Mechanics of Resonance

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Recently, my greatest pleasures have occurred in the company of some beautiful older books I came across while studying the history of vibrations. Chief among them is the history of mechanics from Galileo to Lagrange by Truesdell (1960), in which he points out that *mechanics* belongs to our perennial wisdom. It was associated with our highest pre-modern knowledge, and is not just a bad habit of the modern period. When our post-modern science emerges (if it ever does), I believe we will come to see mechanics resume the place of importance it held in ancient and Renaissance times.

In particular, the *discretization* of continuous systems for the sake of understanding them in mechanical analogy was introduced by Leonardo da Vinci around 1500. Discretization denotes the modeling of a continuous system by a finite number of discrete ones. It is a cognitive strategy inverse to interpolation. For example, a length of flexible cable may be modeled by a length of chain. In fact, this was essentially Leonardo's earliest example of the discretization strategy.

Later, the disenchantment of the Renaissance reduced this cognitive strategy to dogma. This happened in 1600 with the burning of Bruno, according to Berman (1981), or perhaps in 1627 with the dream of Descartes, according to Davis and Hersh (1986). Leonardo's strategy reappears in 1646, in Huyghens' study of suspension bridges, and again in 1675, in Huyghens' study of the vibrating string (Truesdell, 1960, ch. 1, pp. 45–49). By this time, the degeneration of mechanics from cognitive strategy (in the spirit of Hermeneutics and *verstehen*) to dogma (as in Physicalism, Reductionism, etc.) was well under way. Truesdell tells us that the phenomena of resonance were known to the ancient Greeks, that Leonardo resumed its experimental study early in his career, and that Fracastoro gave its correct explanation in 1546 (ch. 1, pp. 16–22).

In this article, I try to give the basic idea of the *nonlinear resonance of vibrations* by extracting a few episodes from the history of mathematical physics in the three-century period beginning with Galileo. One immediate goal is that you should understand how to break a plate of glass by singing, worrying it around its resonant frequency. You might be able to do this without knowing how. The person who figured out how to do this was Duffing (1918), an Austrian engineer. His discovery is the fundamental phenomenon of nonlinear resonance, the *double fold catastrophe*, and it is this which breaks the plate of glass. If you understand this, you can apply it to many other things, such as morphic resonance. A few such applications are suggested in the last sections.

OSCILLATION AND VIBRATION

Oscillation and vibration are two different things. An oscillator is something like a clock; it reproduces its states in a cycle, traversing each cycle in the same period of time. The prototypical examples of oscillation are the rising and setting of the sun, the phases of the moon, the tides, the mammalian reproductive cycle, and the cycle of the seasons. The modern nonlinear resonance concept applies primarily in the domain of oscillators, as studied by Duffing. If two oscillators are nearby and influence each other, a resonance phenomenon may be observed between them. But here, I wish to extend this concept to the classical context of resonant vibrations, of strings for example.

Vibration is a spatially distributed field or family of coupled oscillators. In a vibratory field, cooperative behavior might give the appearance of a wave traveling. Actually nothing is moving, only individually oscillating up and down, like the surf. The cooperative behavior of a field of coupled oscillators is a vibration. The prototypical vibrations are, of course, strings, water waves, and sound waves. No others were known until the relatively recent discovery of wave phenomena in the electromagnetic field, quantum mechanical oscillations, biological systems, and so on.

Resonance between vibrating fields is an extension of the resonance of oscillators. Imagine a vibrating guitar string for example. If you have another vibrating string near it, the resonance phenomenon between these two vibrations or fields of oscillators is a cooperative phenomenon among the individual resonance effects between the oscillators of the one and the oscillators of the other, collectively composing the individual vibrations. This is a much more complex phenomenon than the simple resonance of oscillators. Our goal is to understand this by means of a mechanical analogy or model.

Examples of Oscillators

I think that the first man-made oscillators were models of the natural oscillators. For example, the ancient Egyptian water clock and the pendulum clock of Galileo and Huyghens are self-sustaining oscillators made in imitation of the natural prototypical oscillators.

A pendulum is oscillatory, yet it is not an oscillator in the strict definition I am using here, that of requiring self-sustaining motion, because the widths of the swings of a pendulum die away in a short time. However, a pendulum is oscillatory in the sense that with every pendulum there is associated a certain natural frequency. Around 1588, Galileo had noticed that this frequency is roughly independent of the width of the swing, the so-called isochronous property of the simple pendulum. So although the swinging dies away, as long as it persists one may keep time with it. In fact, Galileo timed his astronomical observations in this way.

