings are by Christopher D. Shaw.

Fig. 1(a). Characteristic curves of the Rayleigh system, R, with k = 1.

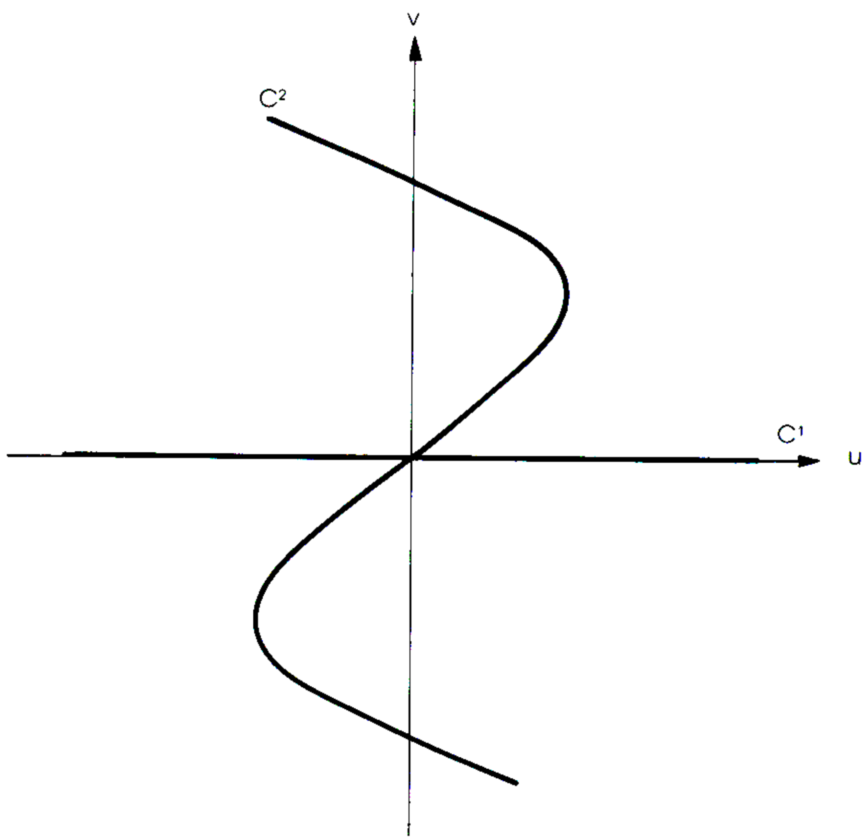


Fig. 1(b). Characteristic curves of the Van der Pol system, P, with k = 1.

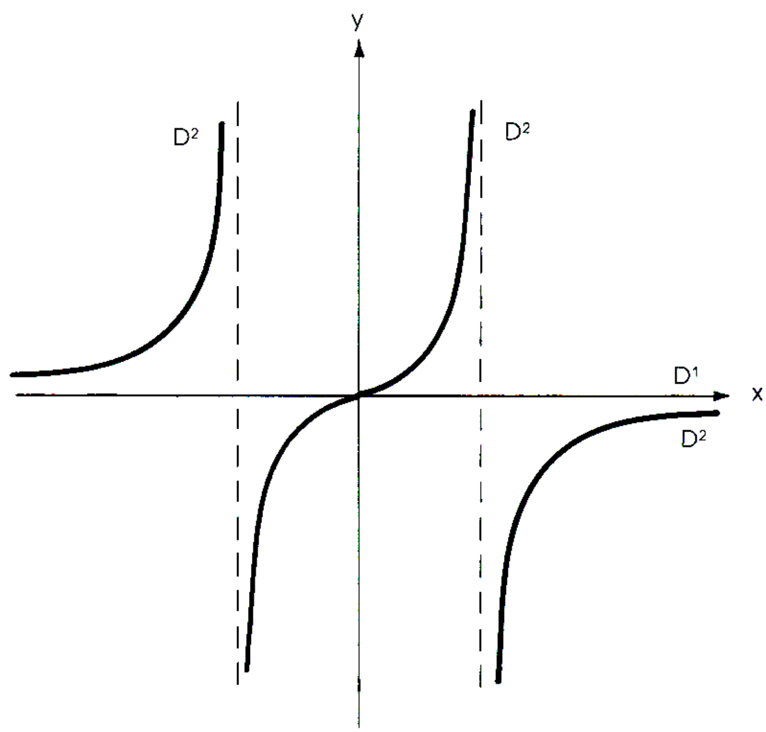


Fig. 2(a). The red cigar, in the response diagram for the acceleration-biased Van der Pol system, PA, or equivalently, the velocity-based Rayleigh system, RV, with k = 1. The control parameter, a, varies from -1 to 1.

