# PHASE REGULATION OF COUPLED OSCILLATORS AND CHAOS

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## ABSTRACT

The phase entrainment of coupled oscillators, often confused with frequency entrainment, was first described by Huygens in 1665 as the "sympathy of clocks". As it refers to the independence of the equilibrium phase difference from variations in initial conditions or the strength of the coupling, we refer to this phenomenon as *phase locking*. In the context of ensembles of coupled oscillators, it has many important applications. Recently, Vassalo Pereira has given a derivation of the sympathy of clocks based on Andronov's model for the pendulum clock, showing that it is the "tick-tock" of the escapements, rather than the swings of the pendula, which are responsible for the phase regulation. Here, we generalize Vassalo Pereira's result to arbitrary coupled oscillators, to obtain a geometric theory of phase regulation due to pulsatile forces. The extension of this geometric theory to chaotic attractors is indicated as well.

## CONTENTS

## 1. INTRODUCTION

The entrainment of coupled oscillators, in frequency and phase, is a fundamental phenomenon in the physical, biological, and social sciences. Both frequency entrainment and phase regulation are classical topics in the theory of nonlinear oscillations. The *entrainment of the frequencies*, restricting the frequencies of two coupled oscillators to equal values (or to rationally related values, called harmonics), is neatly explained by Peixoto's theorem [1]. Consider the isochronous case, in which the entrained frequencies are equal. Calling the two oscillators $A$ and $B$, we arbitrarily choose a point on each limit cycle for its zero phase. Then each has a well-defined phase defined around its limit cycle, $\theta_A$ and $\theta_B$. Let $\Delta$ denote the phase of $B$, when $A$ is at phase zero. Then $\Delta \approx \theta_B - \theta_A$ throughout the isochronous harmonic. If the coupling, or one of the oscillators, depends on a control parameter, then the harmonic will be preserved under small variations of the parameter. The amplitude, frequency, and phase difference, $\Delta$, will vary with this parameter. Plots of these functions, known as *response curves*, abound in the literature of nonlinear oscillations. Here, we are primarily concerned with the *phase response curve*, or *PRC*, of coupled oscillator systems. By *phase regulation*, we mean the design of a coupled oscillator system to obtain a particular PRC, specified in advance. Nature provides us with many such designs.

An astonishing example of such a design is the *Huygens phenomenon*. Around 1664, two self-sustained oscillators (pendulum clocks designed by Huygens and constructed by his clock-maker) were placed on the same table. The coupled system was isochronously frequency-entrained. The phase difference, $\Delta$, was observed to be zero: the pendula swung in unison. After moving one of the clocks, and disturbing their *sympathie*, Huygens observed that the same synchrony was restored after twenty minutes or so (Huygens, 1893). Variation of the control parameter (distance between the clocks) did not change $\Delta$. The PRC of the coupled system was a constant, zero. The clocks were *phase-locked*. Under the influence of Lord Rayleigh, there have been many experiments with coupled pendula. The results of Duffing are particularly well-known, although preceded by Martienssen [2]. They showed that the PRC of such a system is not a constant [3]. So the simple coupling of the pendula of the clocks, through the elastic medium of the table, does not explain the Huygens phenomenon.

After three centuries, we have at last a satisfactory explanation by Vassalo Pereira (Pereira, 1982). He has shown that it is not the swing of the clocks' pendula, but the tick-tock of their escapements, that entrains their phases. His argument, based upon a dynamical model of the escapement mechanism due to Andronov, is simple and convincing (Andronov, 1963). According to this explanation, the phase regulation mechanism is *pulsatile*. The tick of one clock, caused by the fall of its escapement, transmits a soliton through the table, which delivers a sharp, pulsatile forcing to the fulcrum of the pendulum of the other clock, and vice versa. The dynamics of these acceleration pulses, regarded as perturbations of Andronov's model, entrains the phase difference to zero. This *favorite phase difference* is determined by a *hook* in the periodic attractor of Andronov's model, a *geometric feature* shown in Fig. 4.4.

In this paper, we generalize Vassalo Pereira's result to arbitrary coupled oscillators. This makes the explanation of the Huygens phenomenon more convincing, and also provides a simple geometric strategy for predicting, or engineering, the phase regulation of any coupled oscillators. This might have useful applications in the physical, biological, and social sciences. Our main tools, based on the global analysis of Section 2, are the *phase regulation dynamic* of a flexible coupling, described in Section 3, and the *shape form* of an oscillator, in Section 4. These provide a simple expression for the attractive phase differences of forced oscillators, the *favorite phase formula* of Section 5. Evaluated in the context of Andronov's model for the pendulum clock, forced with periodic pulses, this formula reduces to that of Vassalo Pereira. The geometric interpretation of the formula is illustrated in Section 6. The application to phase locking of chaotic attractors is suggested in Section 7. In this paper, we try to standardize, and supply with a new theoretical foundation, ideas which are emerging in numerous applied and experimental studies of pulse-forced oscillators [4]. Those interested primarily in the practical applications of phase regulation may skip the global analysis (comprising two propositions and three corollaries), and proceed directly to Section 6.

ACKNOWLEDGEMENTS

My interest in this subject, and the origin of the ideas presented here, date from extensive talks with Alan Garfinkel in early 1983. My thanks are due to him for inspiration and for his comments on an early draft of this paper; to the late Crump Institute of Medical Engineering, at the Los Angeles campus of the University of California, for the financial support which made these talks possible; and to Tim Poston for extensive comments on several drafts, as well as his work in creating the illustrations with PostScript and an Apple LaserWriter. Finally, it is a great pleasure to acknowledge the inspiration provided by Otto Rössler's work, and his friendship, over these past two decades.

## 2. GLOBAL ANALYSIS OF FLEXIBLY FORCED SYSTEMS

We now turn to forced oscillators, in the context of complex dynamics (Abraham, 1984). In this section, we show how a dynamical scheme representing flexibly forced oscillators, in a very general context, may be represented as a curve of vectorfields, and relate this to classical perturbation theory. Recall that a *dynamical scheme* is a dynamical system (vectorfield) depending on control parameters. Now we consider two dynamical schemes, $A$ and $B$, as in Fig. 2.1, each containing an oscillator. We assume there is a *static coupling* from $A$ to $B$. This means that the instantaneous state of $A$ (a point in its state space $M_A$) influences the control parameters of $B$ (a point in its control space, $C_B$) through a mapping,

$f : M_A \to C_B$. A forced oscillator, the forced Van der Pol system for example, may be represented in this way, as follows.

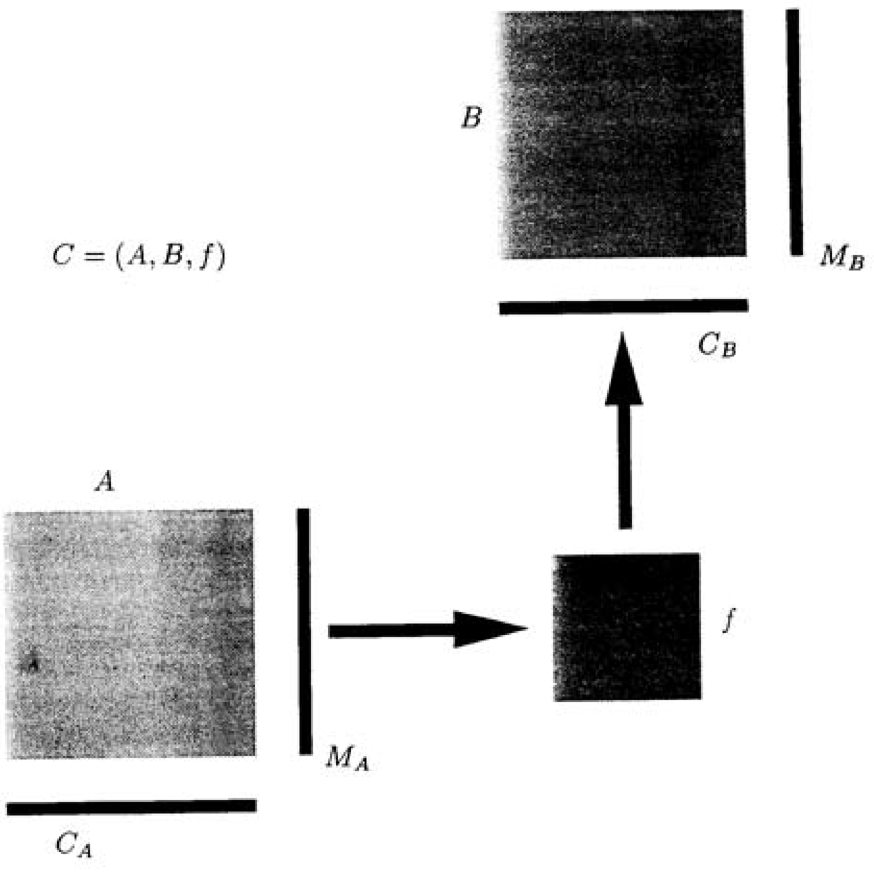


**Fig. 2.1.** A two-scheme complex.

A FORCED OSCILLATOR AS A COMPLEX DYNAMICAL SYSTEM

For the simplest possible case, we will assume that $A$ has no control parameters, or equivalently, that its controls will not be changed. Further, we will assume it runs only in a single periodic attractor, and thus delete the rest of its state space from the model. The state space could consist, then, of a single periodic attractor and its basin of attraction. But we will assume that the system has been running for some time, and that all significant transients have died away. So we may delete the basin as well, leaving only the attractor. Thus, we may model the states of $A$ by a single variable, its *phase*,

