#38

Physica 21D (1986) 394–400
North-Holland, Amsterdam

A CHAOTIC BLUE SKY CATASTROPHE IN FORCED RELAXATION OSCILLATIONS

Ralph H. ABRAHAM
Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

and

H. Bruce STEWART
Applied Mathematics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

Received 12 June 1985
Revised 25 March 1986

The chaotic attractor of a periodically forced Van der Pol oscillator (Shaw variant) is observed in digital simulation, and is made to vanish in a blue sky catastrophe by increasing a constant (bias) term in the force. The detailed bifurcation diagram, based on extensive simulations, reveals the involvement of the homoclinic outset of a nearby limit cycle of saddle type.

1. Introduction

Chaotic attractors have been observed in a number of studies of nonlinear dynamics; see e.g. refs 1, 2. In dynamical systems with dissipation, long-term behavior can settle into irregular post-transient patterns in a low-dimensional subset of phase space. Such attracting sets typically take one of a few basic forms, so that recognizing the occurrence of chaotic attractors can be of fundamental importance in understanding a dynamical system.

Complete analysis of a dynamical system involves construction of a phase space portrait, i.e. a topological model of attractors and their basins in phase space. Since full phase portraits are often difficult to construct, it is useful to have other strategies for identifying chaotic attractors. They are sometimes found by observing qualitative bifurcations of a simple attractor as a control parameter is varied. This is called *onset of chaos* by a *transition scenario*. For example, the observation of a convergent sequence (cascade) of period-doubling bifurcations [3] would be *prima facie* evidence for a chaotic attractor. Such an inference would, however, be indirect at best. For example, period-doubling cascades are known to lead to different basic forms of chaotic attractor [4–9].

In this paper, we present evidence of a more fundamental obstruction to the transition scenario strategy for locating chaotic attractors. Namely, the transition to or from chaos may be abrupt, or catastrophic, in the sense that an infinitesimal change in a control parameter eradicates a chaotic attractor from the phase portrait. Such a discontinuous transition will be called a *blue sky catastrophe* [10]. When such a bifurcation occurs, knowledge of the complete phase portrait is indispensable since the dynamical system will make a finite dynamic jump to a remote attractor, or diverge to infinity [11].

Our evidence for a blue sky catastrophe comes from numerical simulation of differential equations which model forced relaxation oscillations. The existence of this blue sky catastrophe was conjectured previously on theoretical grounds [10].

This bifurcation event coincides with a homoclinic tangency of invariant manifolds, of a type studied previously [12]. However, our interest here focuses on the effect this has on a chaotic attractor (that is, on easily observed dynamic behavior), rather than on the tangled structure of manifolds per se.

After reviewing the basic theory of discontinuous bifurcations in section 2, we describe a relatively simple blue sky catastrophe for a periodic limit cycle attractor in section 3, and then analyze the chaotic attractor bifurcation in section 4.

2. Discontinuous bifurcations

According to Thom [13], the most typical and fundamental manifestations of nonlinearity in dynamical systems are the (generalized) catastrophes. The elementary catastrophes of gradient dynamical systems are rigorously defined and classified in Thom's theory; the catastrophes of more general dynamical systems, involving periodic and chaotic behavior, can be defined with reference to *control-phase space*, as illustrated in fig. 1. In general, a control-phase space is the Cartesian product $\boldsymbol{C} \times \boldsymbol{P}$ of the phase space $\boldsymbol{P} = \{x_1, x_2, \dots, x_n\}$ whose coordinates are the state variables of a dynamical system, with the space $\boldsymbol{C} = \{\mu_1, \mu_2, \dots, \mu_n\}$ of control settings, such as a voltage supplied to a motor, or a parameter in a differential equation. Either $\boldsymbol{C}$ or $\boldsymbol{P}$ might in fact be infinite-dimensional, but for the study of codimension one bifurcations, a single scalar control variable μ is appropriate. Thus fig. 1 represents a 1-d control space $\boldsymbol{C} = \{\mu\}$ crossed with a 1-d phase space, the latter to be interpreted as a projection of a multidimensional phase space onto one dimension for the sake of simplicity.