To make a satisfactory pendulum clock, what was required was a mechanism that would automatically keep winding up the pendulum. Such an escape mechanism was invented by Galileo (who never made it work) and applied by Huyghens (who did).

			1400 1600 1800 2000
Leonardo da Vinci	1452	1519	——A—
Geronimo Fracastoro	1483	1553	———F
John Dee	1527	1608	————
Giordano Bruno	1548	1600	——B
Francis Bacon	1561	1626	———
Galileo Galilei	1564	1642	—H———
Isaac Beeckman	1570	1637	——————
Johannes Kepler	1571	1630	—————
Marin Mersenne	1588	1684	————————
Rene Descartes	1596	1650	—C—
Christiaan Huyghens	1629	1695	—DIE
Isaac Newton	1643	1727	—————
Joseph Saveur	1653	1716	——M—
Leonhard Euler	1707	1783	—————
Jean d'Alembert	1717	1783	—K—
Joseph Lagrange	1736	1813	———
Ernst Chladni	1756	1827	—N—
Joseph Fourier	1768	1830	———
Sophie Germain	1776	1831	—————
Baron von Helmholtz	1821	1894	—J——
Lord Rayleigh	1842	1919	——L—
Georg Duffing	1861	1940?	——G—
Christopher Zeeman	1930		——O—
			1400 1600 1800 2000

Figure 1. The principal participants and events, in order of appearance.

In fact, Huyghens made a great number of clocks. He had a machine shop in downtown Amsterdam make clocks for him, and they were all over the house. He noticed that even though the clocks in separate rooms were keeping time differently, one gaining time every day, another losing time every day, when he put them in the same room, close together, they would keep time at the same rate. If they had an error, it would be the same error. But more than that, the pendulums would actually swing *in phase.* This is the *entrainment phenomenon* discovered by Huyghens in 1665. It is an aspect of resonance.

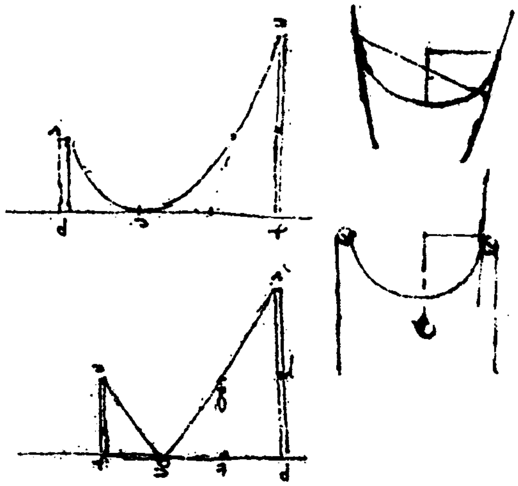

Figure 2. These sketches from Leonardo's notebooks, about 1500 A.D., show the weight of the string concentrated in a single heavy bead near the center. This is the earliest known example of a discrete model for a continuous mechanical system.

The tuning fork interruptor, or door buzzer, is another example. Like the pendulum, a tuning fork is oscillatory but is not an oscillator. One of the first electrical oscillators was Helmholtz's invention of the door buzzer, about 1850. He took a tuning fork, put a nail close to one of its bars, and when the tuning fork vibrated, contact with the nail would complete a circuit with a battery and a coil; the electromagnetic field of this coil would give the impetus to strengthen the vibration. The door buzzer would keep on buzzing.

NONLINEAR RESONANCE OF OSCILLATORS

First of all let us consider linear resonance, a fiction of the imagination because nothing in Nature is truly linear. A tuning fork, for example, might be a linear oscillator if it were infinitesimally thin. A pendulum might be one if all of its mass were concentrated in a small bob at the bottom of a weightless string, if there were no air in the room, or if it swung only slightly, and so on.