$$\theta \equiv \theta_A \in M_A = T^1 \equiv \mathbf{R}/(2\pi) \tag{2.1}$$

and its fixed dynamical system by the constant vectorfield,

$$\theta' = 2\pi/\tau_A \tag{2.2}$$

where $\tau_A$ is the *period* of the oscillation of $A$. Finally, we introduce a static coupling from $A$ to $B$, a mapping

$$f : M_A \to C_B \tag{2.3}$$

from phases of $A$ to controls of $B$. Necessarily, this is a periodic function,

$$f(\theta + \tau_A) = f(\theta). \tag{2.4}$$

The coupled system is a vectorfield, $X$, on the product manifold $N \equiv M_A \times M_B$, represented by

$$\theta' = 2\pi/\tau_A \tag{2.5$a$}$$

$$x' = V_{f(\theta)}(x) \tag{2.5$b$}$$

Here, we use $\theta$ in place of $\theta_A$ or $x_A$, and $x$ in place of $x_B$. Also, for $c \in C_B$, $V_c$ is a vectorfield on $M_B$. Thus, for fixed $\theta$, the map $x| \to V_{f(\theta)}(x)$ is a vectorfield on $M_B$. But adjoining equation (2.5a) to (2.5b), we may regard this as a vectorfield on $N$. This particular suspension will be denoted in this paper by $f^*V$. Thus, $X = f^*V$. All this is summarized in Fig. 2.2.

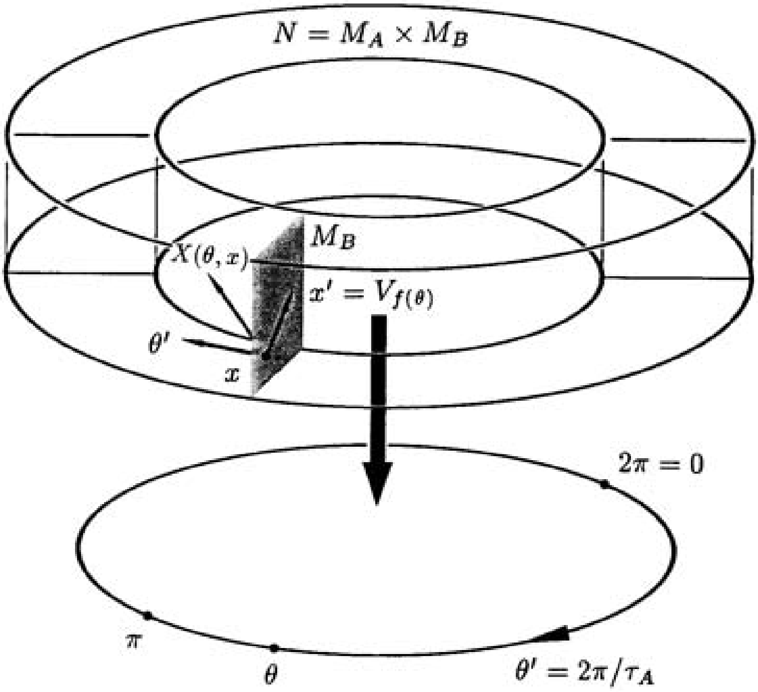


**Fig. 2.2.** A forced oscillator as a two-scheme complex.

FLEXIBLE COUPLINGS

We will be especially interested in the case in which this coupling is *weak*. This means that the image, $f[T^1]$, of the coupling map, $f : M_A \to C_B$, is a small subset of $C_B$, the control manifold of $B$, perhaps within a small neighborhood of a fixed value, say $c_0 \in C_B$. One way this arises in applications is in the context of a *flexible coupling scheme*, in which the coupling map itself depends smoothly upon control parameters,

$$f : D \times M_A \to C_B;\ (d, \theta) \mid\to f_d(\theta) \tag{2.6}$$

where $D$ is the control space for the flexible coupling function, $f$. For fixed $d \in D$, let $X_d$ denote the vectorfield on $N = M_A \times M_B$ defined by (2.5), with $f_d(\theta)$ in place of $f(\theta)$. Let $V = V^k(N)$, the space of all $C^k$ vectorfields on N. Then $X : D \to V;\ d \mid\to X_d$ is a scheme on $N$. See Fig. 2.3.

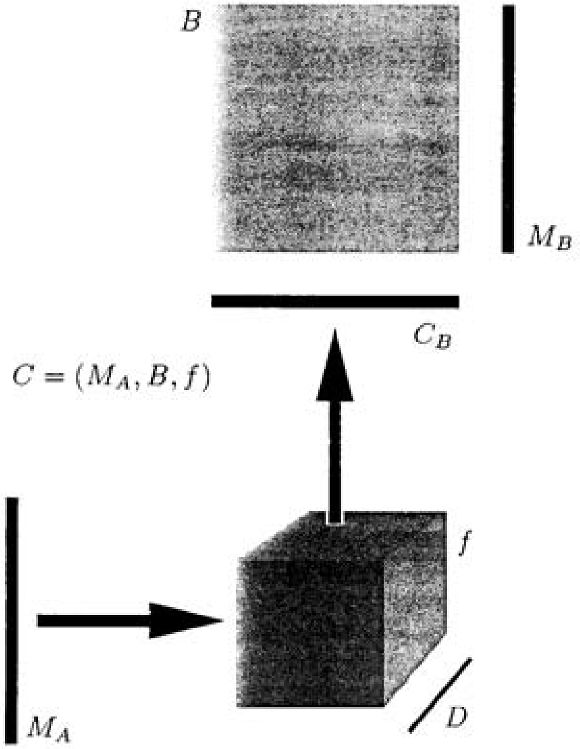


**Fig. 2.3.** A flexible coupling scheme.

In this context, suppose that for a special value of the coupling parameter, say $d_0 \in D$, the coupling map is a constant, or

$$\forall \theta \in T^1, \ f_{d_0}(\theta) \equiv c_0. \tag{2.7}$$

In this case, the coupling with $d = d_0$ may be regarded as the weakest possible, and for $d$ sufficiently near to $d_0$, the coupling is as weak as you wish. The coupling control parameter, $d \in D$, *turns on* the weak coupling *flexibly*, as it moves gradually away from $d_0$, and the control $c = f_d(\theta)$ is perturbed away from $c = c_0$. See Fig. 2.4.

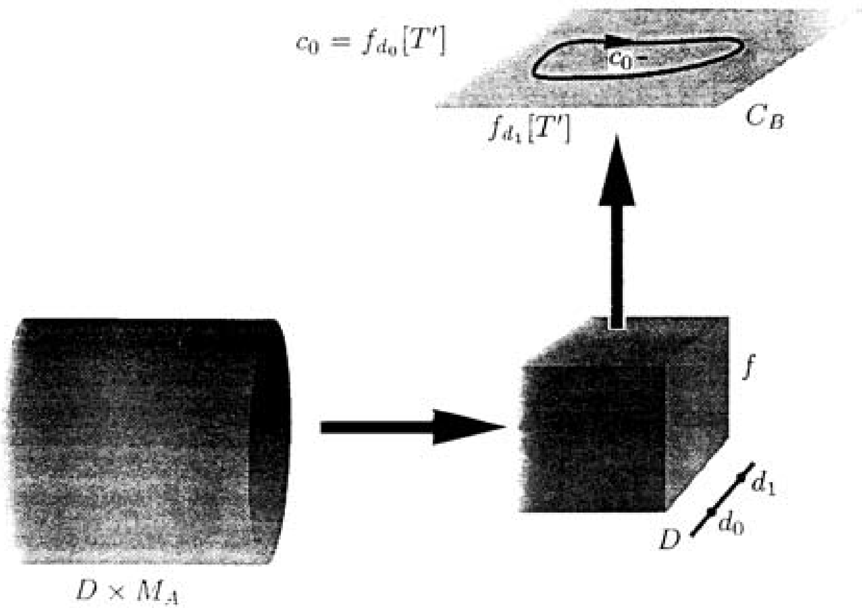


**Fig. 2.4.** A weak flexible coupling.

PERTURBATION FORM

We will now consider such a weak flexible coupling in the special case, common in applications, in which the coupling control parameter is *one-dimensional*, or $D = \mathbf{R}$. If in addition $d_0 = 0$, we call $d$ an *amplitude parameter*. Henceforth, we will consider only this case, and will normally use $a$ to denote the amplitude parameter. Now the coupled system, (2.5) above, becomes a vectorfield $X_a$ on $N$,

$$\theta' = 2\pi/\tau_A \tag{2.8a}$$

$$x' = V_{f_a(\theta)}(x) \tag{2.8b}$$

where $a$ is real, and $f_0(\theta) = c_0$ for all $\theta$. The flexibly coupled system is itself a one-parameter scheme. Letting $V_0 = V_{c_0}$, we may rewrite (2.8b) in classical perturbation form,

$$x' = V_0(x) + W_a(\theta, x) \tag{2.8c}$$

where $W_a(\theta, x)$ is smooth in $a \in \mathbf{R}$ and in $(\theta, x) \in N \equiv T^1 \times M_B$, periodic in $\theta$ with period $2\pi/\tau_A$, and $W_0 = 0$. In fact, as $a$ is real, we may think of $a \mid \rightarrow W_a$ as a curve at the origin in $V$, the space of smooth vectorfields on the product manifold $N$. Likewise, $X_a = V_0 + W_a$ determines a curve at $X_0 = f^*_{d_0} V_0 \in V$, in the suspension notation introduced for (2.5).