The heavy lines and shaded regions in each schematic diagram represent an *attractrix*, or ensemble of attractors (equilibrium, periodic, or chaotic) for various control settings μ. For any given value of μ, a horizontal slice of the attractrix gives the attractor(s) of the dynamical system at

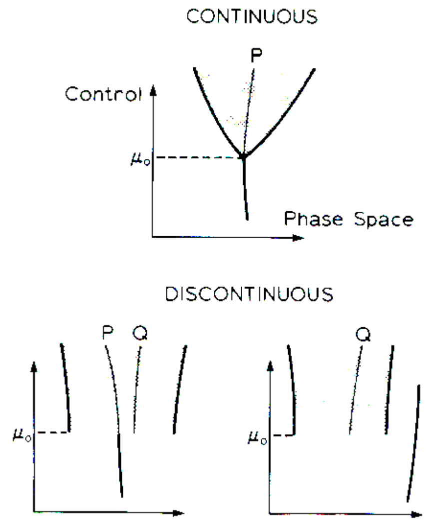

Fig. 1. Schematic control-phase space diagrams of continuous and discontinuous attractrix bifurcations: at least some paths through the attractrix (such as those labelled Q) cannot be continued across μ_0 when the bifurcation is discontinuous, whereas all attractrix paths (such as P) can be continued across a continuous bifurcation.

that control setting. In each diagram, an attractor bifurcation (qualitative change) occurs as μ passes μ_0. Following Zeeman [14], we define a *discontinuous* bifurcation as one in which the locus in phase space of the attractor changes discontinuously. This means that some (or all) attractix paths, such as those labelled Q on fig. 1, cannot be extended continuously across μ_0 without leaving the attractrix.

As shown in fig 1, there are two varieties of discontinuous bifurcation. Bifurcations in which all attractrix paths are interrupted give rise to hysteresis or divergence, and are known as *dangerous boundaries* [15] in control space. (Continuous bifurcations are correspondingly known as *safe boundaries*.) In other cases, some but not all attractrix paths extend continuously across the bifurcation point, as observed in *intermittency* [11, 16]. Such a discontinuous jump in attractor size is neither a safe (continuous) nor a dangerous (hysteretic) boundary.

The term catastrophe was certainly intended by Thom to apply in a general setting to include at least the dangerous boundaries. Zeeman [14] extends this further and defines a catastrophe to be any discontinuous bifurcation. Since we are concerned in this paper only with total disappearance of attractors, we will be more specific and refer to a dangerous boundary as a *blue sky catastrophe* – a bifurcation in which an entire attractor disappears abruptly from the phase portrait as a control is varied.

Blue sky catastrophes do not "just happen": for chaotic attractors they are associated with tangencies of invariant manifolds in phase space. The first two-dimensional diagrams of this phenomenon were constructed by Simó [17] using the Hénon map as an example. Grebogi, Ott and Yorke [18] examined discontinuous bifurcations in the Hénon map and the quadratic map; they named discontinuous bifurcations *crises*. The first to observe a blue sky catastrophe (analogous to a boundary crisis) in differential equations was probably Rössler [19]. Ueda [20] observed a discontinuous jump in size of a chaotic attractor (analogous to an interior crisis) in a forced Duffing oscillator, and constructed detailed phase portraits showing the invariant manifolds involved.

