CUSPOIDAL NETS *

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Abstract

The ubiquitous cusp catastrophe has been pressed into service by Zeeman as a rough qualitative model for many dynamical systems in the sciences, including a democratic nation. The extension to two nations has been made by Kadyrov, who discovered an interesting oscillation in this context. Here we speculate on the properties of connectionist networks of cusps, which might be used to model social and economic systems.

1. Introduction

Complex and cellular dynamical systems provide useful models for morphodynamic systems, such as neural membranes, heart muscle, reaction-diffusion devices, and microeconomic communities [1]. The connectionist neural net is an important example, which uses the simplest standard cell, but achieves complex behavior through a rich net of connections. In this paper, we suggest properties of a network of cusps. The properties are built up through a sequence of examples, beginning with published works of Zeeman and Kadyrov. A later paper will be devoted to the results of simulations [2].

2. One cusp

The cusp catastrophe of elementary catastrophe theory (ECT) is the canonical bifurcation of codimension two. It occurs with static attractors [3], periodic attractors [4], and chaotic attractors [5]. We will adopt the (nonstandard) notations of the Isnard and Zeeman model for hawks and doves (see their Sec. 11) [6]. Let

$$\phi(x, a, b) = -\frac{1}{4}x^4 + \frac{1}{2}bx^2 + ax$$

Then the dynamical scheme of the cusp is

$$x' = \phi_x(x, a, b) = -x^3 + bx + a$$

where ϕ_x denotes the partial derivative of ϕ with respect to x. This scheme has the familiar response diagram shown in Figure 1 (taken from Isnard and Zeeman, Fig. 11). The control parameter b is called the *splitting factor,* while a is the *normal factor*.

3. Two cusps

In the spirit of complex dynamical systems theory, we may couple two cusps in a minimal network. Each of the control parameters of one of the cusps may be expressed as a function of the state of the other. We consider here only one special case, introduced in Kadyrov's model: let the normal factor of each be proportional to the state of the other [7]. Thus,

* Dedicated to: Christopher Zeeman, on his sixtieth birthday.