SUMMARY

By flexibly coupling scheme $A$ (the forcing oscillator) to scheme $B$ (the driven system), we have constructed a complex scheme, $C = (A, B, f)$. Thus, we may bring the methods of global analysis to bear on the problems of coupled oscillators, and periodically forced chaotic attractors as well.

## 3. THE PHASE REGULATION DYNAMIC

We now make use of some simple constructions of global analysis to define a special vectorfield which is natural in this context, called the *phase regulation dynamic*. We begin with an arbitrary one-parameter scheme. At the end of the section, we specialize to the flexibly-forced oscillator, which has been reinterpreted as a one-parameter scheme in the preceding section.

### AN ARBITRARY ONE-PARAMETER SCHEME

Let $X_0$ be a vectorfield of class $C^k, k > 1$, on a manifold, $N$, having a global cross-section, $\Sigma$. Let $V = V^k(N)$, the space of all $C^k$ vectorfields on $N$, and $D = D^k(\Sigma)$, the group-manifold of all $C^k$ diffeomorphisms of $\Sigma$, both with appropriate topologies. Then there is a neighborhood $U$ of $X_0 \in V$ such that all vectorfields $Y \in U$ have $\Sigma$ as a global cross-section. Let $F(Y) \in D$ denote the first-return map of $Y$ on this cross-section. Then

$$F : U \subset V^k(N) \to D^k(\Sigma) \tag{3.1}$$

is a smooth map from vectorfields on $N$ in $U$ to diffeomorphisms of the cross-section $\Sigma$. This is a generalization of the classical *phase return curve*, or PRC (Pavlidis, 1973). We are going to make use of the derivative of this map at $X_0$. Technical details of these methods may be found in Abraham and Robbin (Abraham, 1967).

Let $I \in \mathbf{R}$ be an open interval containing 0, and

$$X : I \to U;\ a \mapsto X_a \tag{3.2}$$

be a smooth curve at $X_0$. This is a one-parameter dynamical scheme. Let $P$ denote the tangent vector to the curve $X$ at $a = 0$. Thus for $a \in I$,

$$X_a = X_0 + aP + a^2 Q(a) \tag{3.3}$$

where $Q(a)$ denotes the remainder in Taylor's formula. For small $a$, we may approximate the curve $X$ by the straight line scheme,

$$X_a \approx X_0 + aP \tag{3.4}$$

which is a common form of perturbation found in applications. Now let $G_a = F(X_a)$, comprising a curve in $D(\Sigma)$ at $G_0$, the first-return map of $X_0$.

**Definition 1.** The unique vectorfield $\xi$ on $\Sigma$ satisfying

$$G_{0*}\xi = \frac{d}{da}G_a|_{a=0} = T_{X_0}F(P) \tag{3.5}$$

is the *phase regulation dynamic* of the scheme $X$ with respect to the cross-section $\Sigma$. Here, $G_*$ denotes the push-forward of vectorfields by the diffeomorphism $G$.

Now, to simplify the discussion, we suppose that the phase regulation vectorfield $\xi$ is complete. Then let $\{\psi_a\}$ denote the flow of $\xi$. For small $a$ we may approximate the true curve of first-return maps with an exponential ray (that is, the exponential of linear ray) translated in the group $D$ to the base-point $F_0$,

$$F_a \approx \psi_a \circ F_0 \tag{3.6}$$

all of which is summarized in Fig. 3.1. This approximation is the basis for our interpretation of $\xi$ as the phase regulation dynamic for the favorite phases of the forced oscillator, to which we now turn.

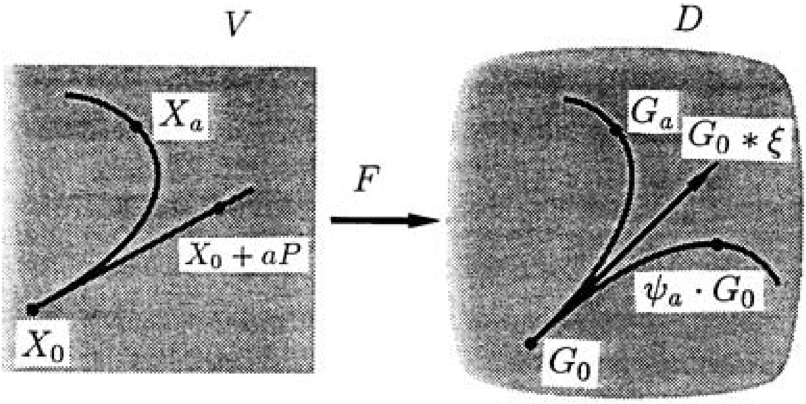


**Fig. 3.1.** The phase regulation dynamic.

### Application to the flexibly forced oscillator

Recall that in the preceding section, we interpreted a forced system as a complex dynamical scheme, consisting of two dynamical schemes, flexibly coupled. One, $A$, is the forcing oscillator. The other, $B$, is the periodically forced scheme. We now apply our theory to this complex dynamical system, $C = (A, B, f)$, regarded as a curve, $X$, at $X_0 \in V = V(N)$. If $a = 0$, there is no coupling. The target system, with vectorfield $V_0$ on $M_B$ before coupling, will be characterized by its portrait of attractors, basins, and separatrices. Note $X_0 = f^*_{d_0}V_0$.

HYPOTHESES

We next assume that this driven system is structurally stable for $a \neq 0$, and that $X$ is a generic arc in the sense of Sotomayor (Sotomayor, 1968). This may be best understood with the aid of Fig. 3.2. Note that the amplitude curve, $k$ passes near to, but not through, the point (1, 0). This point is the vertex

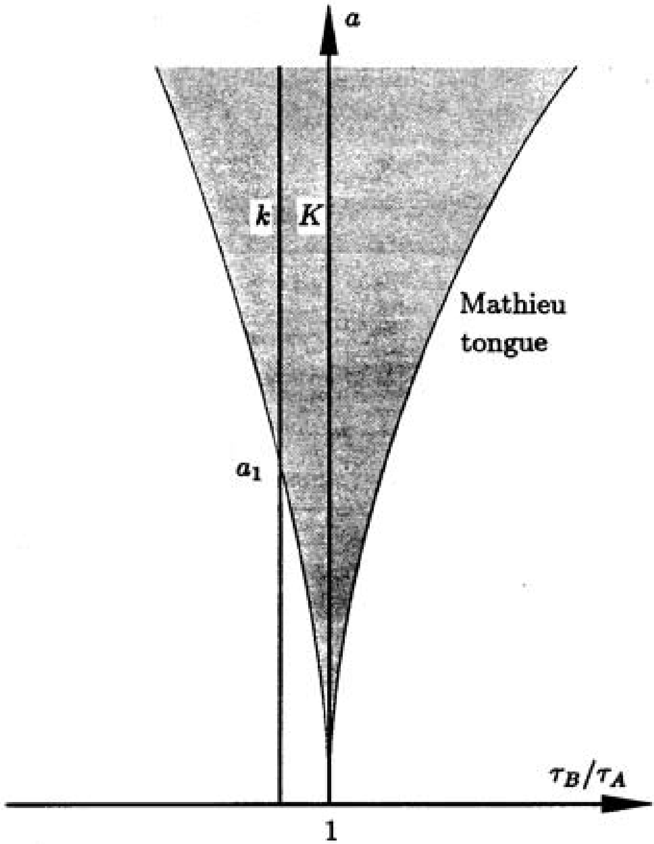


**Fig. 3.2.** Regime of the isochronous harmonic.

of the *Mathieu cusp*, enclosing the two-parameter regime of frequency, $\tau_B/\tau_A$, and amplitude, $a$, in which an isochronous harmonic may be found [5]. Thus, as the amplitude of our flexibly coupled scheme increases from zero, under this hypothesis of genericity, there is no isochronous harmonic until the bifurcation at the value $a_1$ at which the parameter curve, $k$, meets the Mathieu curve, as shown in Fig. 3.2. Meanwhile, with $a \in (0, a_1)$, other periodic attractors, which are called *harmonics*, will come and go on the attractive invariant torus (AIT), $T^2(a)$, in the fractal bifurcation event we have called *Sotomayor vacillation*. Despite the heroic efforts of great analysts such as Chenciner and Hermann, this event is still only partially understood. Nevertheless, the following is known. During the $a$ interval in which each harmonic exists, the rotation number of $G_a$ on the cycle $\Gamma_a$ (the image of a periodic trajectory $\gamma_a$, here moving smoothly with the parameter $a$) is a constant rational multiple of $2\pi$, say $\rho(a)$. As $a$ increases from zero to $a_1$, $\rho(a)$ changes from $\tau_B/\tau_A$ to one, along the *devil's staircase*, as shown in Fig. 3.3.