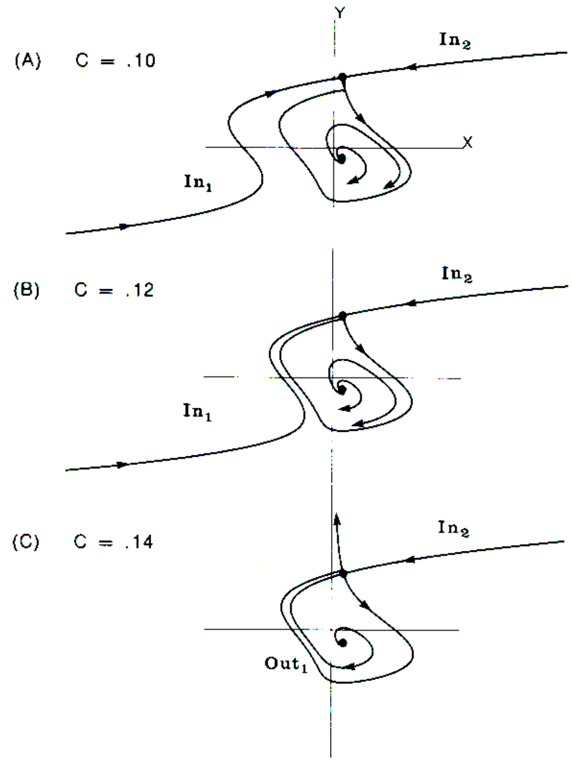

Fig. 2. Phase portraits of eqs. (2), showing the catastrophic disappearance of a limit cycle into the blue.

3. The periodic blue sky catastrophe

The simplest dynamic description of relaxation oscillations is a nonlinear ordinary differential equation introduced by Lord Rayleigh [21] and studied extensively by Van der Pol [22], who recognized the broad range of physical phenomena it encompasses. In the form used by Van der Pol, the autonomous equation can be written

$$\ddot{y} + \alpha(y^2 - b)\dot{y} + ky = 0, \tag{1}$$

where dot denotes time derivative. The linear damping term introduces negative damping for small displacement y, so that the rest state is unstable. For large y, on the other hand, nonlinearity makes damping positive, with the result that all initial conditions in the $(y, \dot{y})$ phase plane settle to a unique limit cycle oscillation.

The same type of self-sustained oscillation occurs in the equations

$$\begin{aligned} \dot{x} &= ky + \alpha x(b - y^2), \\ \dot{y} &= -x + C, \end{aligned} \tag{2}$$

which for $C = 0$ is equivalent to the single second-order equation (1). For $C \neq 0$, eqs. (2) introduce a constant forcing of the velocity $\dot{y}$. This type of forcing is qualitatively different from acceleration forcing, which would result from putting C on the right side of eq. (1). Acceleration forcing is common in mechanical systems represented by second-order differential equations, but velocity forcing may arise naturally in relaxation oscillators of electrical or chemical origin.

By varying the constant C in eqs. (2), the limit cycle response can be made to vanish in a *periodic blue sky catastrophe*. This bifurcation, which is closely related to the blue sky catastrophe for a chaotic attractor discussed below, was investigated by Abraham and Simó [23], and is illustrated in fig. 2, obtained from numerical simulation of eqs. (2) with $k = 0.7$, $\alpha = 10$, $b = 0.1$.

In fig. 2a, trajectories of eqs. (2) are shown for $C = 0.1$. Near the origin a repelling fixed point sends trajectories spiraling out to approach the limit cycle as $t \to +\infty$. Just above the limit cycle is a fixed point of saddle type. The inset (or stable invariant manifold) of the saddle is a pair of trajectories asymptotic to the saddle as $t \to +\infty$. No trajectory can cross this invariant manifold. Initial conditions near the inset are separated into two basins, those which approach the limit cycle, and those for which $y \to +\infty$ as $t \to +\infty$. The inset is thus a separator.

In fig. 2b, with $C = 0.12$, the phase portrait is qualitatively the same, but both the saddle and its inset are closer to the attracting limit cycle. Dynamic observation of simulated trajectories shows that they slow down on the part of the limit cycle nearest to the saddle fixed point.