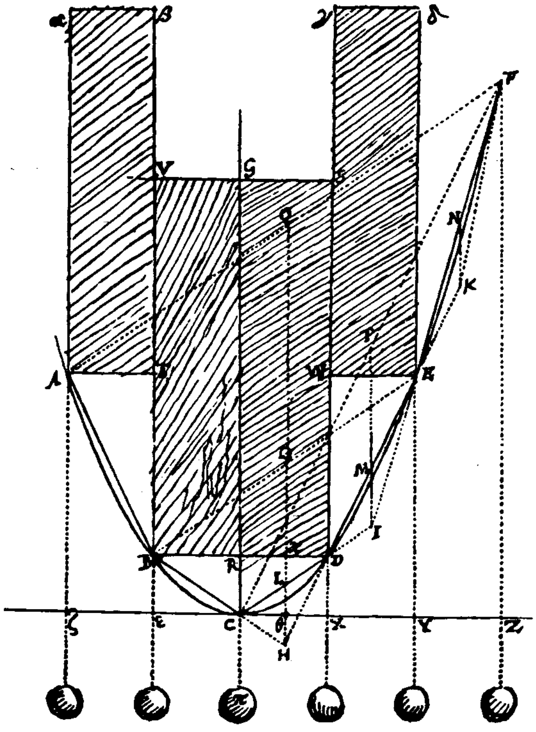

Figure 3. In this study of a suspension bridge from 1646 A.D., Huyghens employs the discretization strategy of neccesity.

The idea of the linear resonator, in the case of a thin tuning fork, is that you sing at it, and that tends to put it into vibration. But if you are at the wrong frequency, its response is nil. When you sing at the right frequency (the natural frequency), the fork will almost instantaneously go into a relatively large oscillation. That is resonance. If you raise the pitch of your voice gradually from below the resonant frequency to above it, the fork will respond only at the one frequency. This behavior is shown by the response curve (see Figure 4).

What happens with a nonlinear (that is to say, real) tuning fork is that this response curve is bent, as shown in Figure 5. Duffing studied this by modeling the forcing oscillator (that is, the nearby voice) as a large pendulum moving very slightly, and modeling the responding oscillator (tuning fork or whatever) as another (smaller) pendulum. The mechanics of each pendulum is inherently nonlinear. For the coupling between them, he hung the smaller pendulum from the bar of the larger one. He was interested in the effect on the little one of the forcing

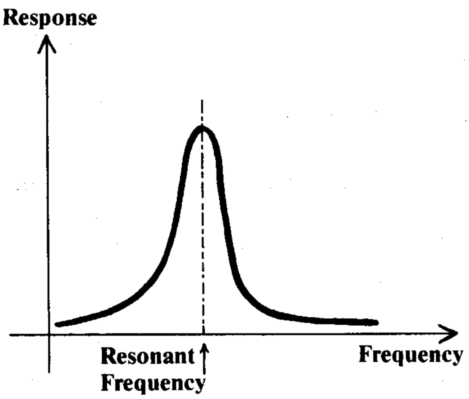

Figure 4. This graph shows the response of the sympathetic oscillator (pendulum, string or fork) concentrated at single frequency, the *resonant frequency*. This is the response diagram for the *linear resonator*.

oscillation of the driving pendulum. This one was so big that it was essentially unaffected by the motion of the smaller one.

Recall that a simple pendulum has a natural frequency at which it likes to swing when left alone. Suppose the big one is forcing the small one at a frequency slightly lower than its natural frequency. Then the small one might respond with a very small oscillation at that same frequency. Now repeat the experiment, increasing the forcing frequency. In the linear case, there would be no significant response of the driven oscillator until the forcing frequency reached the natural (resonant) frequency of the driven oscillator.

But in the nonlinear case, the resonant behavior of the pendulum is bent over; the whole response curve is bent over. As the driving frequency increases, the sympathetic response increases gradually until the driving frequency reaches some critical value well past the natural frequency of the follower. Then abruptly, the response falls to a much lower level. Decreasing the driving frequency again, the smaller response persists until the driving frequency gets to a critical value somewhat above the natural frequency of the follower. Then, again abruptly, the response increases. This complete sequence is called an *hysteresis loop* Abraham & Shaw, 1982). The interval between the two critical frequencies is called the resonant frequency interval or the bimodal regime. This is the nonlinear analogue of the resonant frequency in the linear context.

This is what Duffing discovered with the small pendulum hanging from the bar of a larger pendulum. He examined the complex motions of the system by taking very careful observations with a stroboscope. (According to Lord Rayleigh, the stroboscope was invented by a pendulum scientist, Foucault.) Extended vibrating systems (string, tuning fork, wine goblet, etc.) behave similarly.