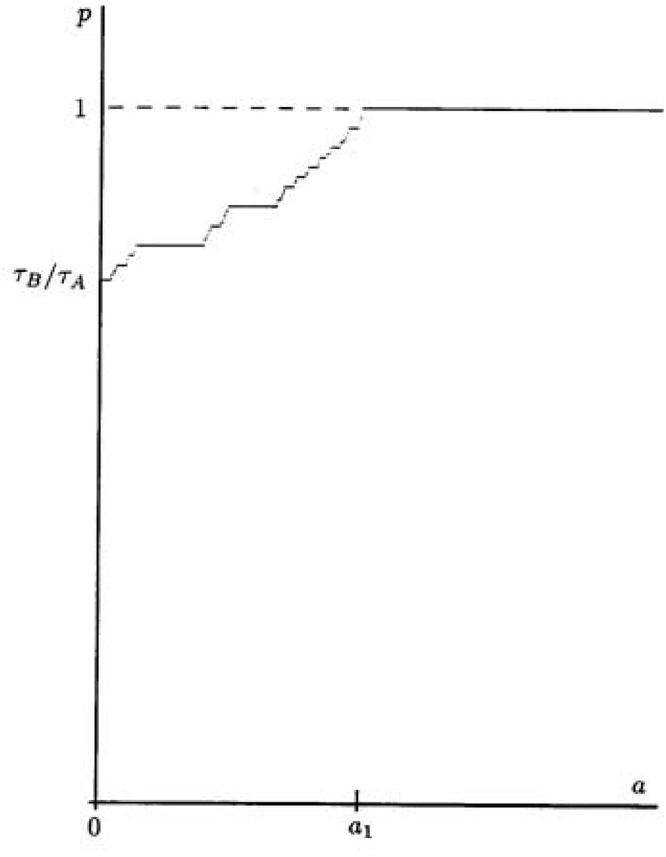


**Fig. 3.3.** The rotation number versus amplitude.

Our analysis, in Section 5 does not apply to the interval $0 \leq a < a_1$, so we must *assume* initial $\tau_B/\tau_A$ close enough to 1 that the $a$ range over which our perturbation expansion for $a_1 = 0$ is useful substantially exceeds $[0, a_1]$. Then behavior along the vertical amplitude curve $k$ (numerically close to the vertical $K$ through the Mathieu vertex, $(1,0)$, shown in Fig. 3.2) will be well predicted by analysis along $k$ for a similar $a$ range, once $a_1$ is passed and the topology of the flows on $N$ become similar, as shown in Fig. 3.3.

Thus, for an approximate result, we must further *assume* now, and throughout this paper, that $\tau_A \approx \tau_B$, so that $a_1$ is sufficiently small. Then the periodic forcing of the individual attractors of $V_0$ may be described in terms of the attractors of $\xi$, which are called the *favorite phases* of the scheme, as follows. Recall the following definition, due to Arthur Winfree [6].

**Definition 2.** Let $\Gamma$ be a periodic attractor of the vectorfield $V_0$ on $M_B$. Corresponding to the CMs of $\Gamma$ (all less than one in absolute value) there is an eigenspace (of codimension one) in the tangent plane $T_pM_B$ for each point $p$ in the image $\Gamma$, complementary to the vector $V_0(p)$. These subspaces extend to a distribution in a neighborhood of $\Gamma$ which is invariant under the flow of $V_0$. The integral manifolds of this distribution comprise the *isochron foliation* of $\Gamma$ [7]. The leaves of the isochron foliation are called *isochrons*.

Intuitively, all of the points of an isochron are asymptotic to the same phase on $\Gamma$. That is, the trajectories of different points on the same isochron come together as their transients die away. Asymptotically, they have the same isochronous phase.

**Proposition 1.** The point attractors of $V_0$ will become periodic attractors of $X_a$, located (approximately) by the point attractors of $\xi$. The *periodic attractors* of $V_0$ will become *attractive invariant tori (AITs)* of $X_a$. The periodic attractors braided around these tori will also be located (approximately) by the points at which the vectors of $\xi$ are tangent to the isochrons of the original unforced periodic attractor. Further, the periodic attractors of $X_a$ on the AIT tend, as $a \to 0$, to the favorite phases of the scheme. Within the zero phase section, $\Sigma$, the attractors describe curves tangent to $\xi$.

This follows from (3.6), and is illustrated in Fig. 3.4. The discussion of the periodic perturbation of the chaotic attractors that $V_0$ may have is postponed to Section 7.

### Summary

In this section, we have defined the phase regulation dynamic of an arbitrary one-parameter scheme as a vectorfield on $M_B$, and related its qualitative features to the harmonics of the flexibly forced scheme on $N$.

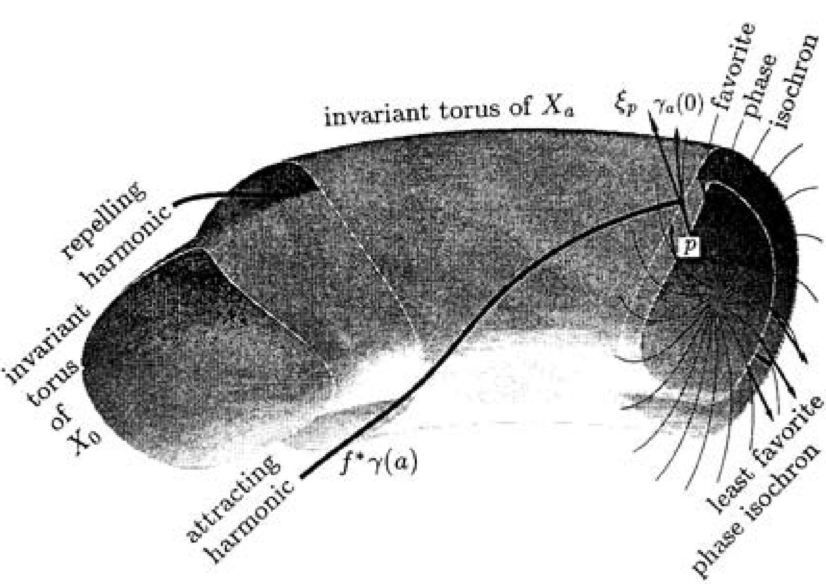


**Fig. 3.4.** The curve of fixed points approaching the favorite phase.

## 4. SHAPE FORM OF AN OSCILLATOR

As above, let $V$ be a vectorfield on a manifold, $M_B$, for which we will now write just $M$. Let $\gamma$ denote a periodic trajectory of period $\tau$,

$$\gamma : \mathbf{R} \to \mathbf{M}, \gamma(t + \frac{\tau}{2\pi}) = \gamma(t), \tag{4.1}$$

$\Gamma = \gamma[\mathbf{R}]$ its image, a limit cycle, and $p_0 = \gamma(0)$ an initial point. With $T^1 = \mathbf{R}/(2\pi)$ the circle with radian measure as above, $\gamma$ determines a unique diffeomorphism, $\phi : \Gamma \to T^1$, by

$$\phi(p_0) = 0, \phi(\gamma(t_1 + t_2)) = \phi(\gamma(t_1)) + \phi(\gamma(t_2)) \quad (mod 2\pi). \tag{4.2}$$

This is the *phase* of $\gamma$, relative to phase zero at the fiducial point, $p_0$. Roughly, $\phi$ is the inverse of $\gamma$, as shown in Fig. 4.1.

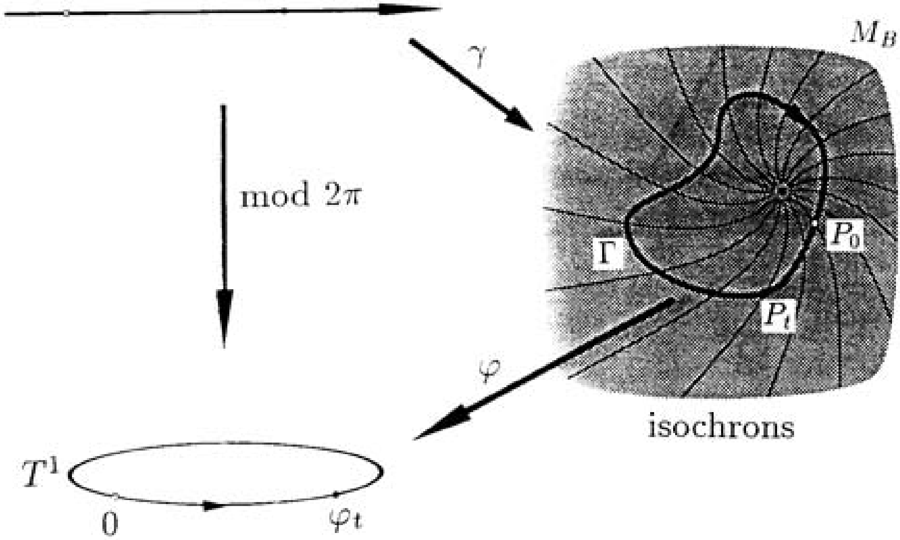


**Fig. 4.1.** Phase and isochrons of a periodic trajectory.

Our goal now is simply to decompose an arbitrary vector, $W_p \in T_pM$, at any point $p \in \Gamma$, into components tangent and complementary to $\Gamma$. We assume now that $\Gamma$ is an hyperbolic periodic attractor: none of its characteristic multipliers (CM's) are on the unit circle. Then the tangent space to $M$ at $p$ splits into a line tangent to $\Gamma$, generated by $\gamma'(t) = V_p \equiv V(p)$, and a complement, $I_p$, defined by the generalized eigenspaces of the CM's, in their aspect as eigenvalues of the derivative, not of the Poincaré map, but of the time $\tau$ map on $M$ [8];