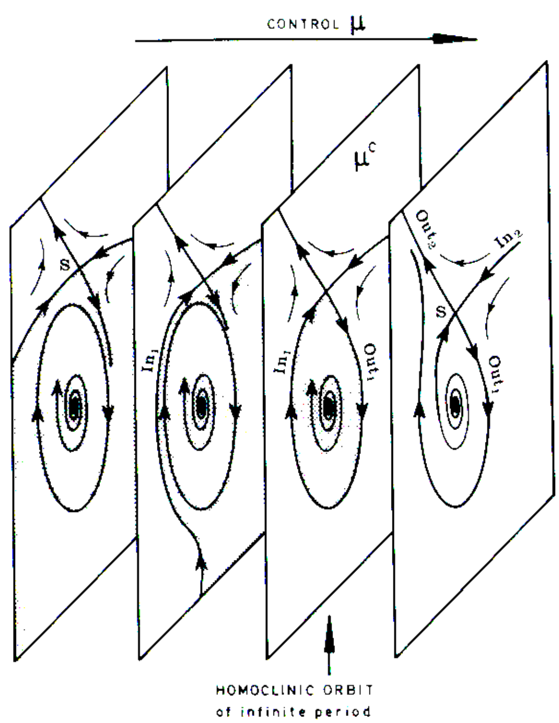

Fig. 3. Sequence of schematic phase portraits of the periodic blue sky catastrophe, showing the structurally unstable homoclinic connection.

Fig. 2c shows a qualitatively different phase portrait at $C = 0.14$. The branch In_1 of the saddle inset curls up inside the branch Out_1 of the outset of S, as is perhaps more clear in the schematic drawing of fig. 3. The schematic also emphasizes that when the limit cycle exists, Out_1 (S) is distinct from, although asymptotic to, the cycle. This is not evident in fig. 2a because of the rapidly attracting nature of this cycle.

The result in fig. 2c is that all trajectories reach the upper part of the phase plane (without crossing the inset) and diverge $y \to +\infty$. There is no attractor in the phase plane; increasing C slowed the limit cycle until it vanished.

In betwen $C = 0.12$ and $C = 0.14$ there must be a homoclinic saddle connection. At some intermediate value of C, the saddle fixed point just touches the limit cycle, which would have infinite period; this value of C is the threshold for stability of the limit cycle.

4. The chaotic blue sky catastrophe

As predicted in [10], we have discovered that a blue sky catastrophe for a chaotic attractor can occur in a relaxation oscillator under time-periodic forcing. Our example is the system

$$\begin{aligned} \dot{x} &= 0.7y + 10x(0.1 - y^2), \\ \dot{y} &= -x + 0.25 \sin 1.5t + C. \end{aligned} \qquad (3)$$

Retaining the constant C makes the sinusoidal forcing asymmetric. Phase portraits of eq. (3) should properly be constructed of trajectories in three-dimensional phase space with coordinates x, y, and $\theta = 1.5t \pmod{2\pi}$, but it is convenient to examine Poincaré sections in the (x, y) plane corresponding to a fixed phase of the driving term, $\theta =$ constant. The results are illustrated in fig. 4, taking $\theta = \pi$.

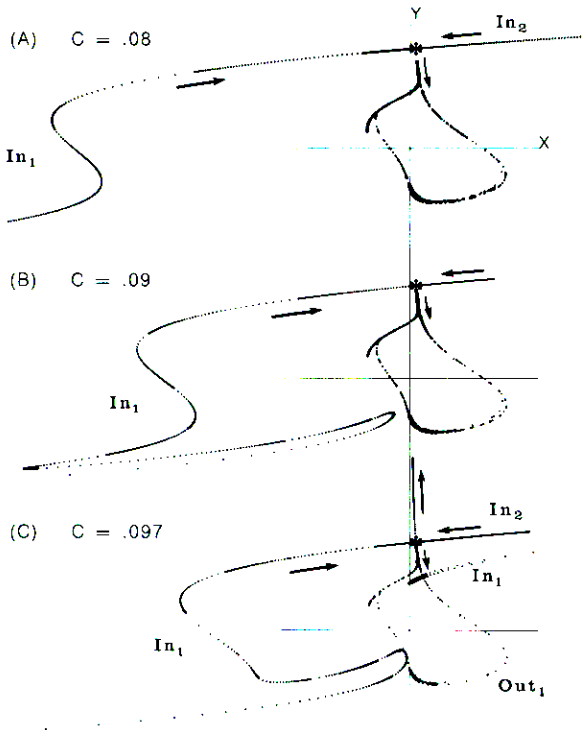

Fig. 4. Phase portraits of the periodically forced equations (3), showing the Birkhoff–Shaw chaotic attractor and its catastrophic disappearance.