You may try this with your own wine glass. You sing at a low frequency, and the glass is not vibrating very much in sympathetic response. You increase your pitch at the same loudness, and it vibrates a lot. You can see it and also hear it because the glass essentially functions as a speaker cone. The sound of the sympathetic vibration gets much louder, although you are changing only the frequency of the forcing sound and not its loudness. When the responding sound gets much louder, you have identified the natural frequency, the resonant interval. You can raise and lower your pitch, keeping the same loudness. When you go *down* through the lower endpoint of the resonant frequency range, there is a snap *up* in the loudness of sympathetic response of the glass. It is this snap or popping that can actually break the glass. As you pop it again and again, the glass weakens. If the glass were a linear vibrator (for example, a very thin plate of glass), you could probably break it by forcing it at the resonant frequency, although you might have to use a somewhat louder forcing sound.

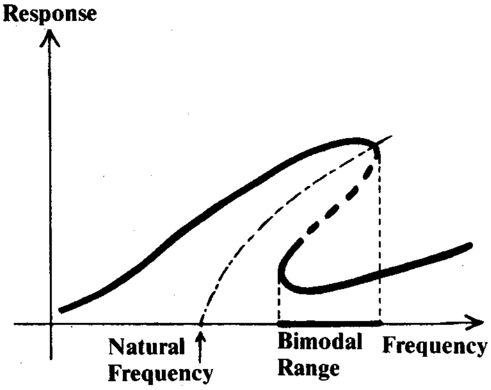

Figure 5. This is the response diagram for a *nonlinear resonator*. The response curve has been bent over towards the higher frequencies. The solid curves represent *attractors*, the dotted curve indicates the *separatrix* between the basins of the two competing attractors. Within the *resonant interval* there are two sympathetic responses, a loud one and a softer one. At the boundaries of this interval are found the *double fold catastrophies*, in which one or the other of the sympathetic vibrations disappears.

We may now apply these concepts of resonance of nonlinear oscillators to a field of oscillators or vibrator. To do this, we may use the discretization strategy invented by Leonardo about 1500. In the case of the tuning fork bar or guitar string, we have to replace it with a discrete mechanical model. We saw one bar of the tuning fork into pieces about an inch long and then put these pieces back together again with springs. There is not much room for the springs, so we take each chunk of aluminum and compress it into a tiny bead of enormous density without losing any mass. Now we have spaces of about an inch between adjacent beads in which to put the springs. This becomes a discrete model in the style introduced by Leonardo and much used in the corpuscular mechanics of the 18th century. We have a discrete mechanical model for one bar of the tuning fork, as a string of dense beads, essentially pendulum oscillators, coupled with small springs.

This model was used by d'Alembert at the dawn of mathematical physics in 1749. He made a model for the vibrating string by discretizing it in this way, imagining that springs came between discrete heavy beads. He then wrote down $f = ma$ (Newton's equation for this discrete model), and continued as though there were more and more beads, lighter and lighter and closer together, making a better and better approximation to the vibrating string. For details of this analysis, see Buckley (1985).

The extreme case of this conceptual simplification of Leonardo is exactly what Rayleigh (1960) did in his analysis of the clarinet reed, in his fundamental book on acoustics in 1882. He replaced the reed by a single heavy bead at the top that was connected to the base of the mouthpiece by a weightless leaf spring. Then it is a simple kind of nonlinear oscillator, like a pendulum.

NONLINEAR RESONANCE OF VIBRATIONS

If you place two tuning fork bars or two vibrating strings side by side and you discretize each in this extreme way into a single bead, and the coupling between them is modeled by another small spring, then you have almost exactly the situation of Duffing's experiment. So the response diagram of Duffing's catastrophe, Figure 5, applies here.

Now consider more realistic discrete mechanical models for the two strings, in which each is modeled by a string of dense beads connected by weightless springs. We may model the coupling between the two strings, the medium for the sympathetic response of one to the motion of the other, by an additional row of even smaller springs, as shown in Figure 6. This is the final goal of our exercise in mechanical modeling. Some adaptation of Duffing's response diagram still applies, as each pair of beads (one from string A, the corresponding one from string B, and the very weak coupling spring between them) has its own double fold catastrophe.

Morphic Resonance

If you want to understand, for example, the phenomenon of memory in the vibratory field, how to store and retrieve memories using *vibratory resonance*, this is how you might do it.