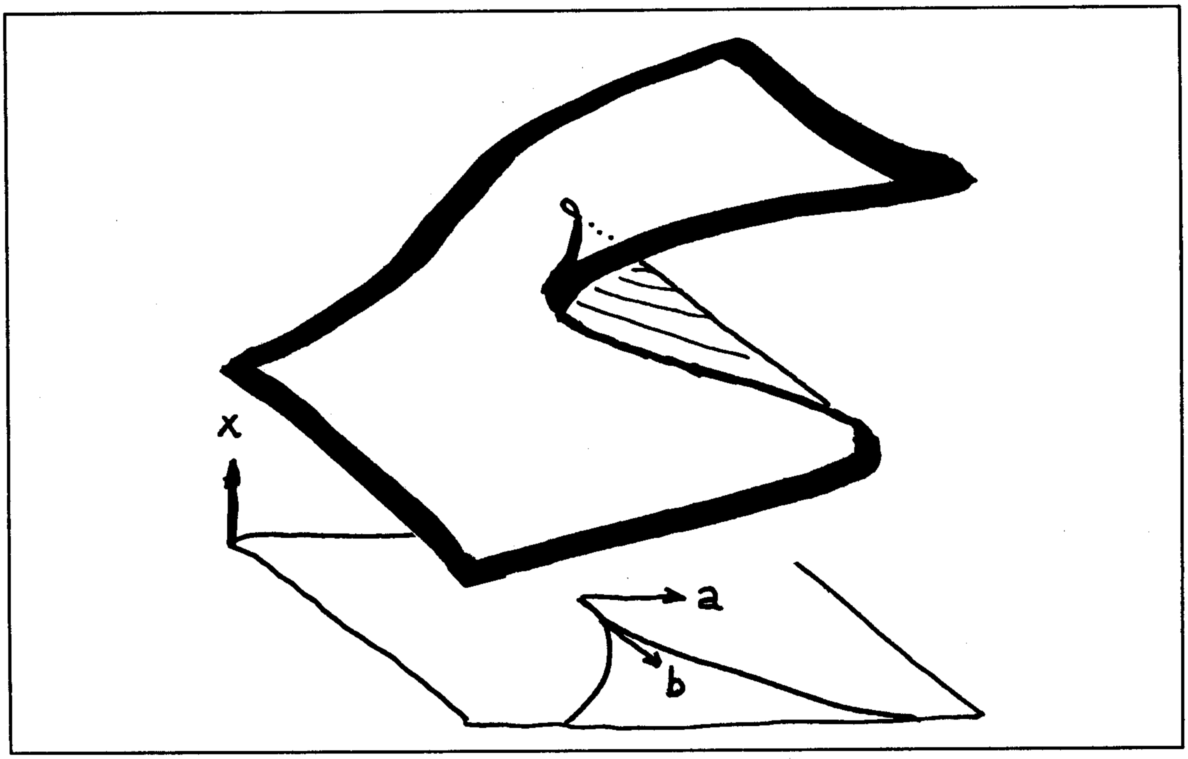

Figure 1. Response diagram of the standard cusp

$$x' = -x^3 + ax + by$$

$$y' = -y^3 + cy + dx$$

where *by* replaces *b* in the preceding equation. This is a dynamical scheme with two-dimensional state space, $(x, y) \varepsilon R^2$, and four-dimensional control space, $(a, b, c, d) \varepsilon R^4$. Due to the source of this dynamical scheme within ECT, we might expect this to be a gradient system. In general, it is not. For the partial derivative of the x' function with respect to y is b, while that of the y' function with respect to x is d. Thus, in the *symmetric case,* in which we set $b = d$, the coupled system is a gradient scheme with three-dimensional control, equivalent to one of the umbilics of ECT (see p. 185 of Poston and Stewart [8]). Its potential function is

$$\phi(x, y) = -\frac{1}{4}(x^4 + y^4) + \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

Its bifurcation set (an algebraic surface in R^3) may be visualized in a plane cross-section by setting any of the three control parameters equal to a constant. For example, with $b = d = 1$, the bifurcation set section consists of two crossed umbilics.

But in the general, *asymmetric* case, we have a non-elementary response diagram.

Kadyrov has obtained, for example, the section of the bifurcation set shown in Fig. 2 (taken from Kadyrov, Fig. 6). This (b, d) plane section is defined by $a = 1$ and $c = 1$ (or any positive values in the wedge defined by $2c > a$ and $2a > c$). The cusps represent degenerate fold catastrophes or blue loops, while the parabolic curves represent basin folds. We may explain these in more detail, by considering the phase portraits in the seven distinct regimes of the section (A through G) in Fig. 2. The jargon from bifurcation theory may be found in *Part Four* of [5].