In fig. 4a with $C = 0.08$, the Poincaré section resembles fig. 2a of eqs. (2), but with wings added. As the angle θ advances through one forcing cycle, points on the attractor generally circulate clockwise; wings are folded flat and new wings emerge periodically. This is essentially the attractor proposed by Birkhoff [24] and first observed by Shaw [25]; cf. [26, 27]. The attractor sections in figs. 4a and 4b were each obtained from a single computed trajectory, discarding the first 50 return points and then recording the next 1000 returns to $\theta = \pi$. The return points were plotted sequentially in the Poincaré section, landing most frequently on the wings but always returning occasionally to the intervening regions. That is, a single transitive attractor is observed in digital simulation.

With periodic forcing, the saddle fixed point S of fig. 2 becomes a limit cycle of saddle type, whose position at $\theta = \pi$ is indicated in fig. 4 by an asterisk (*). Asymptotic to this saddle as $t \to +\infty$ is an inset of infinitely many trajectories, each recurring in the Poincaré section in a sequence of points which approach the saddle cycle. Note that the individual points on the inset in fig. 4 are not successive images of one trajectory, but of many different trajectories. In fact any point shown will jump extremely close to the saddle cycle in one period of the forcing term.

As in fig. 2, this inset is a separator. Initial conditions on one side are in the basin of the chaotic Birkhoff–Shaw attractor, while on the other side all trajectories diverge to $y \to +\infty$ as $t \to +\infty$.

For $C = 0.09$ the phase portrait in fig. 4b is qualitatively similar, but the saddle cycle is now very close to the chaotic attractor, and the branch In_1 of the saddle also comes closer. In the periodically forced system (3), the basin of asymptotic divergence need not approach the chaotic attractor uniformly, but it must do so (by recurrence) at locations in addition to the saddle cycle itself. The intruding finger of In_1 in fig. 4b is the closest approach at $C = 0.09$, $\theta = \pi$, but other fingers could be found by following In_1 further backward in time.

The phase portrait in fig. 4c corresponds to $C = 0.097$. Shown are inset points, together with 500 trajectories started along $Out_1(S)$, that is in the former basin of the chaotic attractor, and plotted at their fifth return to the Poincaré section $\theta = \pi$. These trajectories wander chaotically on a folding structure like the chaotic attractor at $C = 0.09$, but at $C = 0.097$ they remain in this region of the (x, y) plane only for a finite time. The saddle cycle is now just inside the upright wing of the former attractor. Fig. 4c shows some trajectories which started on Out_1 below the saddle cycle, have folded above it, and are on the way to divergence. Numerical evidence indicates that the folding structure – analogous to the invariant set of Smale's horseshoe – is still transitive at $C = 0.097$, and with probability one an observed trajectory will eventually be folded above the sad-

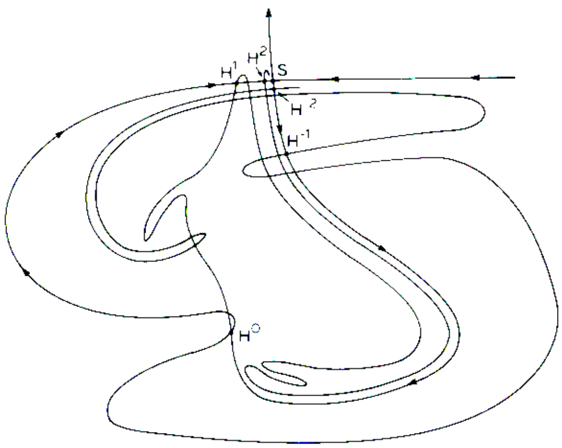

Fig. 5. Schematic diagram of the homoclinic tangle formed after the Birkhoff–Shaw attractor vanishes

dle cycle and diverge to $y \to +\infty$. Thus the Birkhoff–Shaw attractor *has vanished into the blue*.