Consider our basic discrete mechanical model for two coupled strings (Figure). We will suppose that string A is heavier: it is the *driver.* And string B is lighter; it is the *follower.* The coupling strings between the two strings are weightless and very weak. Imagine each string in a state of vibration. Thus, each bead is oscillating across the direction of the parallel strings. The cooperation of the string of driving oscillators might make the appearance of a traveling wave, a standing wave, or whatever. But below is string B, the follower. Each oscillating A bead is forcing a corresponding B bead into a sympathetic, resonant oscillation. And the resonant response is subject to the Duffing diagram (Figure 5). Maybe Pythagoras did this experiment; I don't know. But Saveur did it by 1700 (Cannon & Dostrovsky, 1981).

The driving string is above, the following string below. You pluck the upper one (the driver) and observe the effect on the target string (the follower) below. At a constant amplitude (loudness) of the driving string, you change its frequency (pitch). The lower string will suddenly, at certain places, snap up to the larger amplitude, the second state observed by Duffing.

If you could observe piano strings carefully after they are struck, you might see discrete disjoint segments of the string where it is in the bigger mode or the smaller mode, which differs only by fractions of a millimeter. This is the *memory of a pattern.* Within the vibrating wave, there are two states possible, almost identical, but one has a slightly larger amplitude than the other. Of all the oscillators comprising the coherent phenomenon of vibration, some will be in the loud state, others in the quiet state. If you color the loud beads blue and the quiet beads green, you would see a blue pattern on a green background. That pattern is remem-

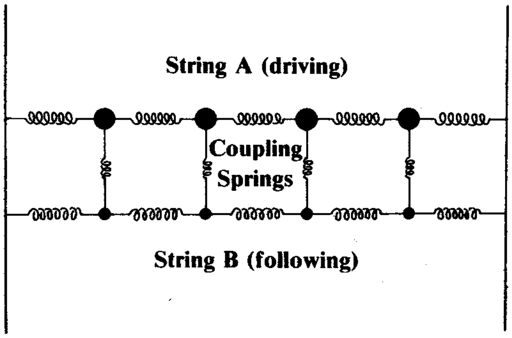

Figure 6. The discrete mechanical model for two coupled vibratory fields.

bered in the vibrating string as long as it continues to resonate.

The pattern in the following string might have been created by intentionally programming the activity of the driving string, by changing its frequency and amplitude patterns. This mechanism would achieve the storage of a selected pattern within the vibratory field of the follower string. To retrieve a vibration memory stored in this way, you would have to drive the system with a very sensitive device that is able to detect which beads are in the loud state and which are in the quiet.

This is a mechanical model for morphic resonance: A pattern in one vibratory field creates a related pattern in another coupled, vibratory field. This particular scenario is an application of just a single phenomenon of nonlinear dynamics, the double fold catastrophe. (There are many others.) With it you can break the glass, or if the glass is more flexible, like an automobile fender, you can impress in it a memory in the shape of a dent, and then retrieve that memory and get its shape out into another vibratory medium. This application was envisioned by Chladni, the father of acoustics, around 1800. He played plates of glass with a cello bow, observing patterns in a thin layer of sand on the plates.

Physiological Resonance

This has been an arduous metaphor to follow, and perhaps not everyone wants to know how to break a wine glass by worrying it to death. However, I believe that this mechanism of morphic resonance may enable us to understand many phenomena involved in brain and mind functions. In fact, it was proposed explicitly by Zeeman (1977), around 1970, as a mathematical model for memory traces in the brain.

To understand the occurrence of this kind of vibratory pattern in physiology, we must observe that nature has designed biological organs somewhat in the style of Leonardo's discrete mechanical model. Take a liver for example. The liver is a mammalian organ that consists primarily of one kind of cell. (Most organs have many different types of cells.) There is a lot of structure besides a homogeneous mass of liver cells. We will just try to imagine what kind of behavior we would expect from a mass of liver cells.

First, there may be an oscillatory process in each cell. Second, these oscillatory processes may communicate with each other through different kinds of messages (which are not entirely understood) passing between cells. There is a whole universe of life in the extra-cellular space, involving electrolyte physics and biomolecular processes. Third, the response diagram of each cell to an exogenous forcing oscillation may contain double fold catastrophes or even more complicated behavior. Thus, there may be amplitude patterns spread over the liver. Finally, there is a cooperative mechanism among the cells, which is more or less predictable from this kind of dynamics of coupled populations of oscillators, based upon phase regulation (Abraham, 1986). The oscillations are of approximately the same frequency for the different cells, and their relative phases organize into patterns. In sum, there may be amplitude patterns, phase patterns, and frequency patterns, as in radio communication.