$$T_pM = < V_p > + I_p. \tag{4.3}$$

We use this splitting to decompose the tangent vector, $W_p = Z_p + Y_p$, where $Y_p \in I_p$, and $Z_p \in < V_p >$, so $Z_p = \alpha V_p$ for some unique $\alpha \in \mathbf{R}$, as shown in Fig. 4.2. As this construction may be carried out for any vector $W_p \in T_pM$, we may define a one-form, $\alpha_p \in T_p^*M$, by $\alpha_p(W_p) = \alpha$, as constructed above. This is the *shape form* of $\gamma$ at $p$. Note that it depends on the shape of $\gamma$, and the configuration of its CM distribution and nearby isochron foliation, all of which we call the *geometry* of $\gamma$.

**Definition 3**. Varying $p$ around $\Gamma$, we obtain a section of the cotangent bundle of $M$ restricted to $\Gamma, T^*M|\Gamma$. This section,

$$\alpha : \Gamma \to T^*M;\, p \mid\to \alpha_p \tag{4.4}$$

is called the *shape form* of $\gamma$. Replacing $p$ by its phase, $\phi = \phi(p)$, we may relabel the shape form, $\beta_\phi = \alpha_p$, obtaining a section of $\phi$,

$$\beta : T^1 \to T^*M|\Gamma;\ \phi| \to \alpha_p \tag{4.5}$$

This may also be called the shape form of $\gamma$.

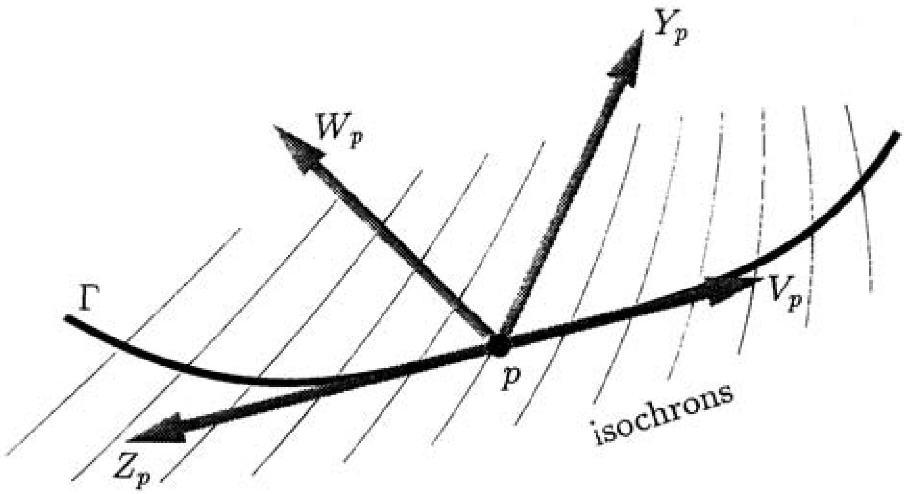


**Fig. 4.2.** The splitting of the tangent space of $M$.

Either way, the shape form is related to the *normal coordinates* of the hyperbolic periodic attractor, $\Gamma$. Any hyperbolic invariant manifold admits normal coordinates (Hirsch, 1977). In the case of a periodic attractor, normal coordinates are of the form $(R, \Phi)$, where $R$ is a coordinate chart on the normal foliation, and $\Phi$ is the asymptotic (isochronic) phase, an extension of $\phi$ to a neighborhood of $\Gamma$. That is, a point $x_0$ near $\Gamma$ with normal coordinates $(R(x_0), \Phi(x_0))$ will move (under the flow of $V$) to a point $x_t$ at a later time $t$, with normal coordinates $(R(x_t), \Phi(x_t))$. As $x_t$ is attracted to $\Gamma$, the normal coordinates tend to zero, $R(x_t) \to 0$, as $t$ increases. Meanwhile, for all times $t$, if $\Phi(x_0) = \Phi(p_0)$ (that is, $x_0$ is on the isochron of zero phase), then

$$\Phi(x_t) = \Phi(p_t) = \Phi(p_0) + \phi(p_t) = t\ (mod 2\pi). \tag{4.6}$$

That is, the isochrons (coordinate hyperplanes defined by $\Phi = \Phi_0$, a constant) are permuted among themselves by the flow of $V$, and all points in the same isochron (such as $(R,\ \Phi_0)$ with $\Phi_0$ fixed) tend to the same trajectory on $\Gamma$, the trajectory of $(0, \Phi_0)$. See Fig. 4.3.

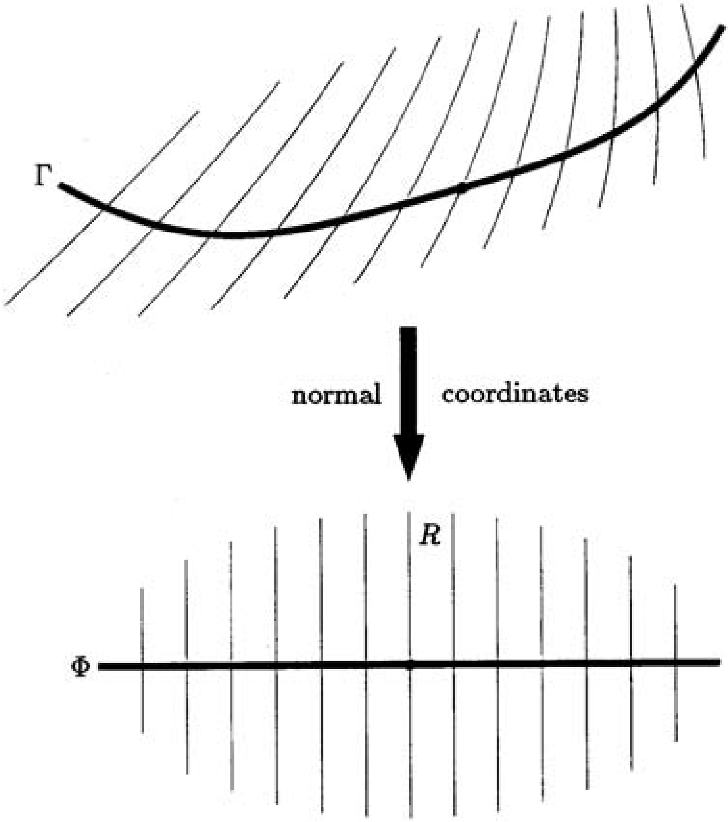


**Fig. 4.3.** Normal coordinates of an hyperbolic periodic attractor.

Returning to the shape form of $\gamma$, the factors of the decomposition $W_p = I_p + < V_p >$ correspond to the normal (within an isochron) and tangential (angular or phase) coordinates, $R$ and $\Phi$, respectively. The one-dimensional factor in the $\Phi$ direction (along $\Gamma$) is rescaled by the shape form, using the speed (length of $V_p$) as a unit. Thus,

$$W_p = \alpha_p(W_p)V_p + U_p, \tag{4.7}$$

where $\alpha_p(W_p)V_p$ is the angular component of $W_p$, and the normal component $U_p \in I_p$, is the CM eigenspace tangent to the isochron at $p \in \Gamma$. Note that $\alpha_p(W_p) = 0$ iff $W_p \in I_p$.

As we are favoring phase rather than time as the angular variable, the representation of $V_p$ itself, in normal coordinates, is $(0, 2\pi/\tau)$, rather than $(0, 1)$. But $\alpha_p(V_p) = 1$. And if an arbitrary vector, $W_p \in T_pM$, is represented by $(W_R, W_\Phi)$

in normal coordinates, then

$$a_p(W_p) = \frac{\tau}{2\pi} W_\Phi. \tag{4.8}$$

In the case of Andronov's model, Fig. 4.4, the isochrons are rays orthogonal to the circular arcs, as Vassalo Pereira has shown, and the shape form of this oscillator corresponds to the Euclidean orthogonal complement (Pereira, 1982). The analysis of this section is comparable to that of earlier authors, simplified through the use of normal coordinates.