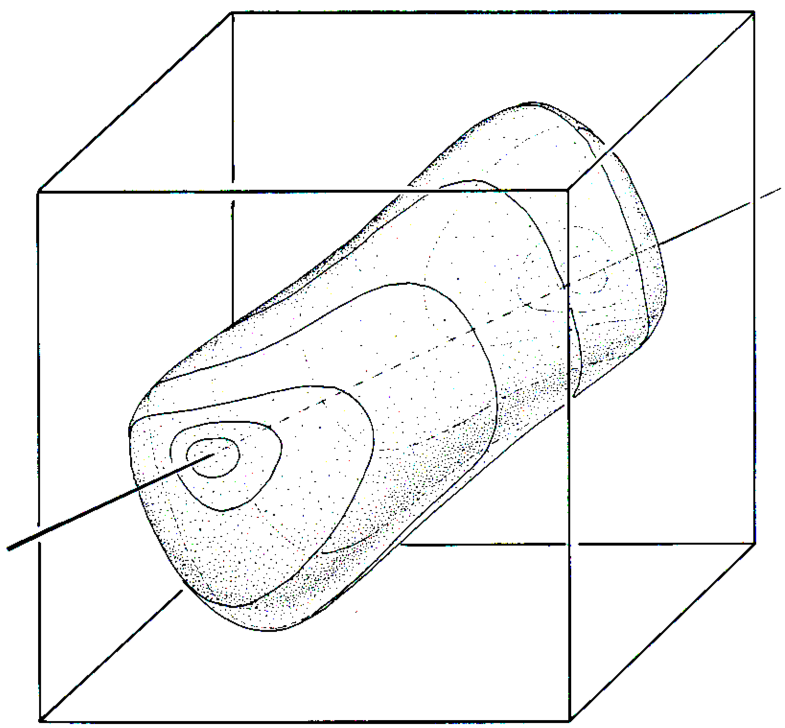
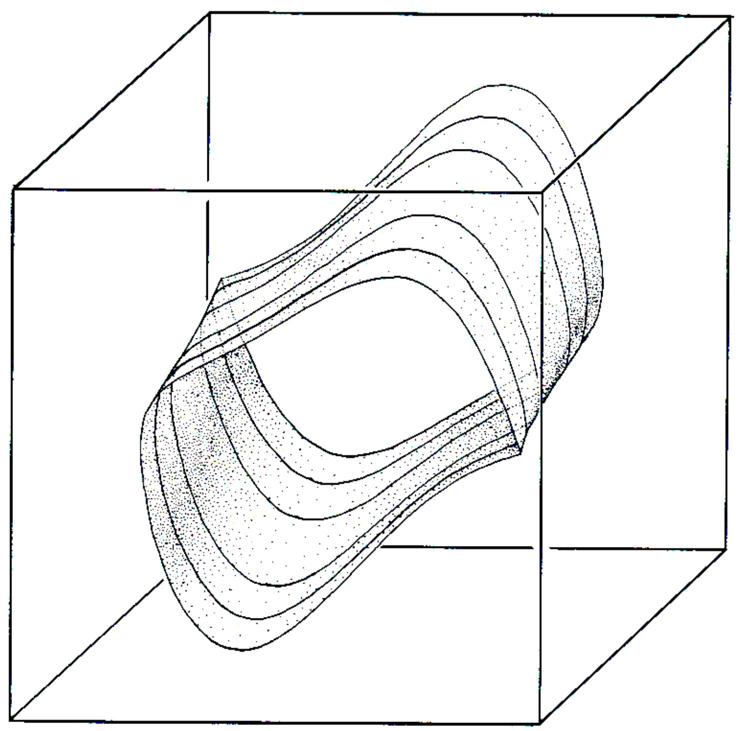


Fig. 2(b) The blue sleeve, in the response diagram for the velocity-biased Van der Pol system, PV, with k = 1. The control parameter, c, varies from -1 to 1.



Bibliography

[1] Birkhoff, G. D., Sur quelques courbes fermees remarquables, Bull. Soc. Math. d. France 60 (1932) 1-26.

[2] Levinson, N., Transformation theory of non-linear differential equations of the second order, Ann. Math. 45 (1944) 723-737.

[3] Cartwright, M. L. and Littlewood, J. E., On non-linear differential equations of the second order, J. London Math. Soc. 20 (1945) 180-189.

[4] Van der Pol, B. and Van der Mark, J., Frequency demultiplication, Nature 120 (1927) 363-364.

[5] Shaw, R. S., Strange attractors, chaotic behavior, and information flows, J. Naturforsch. 36a (1981) 80.

[6] Abraham, R. H. and Simo, C., Bifurcations and chaos in forced Van der Pol systems, this volume.

[7] Abraham, R. H., Catastrophes, intermittency, and noise, in Fischer, P. and Smith, W. R. (eds.), Chaos in Dynamics (Dekker, New York, 1984).

[8] Abraham, R. H. and Scott, K. A., Chaostrophes of forced Van der Pol systems, in Fischer, P. and Smith, W. R. (eds.), Chaos in Dynamics (Dekker, New York, 1984).

[9] Chenciner, A., Courbes invariantes, orbites periodiques, ensembles de Cantor invariants dans les bifurcations des points fixes elliptiques non conservatifs, this volume.

[10] Diener, M., Canards, ou comment bifurquent les systemes differentiels lents-rapides, preprint, CERDRO, BP1510 Saint-Charles, Oran.

[11] Roux, J.-C., Bifurcations and complex behavior in physico-chemical systems: recent experimental results, this volume.

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