If you observe phase patterns in the right way, for example, the cells in phase with each other would all appear blue to you, and the cells out of phase with these would seem green. Some cells would change from blue to green and back again according to their phase relationships, under the influence of some external driving field of bioelectrochemical vibration. Then you would observe this as a green pattern moving on a blue background. This is how physiological vibrations might be mechanically modeled. A lot of functions might be understood this way, particularly of the pituitary, where clocks have to be in phase; or in the reproductive cycle, where there is the mysterious phenomenon of the luteinizing hormone (LH) spike.

In the middle of the reproductive cycle, before ovulation, the luteinizing hormone LH concentration in the blood suddenly rises to astronomical levels. This LH is released by the pituitary on receiving a message from the hypothalamus of LH releasing hormone (LHRH). Now imagine you are a pituitary

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cell, and you have around your periphery a bunch of vesicles full of this LH, which you have been saving up for your moment, your place in history. You must release your whole store at the proper time and synchronously with all your sister pituitary cells. If all these pituitary cells let go their LH stores on the same day, then your owner has a proper LH spike and ovulation is possible. If you get it a little bit wrong then there is no LH spike and no ovulation and no subsequent reproduction. Life itself depends on strict cooperation!

How do all these pituitary cells know their exact circumstances? Is it simultaneous arrival of the LHRH message? If so, how does the hypothalamus know how to do this, and so on? I am suggesting that the answers to these questions may be sought in the behavior of discrete mechanical models, particularly in models of resonant vibrations. The mechanisms of morphic resonance, applied to physiological models, may increase our understanding of life processes.

There is a universal strategy in mathematical modeling, including all of mathematical physics, mathematical biology, and mathematical sociology up to the present time, with very few exceptions (Abraham, 1984). It is the exercise of this strategy, in combination with participatory experiments and observations that advances our grokking of the world of phenomena and process. This is the hermeneutical view of the history of the mathematical sciences, from Cro-Magnon times to the present.

Applied to the vibrating string, following Leonardo, Galileo, Huyghens, d'Alembert, and Euler, it is *mathematical physics*. Applied to the pituitary, the liver, and the other organismic vibrators (populations of oscillating cells) it is *mathematical biology*. Applied to social structures and ecosystems, it is *mathematical sociology*. *The certain sure sign of life is vibration*, and the mathematics of vibration (including mechanical models for morphic resonance) is a valuable strategy for grokking life.

REFERENCES

Abraham, R. H. (1984). Complex dynamical systems. In X. J. R. Avula, R. E. Kalman, A. I. Leapis, & E. Y. Rodin (Eds.), *Mathematical modelling in science and technology* (pp. 82–86). Elmsford, NY: Pergamon Press.

Abraham, R. H. (1986). *Phase regulation engineering*. Unpublished manuscript, University of California, Santa Cruz.

Abraham, R. H., & Shaw, C. D. (1982). *Dynamics, the geometry of behavior, part one: Periodic behavior*. Santa Cruz, CA: Aerial Press.

Berman, M. (1981). *The reenchantment of the world*. Ithaca, NY: Cornell University Press.

Buckley, R. (1985). *Oscillations and waves*. Bristol, England: Adam Hilger.

Cannon, J. T., & Dostrovsky, S. (1981). *The evolution of dynamics: Vibration theory from 1687 to 1742* (p. 26). Berlin, W. Germany: Springer-Verlag.

Davis, P. J., & Hersh, R. (1986). *Descartes' dream: The world according to mathematics*. New York: Harcourt Brace Jovanovich.

Duffing, G. (1918). *Erzwungene schwingungen bei veranderlicher eigenfrequenz und ihre technische bedeuting*. Berlin, W. Germany: Braunscheig Vieweg Verlag.

Rayleigh, W. S. (1960). *Theory of sound*. New York: Dover. (Original work published 1882).

Truesdell, C. (1960). *The rational mechanics of flexible or elastic bodies: 1638–1788, introduction to Leonhardi Euleri opera omnia, Vol. X et XI seriei secundae*. Turici, Italy: Orell Fuessli.

Zeeman, E. C. (1977). *Catastrophe theory*. Reading, MA: Addison-Wesley.