The phase portrait in fig. 4c shows that at the same time the saddle cycle touches the wing, the inset In_1 passes homoclonic tangency with Out_1. The points cut from Out_1 by the first homoclinic intersection will be carried above the saddle within one forcing cycle, while other points must wait more than one cycle. Mapping this intersection backward in time by one forcing period, we construct the second, thinner finger of In_1, visible in the first quadrant. Points cut out by this finger leave the folding structure after two forcing cycles. Since all points eventually leave the folding structure, it must be cut everywhere by infinitely recurring fingers. The tangled inset has an intricate structure, some of which is drawn schematically in fig. 5. The full infinite tangle can be deduced from fig. 5 by the mathematics of recurrence [28, 29].

The basic configuration, or signature of the homoclinic tangency, is a type already studied mathematically. Gavrilov and Shilnikov [12] proved that such a bifurcation point is in their terminology "inaccessible from both sides," meaning that there are infinite cascades of bifurcation, accumulating from either side at the critical control threshold. These bifurcations include saddle-node bifurcations of subharmonics of increasingly higher order.

It might be inferred that the homoclinic tangency and associated attractor disappearance might be so masked by these subharmonic bifurcations as to be an indistinct event. But on the contrary, in extensive simulations we have observed no evidence of the subharmonics. This is perhaps explained by the fact, noticeable in fig. 4, that In_1 passes the threshold of tangency with Out_1 very rapidly as C is varied. Furthermore, by analogy with a result on the basin size of such subharmonics in another forced oscillator [28, p. 91] we may conjecture that the control ranges over which subharmonics exist will decrease exponentially with increasing subharmonic number.

The rapidly attracting nature of the Van der Pol system probably plays a role here as well. Because the volume contraction rates in phase space are so great, the section of attractor in fig. 4 is nearly one-dimensional. In other, weakly dissipative systems, fractal layers of a chaotic attractor would be more evident, and subharmonic bifurcations might be easier to detect.

In sum, the chaotic attractor disappearance is the only bifurcation apparent in numerical simulations. This bifurcation coincides with the homoclinic tangency of invariant manifolds.

5. Conclusion

We have examined the invariant manifold geometry associated with the abrupt disappearance of a chaotic attractor in a simple relaxation oscillator with periodic, asymmetric forcing. This chaotic attractor, predicted by Birkhoff [24], was first observed by Shaw [25]. The blue sky disappearance of the Hénon-Pomeau attractor, painstakingly drawn by Simó, is closely related to the event described in this paper. The present case is important to dynamical systems theory because of its relatively simple geometry. In view of the many manifestations of relaxation oscillations, the blue sky catastrophe for the Birkhoff–Shaw attractor will undoubtedly arise in applications as well.

This same bifurcation topology could also occur in a dynamical system having an additional attractor, say a periodic limit cycle, lying above the separator. If the additional attractor remains at a positive distance while the chaotic attractor disappears catastrophically, then the dynamical system has one leg of a hysteresis loop. For example, the transition to chaos in a turbulent fluid, such as Couette flow, could occur through a blue sky event involving only two modes of excited oscillation.

Acknowledgements

This work was supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy, under contract DE-AC02-76CH00016.

