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CONCEPTS OF DYNAMICAL SYSTEMS THEORY

Prologue

by

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Dynamical systems theory. Dynamical systems theory, a relatively new branch of mathematics, contains an extensive strategies for building models for complex processes, These strategies utilize the new theories of chaos and bifurcations. The basic dynamical concepts — state spaces, attractors, basins, and bifurcations — will be exhibited in this show.

Dynamical models. In modern mathematics and science, several words are nearly synonymous with *dynamics*. Among them, historically, have been the words *mechanics* (the art of making models, especially mathematical models for dynamical processes), *dynamics* (the physics of forces and motions of discrete or continuous mass systems), *kinematics* (the abstract description of motion exclusive of mass and forces), and *kinetics* (the study of motion, including dynamics and kinematics). These denote classsical subjects, with important contributions made by Archimedes, Galileo, and Newton. Since Poincaré. However, *dynamical systems theory,* or *dynamics* for short, has recently come into standard use among mathematicians, in place of mechanics and kinematics, to denote a new subject, synthesizing the historical subjects. Dynamics, in the new mathematical sense, means the abstract mathematical theory of motion, without reference to mass, force, or any other physical property of the moving system. That is, a *dynamical model* is an abstract mathematical model for a dynamical process. For example, a point might model a process at rest, a circle might model a cyclically recuring process. Chaotic attractors are new models for noisy or irregular motions. There are dynamical models in this sense for many physical systems (for example, a dripping faucet), and also for biological (such as the mammalian immune system) and social systems (such as an arms race) which have no physical forces. These models, together with their simulation on digital computers with color graphics, comprise a new style in the traditional model-building art of mechanics.

State spaces. A dynamical model for a natural system begins with a geometric model for its virtual states, called the *state space.* For example, the state space for a model of the dripping faucet might be a straight line, on which we record the time interval between drips.

Dynamical rules. Another basic ingredient of a dynamical model, called its *dynamical rules,* express the fundamental rules of the game, which determine the motion of the model system within its state space. At each point of the state space, the rules specify the vector of motion (direction and speed) which the model must obey whenever it is in that state. The qualitative analysis of these model systems emphasizes the long-term behavior, expressed in terms of the final behavior after transient effects have died away. These final behaviors are described in terms of attractors and basins.

Attractors. From the viewpoint of dynamics, processes are observed in special forms of dynamical behavior called *attractors.* These represent the stable (or observable) states of a system or process. There are three types of attractors. *Static attractors* (also called *point attractors)* model the stationary states of a system. They represent the system at rest. *Periodic attractors* consist of a cycle of states, repeated again and again, always in the same period of time. They represent the system in oscillation. *Chaotic attractors* consist of fractal (infinitely folded) sets of states, over which the model system moves, apparently at random. They represent the system in state of agitation, or turbulence. These three mathematical objects were not discovered all at once. First, the point attractor emerged, soon after Newton developed the basic mathematics for dynamical systems (the branch of mathematics now known as *ordinary differential equations)* circa 1700. The periodic attractor came into general use around 1850, spawning a new branch of mathematics known as *nonlinear oscillation theory.* And the chaotic attractor, although known in one form (the so-called *homoclinic tangle)* to Poincaré in 1885, emerged into scientific consciousness only around 1963. Thus, these three dates, 1700, 1850, and 1963, marked quantum leaps in the mathematical understanding of dynamical processes.

Basins. All states tend, as the process evolves, to an attractor. But there are usually several attractors. For one attractor in particular, all of the states which end up going around it comprise its *basin of attraction.* The basins, one for each attractor, provide the most important information about the model system, in that the enable the prediction of the result (terminal attractor) of a given cause (initial state). The basins are bounded by the *separators,* which are the target of much mathematical study.

Bifurcations. In many applications, the target system depends on *control parameters* which affect the dynamical process of the system. For each set of values of these parameters, we may construct a different dynamical system. The dynamical system depending smoothly upon parameters is called a *dynamical scheme.* As the parameters are changed, the attractors and basins are usually affected, sometimes critically. The radical change of the attractor and basin portrait of a such a system is called a *bifurcation* *event.* Many of these are known, and are used to model special transformations such as phase transitions in physical systems, radical changes of behavior in biological systems, or revolutions in social systems. Other bifurcation events are as yet unknown, and experimental dynamicists are searching for them.

Throughout the history of dynamics, discoveries were made by scientists in laboratories, leading to theorems expressed by mathematicians. Today, the advent of computers, scientific computation, and computer graphics have revolutionized the conduct of dynamics research. The nature of the subject makes graphical representations particularly effective in the study, exploration, and demonstration of dynamical models concepts. My own books, *Dynamics, The Geometry of Behavior* (Four Vols., Aerial Press, Santa Cruz, CA 95061, 1982-88), written with artist Chris Shaw, have made extensive use of graphics.

In this exhibition, you may see images from the research frontiers of experimental dynamics, relating to chaotic attractors and their bifurcations. Thanks to computer graphics, many esoteric objects from the world of mathematics have become visible.