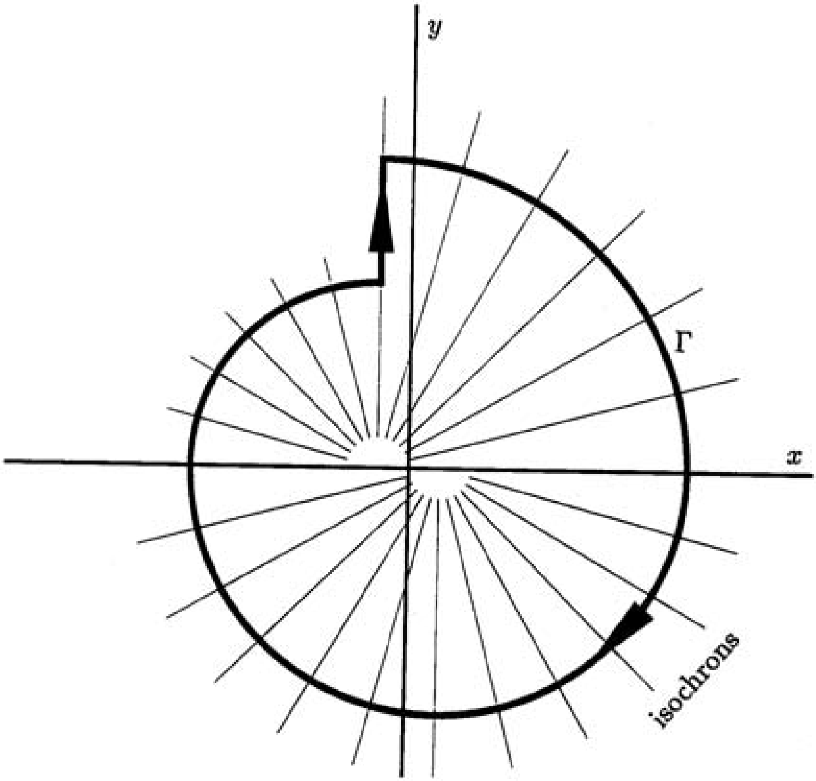


**Fig. 4.4.** Andronov's model.

## 5. THE FAVORITE PHASE FORMULA

We now apply the global analysis of Section 2 to the flexibly forced oscillator of Section 3, using the shape form of Section 4 to obtain a useful formula for the favorite phases of the weakly-coupled system. We now consider a one-dimensional scheme, $C = (A, B, f)$, as in Section 2 (the $f$ of 2.6). Thus we have:

$I \subset \mathbf{R}$, an open interval about zero,
$M = M_B$, the state space of the forced scheme,
$V_0$, a vectorfield on $M$, the driven system
$N \equiv T^1 \times M$, the product manifold,
$V$, the space of smooth vectorfields on $N$,
$X_0 = f^*_{d_0} V_0 \in V$, the suspension of $V_0$,
$\Sigma \subset N$, a global cross section of $X_0$,
$U$, a neighborhood of $X_0 \in V$,
with all its vectorfields having $\Sigma$ as a global cross-section, and
$X : I \to U \subset V;\ a \mid\to X_a$, a curve of the form

$$\theta' = 2\pi/\tau_A \tag{5.1a}$$

$$x' = V_0(x) + W_a(\theta, x), \tag{5.1b}$$

where $W_a(\theta, x)$ is periodic in $\theta$ with period $2\pi/\tau_A$, and $W_0 \equiv 0$. The parameter $a$ is called the amplitude of the coupling.

### HYPOTHESES

We suppose that $V_0$ has a unique attractor, $\gamma$ of period $\tau_B$, hyperbolic and periodic, and that $I$ is sufficiently small so that for all $a \in I$, $X_a$ has an attractive invariant torus (AIT), $T^2(a)$ [9]. On this AIT, for most $a$, there will be complementary braids of periodic attractors and periodic saddles. These may bifurcate as the amplitude $a$ changes [10]. Here, we make the *isochronous hypothesis*: $\tau_A \approx \tau_B$, and for all $a \in I$, all the periodic trajectories on the AIT are isochronous, $\tau_A = \tau_C$. Thus, fixing the point $0 \in T^1 = M_A$, and a point $p_0 \in \Gamma$ as the fiducial points for phase reference (see Section 2), each isochronous harmonic (limit cycle of $C$) on the AIT of $X_a$ ($T^2(a)$) has a well-defined phase difference, with respect to the forcing oscillator, $A$. We shall now give a precise definition of this phase difference, in the form of a relative phase function, $\Delta$.

Fixing $a$, choose one of the harmonics, say $\tilde{\gamma}$, with image $\tilde{\Gamma} \subset T^2(a)$ tending to the image $\Gamma \subset T^2(0)$ of $\gamma$ as $a \to 0$. This perturbed harmonic pierces the phase zero section, $\Sigma = \{0\} \times M_B$, at a unique point, $\tilde{p}_0$. This point, if $a$ is sufficiently small so that $\tilde{p}_0$ is close to $p_0$, lies on a unique isochron of $\Gamma$ in $M$.

**Definition 4.** The *relative phase* of $\tilde{\gamma}$ with respect to $\gamma$ is the phase of the isochron of $\gamma$ containing $\tilde{p}_0$. The *favorite phases* of the scheme are the relative phases of the attractive harmonics. See Fig. 5.1.

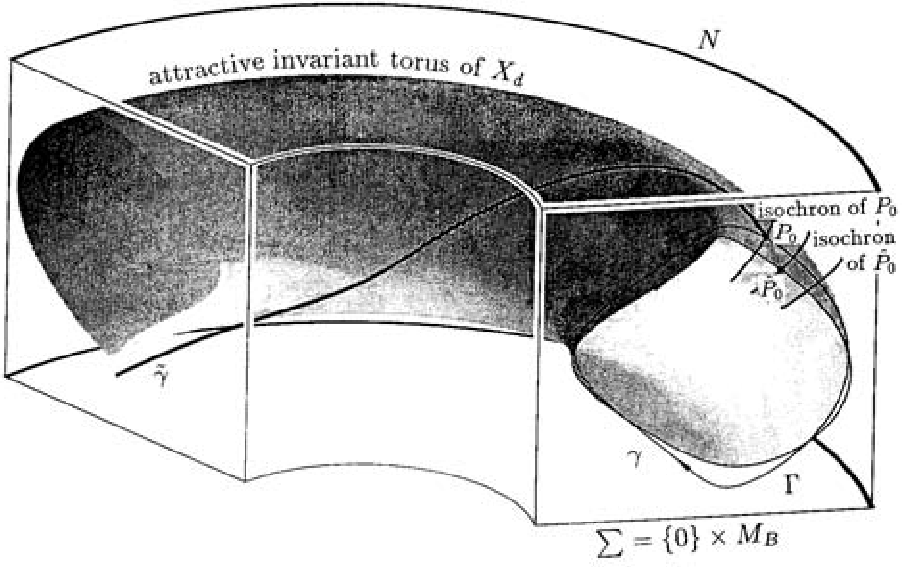


**Fig. 5.1.** Determination of the phase difference.

THE INTEGRAL APPROXIMATION

The submanifold $\Sigma = \{0\} \times M_B$ (which we may identify with $M$) is a global cross-section for $X_0$. There is a unique vectorfield $\xi$ on $M$ which is tangent to the curve of first-return maps, $G_a = F(X_a)$ in $D(M)$. This is the phase regulation dynamic of Section 3, which approximates the PRC of the forced system. And according to Proposition 1 of Section 3, the critical points of $\xi$ locate, in an approximation improving as $a$ decreases, the favorite phases of the isochronous harmonics. Thus the dependence of these critical phases upon the amplitude $a$ approximate the PRC of the flexibly forced oscillator scheme, $C$. This approximation may be useful because, in general, there will be no explicit formula for the exact favorite phases. Thus, we seek an explicit formula for $\xi$, the *favorite-phase formula*, based upon the shape form, $\alpha$, of the original periodic attractor of $V_0$, $\gamma$ in $M$. First, we will need an integral approximation for the first-return map.

Let $\alpha$ be the shape form of $\gamma$, an hyperbolic, periodic attractor of the vectorfield $V_0$ on $M_B$, of period $\tau_B$. Let $W$ be a periodic perturbing vectorfield on $N$, that is, $W(\theta, x)$ is periodic in $\theta$, with period $\tau_A$. Suppose $W$ sufficiently small so that $X = f^*(V_0 + W) \in U$, thus has $\Sigma$ as a global section. Let $G = F(X)$ be the first return map of $X$. For a point $x$ near $\Gamma$, let $(R(x), \Phi(x))$ be the normal coordinates. Then the map

$$\Phi \circ G : M_B \to T^1;\ x \mid\to \Phi(G(x)) = \Phi(G(R(x), \Phi(x)) \tag{5.2}$$

is a measure of the phase shift of the perturbed system, from the point of view of the isochrons of the unperturbed system. We should like to apply this map to the points of the perturbed isochronous harmonic. But we do not know where it is, except that it is close to $\Gamma$. So instead, as an approximation, we apply this map to $\Gamma$ (defined by $R(x) = 0$). Thus, we consider the *phase shift map*,

$$\Delta : T^1 \to T^1;\ \phi \mid\to \Phi(F((0, \phi))). \tag{5.3}$$

See Fig. 5.2. Here is an integral approximation formula for this map.