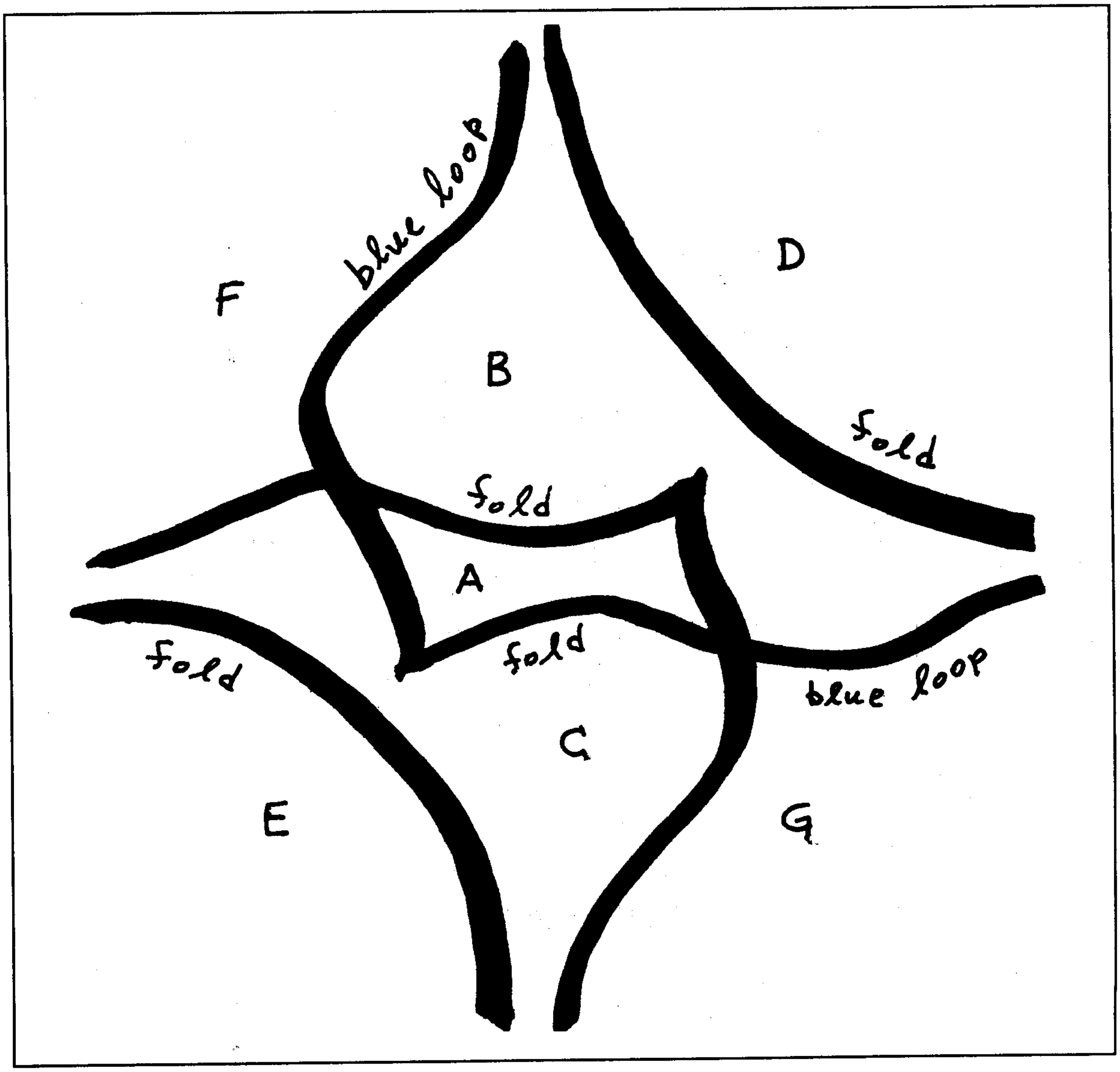

Figure 2. Section of the bifurcation set.

(A) In the central region enclosed by the two cusps, there are four static, nodal attractors (one in each quadrant, I, II, III, and IV), four saddles (their insets comprising the separators of the four basins), and one central, nodal repellor.

(B) Here there are two point attractors (in quadrants I and III), two saddles, and one central repellor. Across the bifurcation curve between A and B there are two fold catastrophes, in each of which a point attractor and a saddle mutually annihilate. (If nondegenerate, these would occur one at a time across disjoint curves.)

(C) Two attractors as in B (but in quadrants II and IV).

(D) Two attractors as in B, but across the bifurcation curve between B and D there is a fold in which one saddle and the central repellor annihilate. This is a basin bifurcation.

(E) Two attractors as in D (but in quadrants II and IV).

(F) In this regime there is a single attractor (periodic) and a central, spiral repellor. Across the bifurcation curve separating B (or C) from F, there is a degenerate blue loop explosion, in which both point attractors (in B or C) explode simulteneously. (In a nondegenerate analogue, each blue loop event would occur across its own, distinct, curve. Alternatively, there might be a fold catastrophe, followed by a single, nondegenerate blue loop.)

(G) One periodic attractor, like F. In these two regimes, F and G, we call the attractor a *Kadyrov oscillator*. The existence of this oscillation in a system of coupled ECT schemes is crucial for the applications we envision, and discuss below. It is reminiscent of the oscillator found by Smale in the context of two coupled cells [9]. We ignore the degeneracy of this scheme for the present, but will return to this in a future paper, in which a complete unfolding will be suggested [2].

4. Cuspoidal nets

We may easily extend our definitions to an ensemble of N cusps. Let A be a real matrix of size N by N, and consider the vectorfield on R^N defined by

$$x_i = -x_i^3 + A_{ij} x_j \,, \quad i = 1,,, N$$

where the Einstein sum convention is implied by the repeated subscripts. This is a dynamical scheme with N-dimensional state space, $x \, \varepsilon R^N$, and N^2-dimensional control space, $A \, \varepsilon R^{N \times N}$. From the connectionist point-of-view, this is a neural net, slightly generalized from the usual linear one by the addition of the cubic terms. In the symmetric case, $A_{ij} = A_{ji}$, the system is of gradient type, with potential function,

$$\phi(x_1, \ldots, x_N) = -\frac{1}{4}(x_1^4 + \ldots + x_N^4) + \frac{1}{2} A_{ij} x_i x_j$$

The interpretation of the attractors of the scheme as the local minima of this function, common to ECT and to neural net theory alike, is valid and useful here [10]. Further, between the symmetric and the general cases, intermediate cases may be of interest, such as the case in which A is symplectic. We might suggest to the neural net community to experiment with these systems, which have long-term memory properties in their bistable regimes.

An example of a cuspoidal net is illustrated in Fig. 3, in which each cusp is reduced to a double fold by fixing its splitting factor, $b_i = +1$. This net exhibits *memory*, in that the temporary displacement of the local (normal factor) controls a_i from neutral values may effect a catastrophic shift of state, which is remembered after the return to neutral.