References

[1] H.L. Swinney. Observations of order and chaos in nonlinear systems, Physica 7D (1983) 3–15.
[2] N.B. Abraham, J.P. Gollub and H.L. Swinney, Meeting report: testing nonlinear dynamics, Physica 11D (1984), 252–264.
[3] M.J. Feigenbaum, Universal behavior in nonlinear systems, Physica 7D (1983) 16–39.
[4] Y. Ueda, Steady motions exhibited by Duffing's equation, in: New Approaches to Nonlinear Problems in Dynamics, P.J. Holmes, ed. (SIAM, Philadelphia, 1980).
[5] B. Huberman and J. Crutchfield, Chaotic states of anharmonic systems in periodic fields, Phys. Rev. Lett. 43 (1979) 1743–1747.
[6] J.C. Roux, R.H. Simoyi and H.L. Swinney, Observation of a strange attractor, Physica 8D (1983) 257–266.
[7] C.J. Marzec and E.A. Speigel, Ordinary differential equations with strange attractors, SIAM J. Appl. Math. 38 (1980) 403–421.
[8] J.M.T. Thompson, Complex dynamics of complaint offshore structures, Proc. Roy. Soc. London A 387 (1983) 404–427.
[9] K. Tomita, Chaotic response of nonlinear oscillators, Phys. Rep. 86 (1982) 113–167.
[10] R.H. Abraham, Chaostrophes, intermittency, and noise, in: Chaos, Fractals and Dynamics, P. Fischer and W.R. Smith, eds. (Dekker, New York, 1985).
[11] H.B. Stewart and J.M.T. Thompson, Towards a classification of generic bifurcations of dissipative dynamical systems, Dyn. Stab. Syst. 1 (1986), in press.
[12] N.K. Gavrilov and L.P. Shilnikov, On three-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve, Math. USSR Sb. 17 (1972) 467–485; 19 (1973) 139–156.
[13] R. Thom, Structural Stability and Morphogenesis (Benjamin, Reading, MA, 1975).
[14] E.C. Zeeman, Bifurcation and catastrophe theory, in: Papers in Algebra, Analysis, and Statistics, R. Lidl, ed. (Amer. Math. Soc., Providence, RI 1982).
[15] L.P. Shilnikov, Theory of the bifurcation of dynamical systems and dangerous boundaries, Sov. Phys. Dokl. 20 (1976) 674–676.
[16] Y. Pomeau and P. Mannevile, Intermittent transition to turbulence in dissipative dynamical systems, Commun. Math. Phys. 74 (1980) 189–197.
[17] C. Simó, On the Hénon–Pomeau attractor, J. Stat. Phys. 21 (1979) 465–494.
[18] C. Grebogi, E. Ott and J.A. Yorke, Crises, sudden changes in chaotic attractors, and transient chaos, Physica 7D (1983) 181–200.
[19] O.E. Rössler, Different types of chaos in two simple differential equations, Z. Naturforsch 31a (1976) 1664–1670.
[20] Y. Ueda, Explosion of strange attractors exhibited by Duffing's equation, in: Nonlinear Dynamics, R.H.G. Helleman, ed. (NY Acad. Sci., New York, 1981).
[21] Lord Rayleign, On maintained vibrations, Phil. Mag., Ser. 5, Vol. 15 (1983) 229–235; The Theory of Sound, vol. 1, pp. 79ff. (Dover, New York, 1945).
[22] B. Van der Pol and J. van der Mark, The heart beat considered as a relaxation oscillation, and an electrical model of the heart, Phil. Mag., Ser. 7, Vol. 6 (1982) 763–775.
[23] R.H. Abraham and C. Simó, Bifurcations and Chaos in Forced Van der Pol Systems, UCSC preprint (1984).
[24] G.D. Birkhoff, Sur quelques courbes fermées remarquables, Bull. Soc. Math. d. France 60 (1932) 1–26; Collected Mathematical Papers, vol. 2, 418–444.
[25] R. Shaw, Strange attractors, chaotic behavior, and information flow, Z. Naturforsch. 36a (1981) 80–112.
[26] R.H. Abraham and C.D. Shaw, Dynamics: The Geometry of Behavior, Part Two (Aerial Press, Santa Cruz, CA, 1983).
[27] J.M.T. Thompson and H.B. Stewart, Nonlinear Dynamics and Chaos (J. Wiley, Chichester, 1986).
[28] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Springer, New York, 1983).
[29] C. Hayashi, Selected Papers on Nonlinear Oscillations (Kyoto University Japan, 1975); The method of mapping with reference to the doubly asymptotic structure of invariant curves, Int. J. Non-linear Mech. 15 (1980) 341–348.