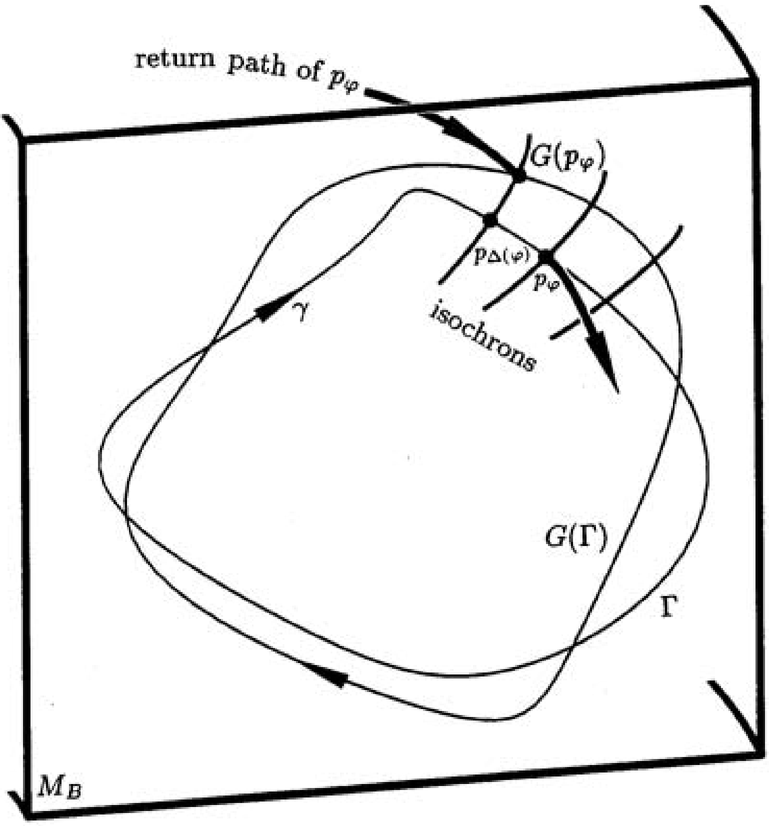


**Fig. 5.2.** The phase shift map, $\Delta$.

**Proposition 2**. With these notations,

$$\Delta(\phi_0) \approx \int_0^{\tau_A} \alpha_{p_t} W(\phi_t, p_t) dt \tag{5.4}$$

where $t \mid \to p_t$ is the trajectory of $\Gamma$ (from the chosen initial point, $p_0$) and $\phi_t = \phi_0 + 2\pi t/\tau_A$. The approximation improves as $W$ decreases in the $C^0$ norm.

**Proof.** The estimation of this approximation is a classical exercise, using the mean value theorem, in normal coordinates for $\gamma$ in $M$. We simply approximate the actual trajectory of $X_a$ in $N$, starting from $(\phi_0, p_0)$, by the known trajectory of $X_0$, from the same point.

To obtain the corresponding estimate for the phase regulation dynamic, $\xi$, we must replace $W$ by $W_a$ in the integral formula, and differentiate with respect to $a$. First we replace $W$ by $W_a$ in (5.4). Proposition 2 then becomes:

$$\Delta_a(\phi_0) \approx \int_0^{\tau_A} \alpha_{p_t} W_a(\phi_t, p_t) dt \tag{5.4a}$$

Then differentiating with respect to $a$ and using the Definition of Section 3, we obtain the following.

**Corollary 1.** Let

$$\delta = \frac{d}{da} \Delta_a |_{a=0} \quad .$$

Then $\delta = \alpha\xi$, that is,

$$\delta(p) = \alpha_p(\xi_p). \tag{5.5}$$

Note this is the isochronic (or $\Phi$) component of $\xi_p$, which is zero at the favorite phases, according to Proposition 1.

We now expand $W_a$ in Maclaurin's formula (as $W_0 = 0$), and drop the remainder:

$$W_a \approx a\, P \tag{5.6}$$

as in (3.4). This means that the flexible coupling is approximated by the *amplitude scheme*,

$$V_0(x) + W_a(\theta, x) \approx V_0(x) + a\, P(\theta, x) \tag{5.7}$$

in which the amplitude parameter, $a$, turns on the perturbation, P, in a linear way. Making this approximation for $W$ in the integral approximation formula, differentiating inside the integral as usual, and using (5.5), we obtain immediately the following *favorite phase formula*.

**Corollary 2**.

$$\delta(\phi_0) = \alpha\xi(\phi_0) \approx \int_0^{\tau_A} \alpha_{p_t} P(\phi_t, p_t) dt \tag{5.8}$$

Note that $\Delta_a(\phi_0) \approx a\delta(\phi_0)$.

In many applications, the generator of the perturbation, $P$, is indifferent to the state of the driven system. That is, $P(\theta_A, x_B)$ depends on $\theta_A$ only. This results in a simpler favorite phase formula, with *separation of variables*.

**Corollary 3**. If $P(\theta, x)$ is independent of $x$, then

$$\delta(\phi_0) \approx \int_0^{\tau_A} \alpha_{p_t} P(\phi_t) dt \tag{5.9}$$

This is a simple generalization of the formula of Vassalo Pereira, for Andronov's clock. For if $P(\theta)$ is a rectangular pulse:

$P(\theta) = P_1$ if $\theta_1 \leq \theta \leq \theta_1 + \epsilon$

$P(\theta) = 0$ otherwise,

then (5.9) becomes

$$\delta(\phi_0) \approx \frac{\tau_A \epsilon}{2\pi} \alpha_{p_1} P_1 \tag{5.10}$$

where $t_1 = \tau_A / 2\pi \, \phi_1$, $p_1 = \gamma(t_1)$.
Here, $\phi_1$ is the phase of $A$ when it ticks, $t_1$ is the time when $A$ ticks, $p_1$ is the state of $B$ when $A$ ticks, $\alpha_p$ is the shape form at the point $p$, which measures the length of the projection of a vector (along isochrons) onto the tangent of $\Gamma$ at $p$ relative to $V$, and $P_1$ is the constant force vector exerted by $A$ during the pulse.

Summary

In this section we have developed some methods to compute an approximate PRC, or phase shift map, in terms of the phase component, $\xi_\Phi = \alpha\xi = \delta$, of the phase regulation dynamic. It turns out that the one-dimensional vectorfield $\delta V_0$ on $\Gamma$ locates the harmonics on the AIT of the forced system.

## 6. A GEOMETRIC INTERPRETATION

We wish now to emphasize the geometric basis of the favorite phase dynamics. First, we collect the theory in a compact summary.

### The geometry of a periodic attractor

Let $V_0$ be a vectorfield on a manifold $M_B$, and $\gamma$ an hyperbolic, periodic trajectory of period $\tau_B$. Then the CM's of $\gamma$ define a distribution of tangent subspaces on $\gamma$. For $p \in M_B$, the CM subspace, $I_p$, is complementary to the tangent vector along $\gamma$, $V_0(p)$. Also, the CM distribution is tangent to the normal foliation, by hypersurfaces of constant isochronous phase, called the *isochrons* by Arthur Winfree (Winfree, 1980). All this comprises what we mean by the geometry of a periodic attractor. A useful aspect of this geomtery is the *shape form*, which measures the component of a vector in the direction of $V_0$ by projection along $I_p$.

PERIODIC FORCING

We now apply periodic forcing to the situation above, of period $\tau_A$, by increasing an amplitude parameter $a$ from zero. We view this in suspension, that is, in the ring model of Fig. 3.4. The original unforced oscillation, $\gamma$ in $M_B$, then corresponds to an attractive invariant torus, $T^2(0)$, of the suspended vectorfield, $X_0 = f^*_{d_0} V_0$. Every point of $\gamma$ in $M_B$ (identified with $\Sigma$, the Poincaré cross-section defined by driving phase, $\theta_A$, equal to zero) then sweeps out a spiralling trajectory on $T^2(0)$. After turning on the perturbation ($a > 0$) we still find an attractive invariant torus, $T^2(a)$. Generically, this will have a number of periodic attractors, and an equal number of periodic saddles, braided around it. We now assume that $\tau_A \approx \tau_B$, and that these braids are *isochronous harmonics*. That is, they all wrap exactly once around $T^2(a)$ in a single turn around the ring.

FAVORITE PHASES OF THE FORCED SCHEME

Choose one of the isochronous, periodic attractors braided around $T^2(a)$ and let $p_0(a)$ denote the unique point in which it pierces the Poincaré section, $\Sigma$. As $a$ decreases to zero, this strobe point, $p_0(a)$, approaches a point $p = p_0(0)$ in the direction of the CM subspace $I_p$. This point, $p$, is a *favorite phase* of the forced scheme. The strobe curvelike function $a \mid\rightarrow p_0(a)$ is roughly tangent, at its endpoint $p = p_0(0)$, to the CM subspace, $I_p$, as shown in Fig. 3.4.

THE PHASE REGULATION DYNAMIC

Using global analysis, we have derived a vectorfield of infinitesimal perturbation, $\xi$ on $M_B$. Restricting it to $\gamma$ and projecting along the CM distribution, we obtain the *favorite phase dynamic*, $\xi_\Phi = (\alpha\xi)V_0$, a vectorfield on the cycle $\Gamma$, (or equivalently, $\phi_*\xi_\Phi$, a vectorfield on the cycle of phases, $T^1$). The attractors of this one-dimensional vectorfield determine the favorite phases of the weakly forced system, while its repellors determine the periodic repellors. Some approximate formulas help in the evaluation of this favorite phase dynamic.