Note that if two neighboring cells are coupled by the Kadyrov scheme of the preceding section, Kadyrov oscillations may result from their interaction. There may even be two or more Kadyrov oscillators within the network. And the coupling of two

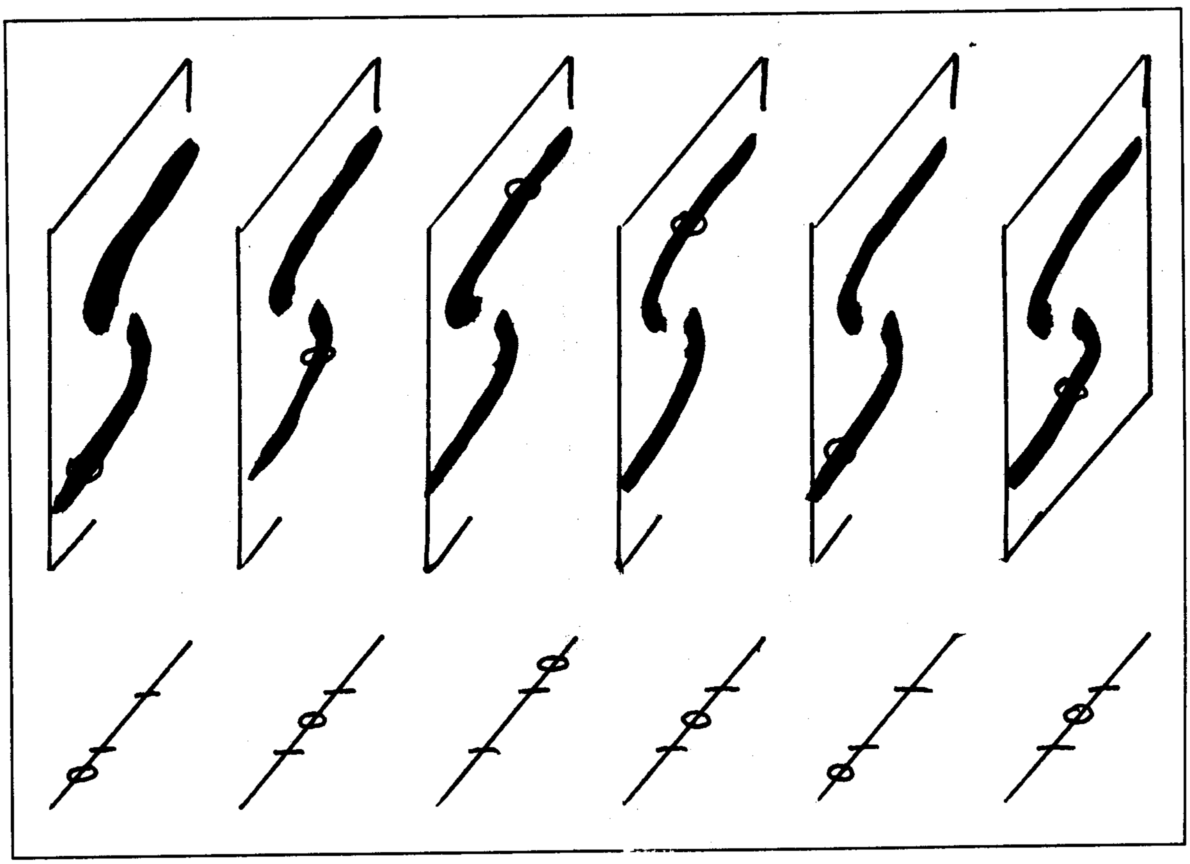

Figure 3. One-dimensional array of double folds.

Kadyrov oscillators may easily result in chaotic behavior. Thus, a cuspoidal net with four cells and Kadyrov (that is, linear) coupling is capable of multistable behavior, with static, periodic, and chaotic regimes. We may conclude that models for social and economic behavior, such as business cycles, may be constructed of cuspoidal nets. These may be *the simplest models* exhibiting such behavior.

5. Conclusion

Complex dynamical systems, made from a finite set of identical dynamical schemes by a complete graph of coupling functions of adjustable strength, may be regarded as a potentially useful generalization of neural nets and excitable media. Taking the standard cell from ECT defines an *ECT net,* and provides us with a head start in the understanding of the global behavior of the network, and the geometry of its response diagram. In this paper, we have described only one example of an ECT net, *the cuspoidal net,* based on the cusp catastrophe of ECT. The oscillatory cusp of the Duffing system also suggests itself for this treatment [4]. Exploration of these nets will require extensive computation, and will provide serious challenge to our computer-graphic visualization skills. Massively parallel machines would be particularly well suited for this research.

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