EXAMPLES

We now want to illustrate this geometric method of locating the favorite phases, in a phase locked system of forced oscillation. Three simple examples will suffice. In all three, the periodic force is pulsatile, and independent of the point of application, as in Corollary 3. The original state space is the plane, $M_B = \mathbf{R}^2$.

**Case 1. The clock.** We begin with Andronov's clock, in which case our theory reduces to that of Vasallo Pereira. The geometry of the original oscillation is shown in Fig. 4.4. The periodic force is always upward. Resolving this into tangent and normal (isochronous) components, we have the favorite phase dynamic shown in Fig. 6.1.

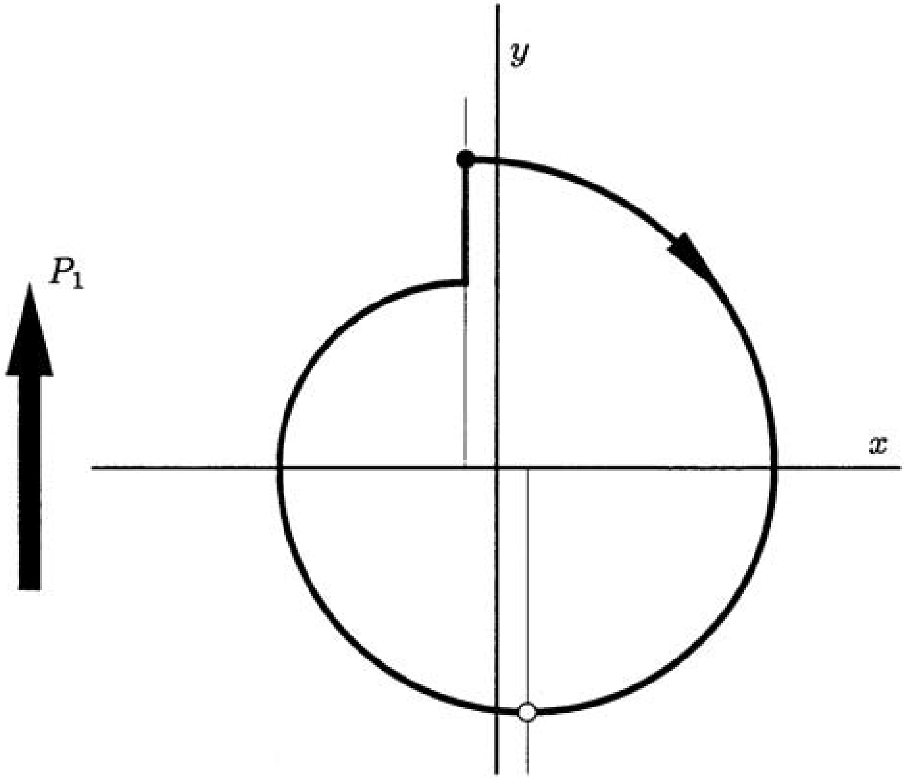


**Fig. 6.1.** Favorite phases of Andronov's clock (• favorite, ○ least favorite).

Here, the solid dot is an attractor, the small circle a repellor.

**Case 2. The butch cut**. Here is a periodic attractor with a flat top. The periodic force is again upwards, as in Case 1. Look carefully at the isochron foliation, which turns back and forth. Favorite phase attractors and repellors alternate along the flat stretch, because of the variation of the isochrons, as shown in Fig. 6.2.

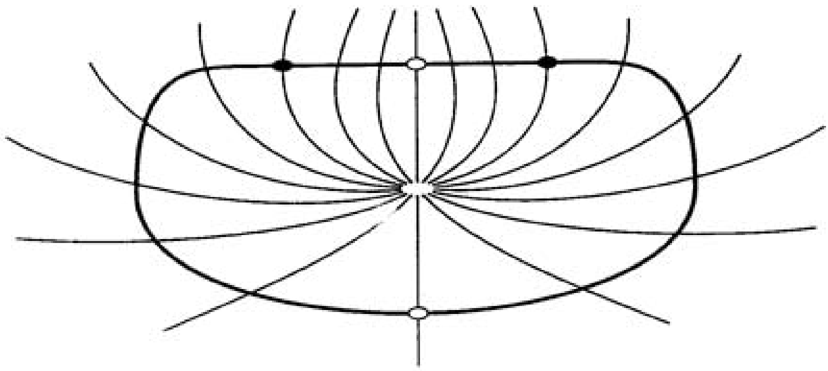

**Fig. 6.2.** The butch cut.

**Case 3. The camel's humps.** Here is a periodic attractor with two humps. Each contributes an attractive favorite phase, while between them is a repellor, as shown in Fig. 6.3.

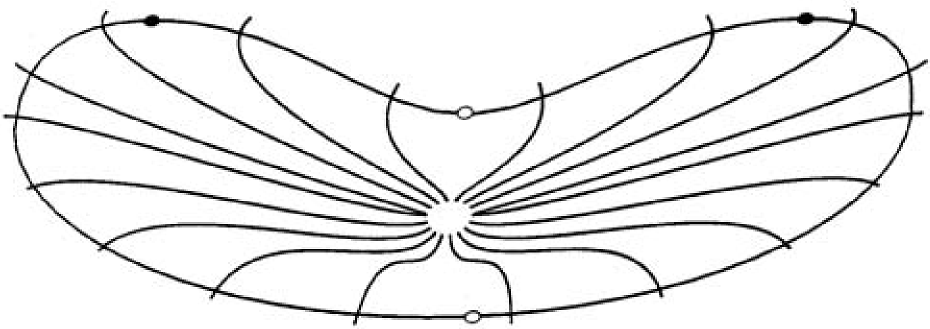

**Fig. 6.3.** The camel's humps.

Probably these examples suffice to get the full idea of the geometric theory of phase regulation. In principle, it should be possible to *engineer* a forced oscillator system to phase lock at desired favorite phases. Certainly this is easiest in the case of pulsatile forcing. Perhaps this is the reason that nature loves pulsatile communication. We then must wonder, as nature also loves chaos, if chaotic attractors also admit favorite phases. We end with a single example, which does permit phase regulation.

## 7. PHASE REGULATION OF RÖSSLER'S BAND

As the original Rössler attractor appears, in its power spectrum, as a noisy oscillator, it is a good candidate for phase regulation (Crutchfield, 1980). Further, it has been observed to preserve a cross-section, with very gradual dispersion along the attractor (Farmer, 1986). Thus, we may surmise that there is an *approximate isochron foliation*. Furthermore, there is a distinct hook at the top of the band, rather like Andronov's clock. This suggests that a periodic, pulsatile force upward, with the right period, might be able to synchronize the chaotic trajectory. In fact, this experiment has been tried, and synchronized successfully (Farmer, 1986). Other chaotic attractors, such as Rössler's funnel, also have hooks, but may not have approximate isochron foliations. Thus, they may not be such good candidates for synchrony.

## 8. CONCLUSION

Forced oscillators generically produce AITs with braided harmonics. Peixoto's theorem on structurally stable dynamics in two dimensions explains this nicely [11]. But where are they located? In many cases, such as Duffing's catastrophe, the phase relationship between the forcing oscillator and the isochronous harmonic depends sensitively on various parameters. This is very counterproductive in many practical situations. For example, if the LH surge of the human female reproductive endocrine cycle is not in the correct phase of the cycle, reproduction fails. Thus, a strategy to guarentee phase locking is essential to nature. In this paper, one such strategy has been presented. It depends, in its simplest version, upon a pulsatile periodic force, applied in a constant direction, to a periodic attractor characterized by a distinct hook (or hooks) in its isochron geometry. As it applies equally well to a forced chaotic attractor having approximate isochrons (unlike periodic attractors, not all do) it may have simple applications in the biological and social sciences.

## NOTES

[1] See Ch. 5 of *Part One*, (Abraham, 1982–88)
[2] The original papers are (Martienssen, 1910) and (Duffing, 1918) while an excellent summary may be found in (Stoker, 1950)
[3] See Section 5.5 in *Part One* of (Abraham, 1982–88)
[4] See (Rapp, 1984, Glass, 1979, Hayashi, 1986, Grasman, 1984, Hoppensteadt, 1982, Kopell, 1983, Cohen, 1982) and the references therein.
[5] These bifurcation curves, frequently called *tongues*, are described in many texts, for example (Arnold, 1973/1978)
[6] See (Winfree, 1967) for the original description, also (Guckenheimer, 1975, Winfree, 1980) or (Winfree, 1987)

[7] For the full theory of this foliation, and its associated *normal coordinates*, see (Hirsch, 1977)
[8] For an explanation of these equivalent aspects, see p. 523 of (Abraham, 1978/1982)
[9] See *Part One*, Section 5.5 of (Abraham, 1982–88)
[10] See *Part Four*, of (Abraham, 1982–88)
[11] See, for example, *Part One* of (Abraham, 1982–88)

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