13

Visualization Techniques for Cellular Dynamata

Ralph H. Abraham

13.1 Historical Introduction

Complex dynamical systems (CDSs) [1] and cellular dynamata (CDs) [2] are classes of dynamical models that have evolved in the context of applied work over a period of years. The literature on one-dimensional cellular dynamata is extensive [3]. Recently, two-dimensional lattices have become the focus of several exploratory studies [4]. In this chapter, we describe some methods for creating visual displays of important qualitative features of cellular dynamata in one and two spatial dimensions.

Reaction/diffusion equations constitute a special class of partial differential equation (PDE) systems of evolution type. They were introduced and used by the pioneers of biological morphogenesis: Fisher (1930), Kolmogorov-Petrovsky-Piscounov (1937), Rashevsky (1940), and Turing (1952). Fisher introduced the logistic/diffusion equation, particularly important for CD research, as a model for the diffusion of mutants in a population of flys [5]. Rashevsky introduced spatial discretization corresponding to biological cells in his work on embryogenesis [6]. Discretized reaction/diffusion systems provide examples of cellular dynamata, probably the first in the literature. Reaction/wave systems are another source of important CDs. Further developments were made by Southwell (1940–1945), Turing (1952), Thom (1966–1972), and Zeeman (1972–1977). The latter includes a heart model [7], and a simple brain model exhibiting short- and long-term memory [8]. The arrival of scientific computation in Los Alamos stimulated Ulam and Von Neumann to develop cellular automata (CAs) in connection with their efforts to solve the heat equation. During the 1980s, under the influence of the growing availability of computer graphic workstations in the scientific community, experimental work on 1D/CDs (one spatial dimension) began. Later, as computational power became adequate, 2D/CD studies began to appear in the literature. The ideas of cellular dynamaton theory, inspired by these pioneers, are summarized in the next section.

13.2 Cellular Dynamata

Cellular dynamata are complex dynamical systems characterized by two special features: the nodes (component schemes) are all identical copies of one scheme, and they are arranged in a regular spatial lattice. We now recall the basic definitions.

13.2.1 Dynamical Schemes

By *dynamical system* we mean an autonomous system of coupled ordinary differential equations of the first order. More generally, we include vector-fields on manifolds, both finite and infinite dimensional, which we call *state spaces*. The *phase portrait* is a visualization of the dynamical system within its state space. Thus, systems of coupled partial differential equations of evolution type are included, along with integro-differential-delay equations, and so on. By *dynamical scheme* we mean a dynamical system depending upon parameters in a supplementary manifold, the *control space*. The visualization of a dynamical scheme (in low dimensions) is provided by its *response diagram*. The familiar pictures from elementary catastrophe theory (ECT) [9], dynamical bifurcation theory (DBT) [10], and geometric function theory (GFT) [11], are exemplary response diagrams. The chief features are the *attractrix* (or locus of attractors) and the *separatrix* (or locus of separators). The most widely known of these diagrams are the *cusp* (shown in Fig. 13.1), and the *logistic period doubling sequence* (shown in Fig. 13.2). We will make use of these to illustrate visualization techniques for cellular dynamata.

13.2.2 Complex Dynamical Systems

Dynamical schemes may be serially coupled in various ways. The simplest, which suffices for most applications, is called a *static coupling*. This is a function from the state space of one dynamical scheme to the control space of another. The canonical example is the driven pendulum. In this way, a finite set of dynamical schemes (nodes) may be serially coupled by an appropriate set of static couplings (directed edges) in a network (directed graph). This is the definition of a *complex dynamical system*, the primary object of complex dynamical systems theory. Exemplary models for several physiological systems have been developed and run, producing convincing simulated data [12].

13.2.3 CD Definitions

By a *cellular dynamical system, cellular dynamaton*, or *CD*, we mean a complex dynamical system in which the nodes are all identical copies of a single dynamical scheme, the *standard cell*, and are associated with specific loca-

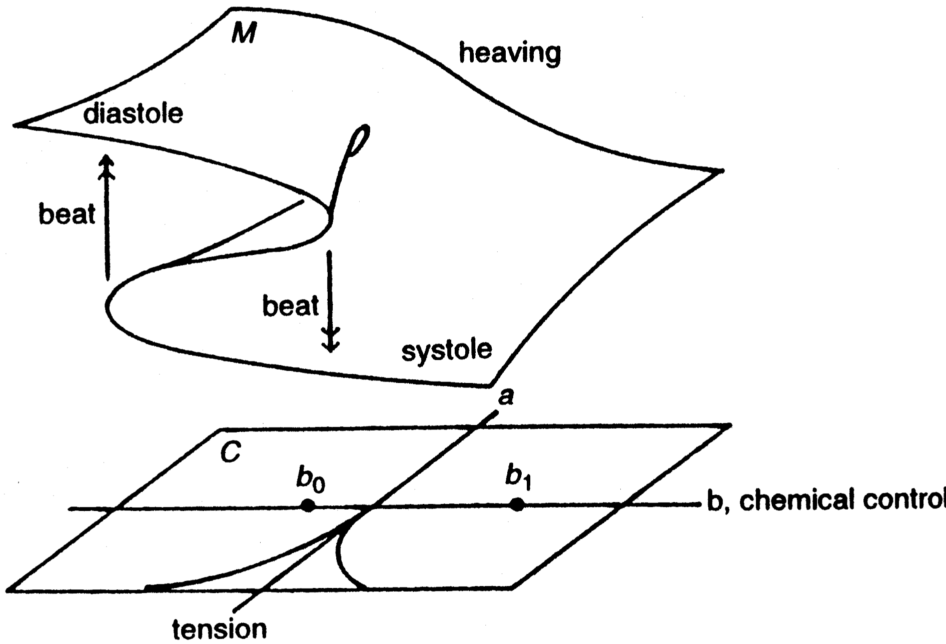

FIGURE 13.1. The cusp.

tions in a supplementary space, the *physical space*. Exemplary systems have been developed from reaction/diffusion systems by numerical methods, such as Southwell's relaxation method, which proceed by discretization of the spatial variables. Other important examples of this construction are the heart and brain models of Zeeman. CDs have something in common with the *cellular automata*, or *CAs*, of Ulam and Von Neumann, but possess more structure, and are in some ways more general.

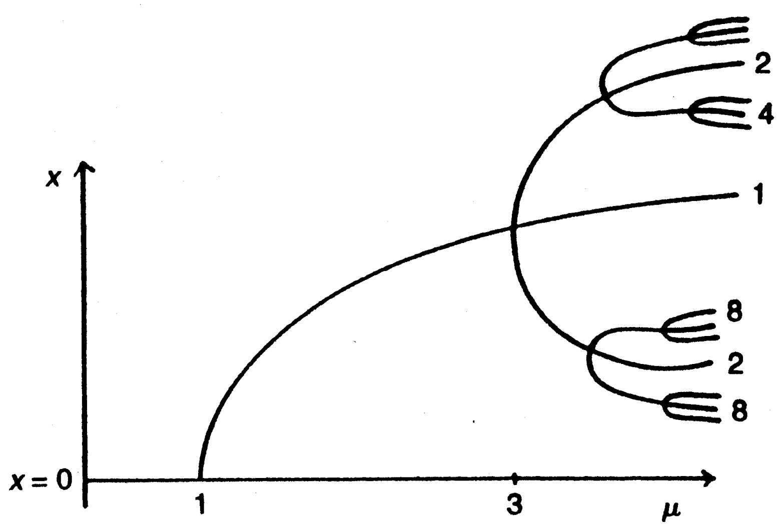

FIGURE 13.2. The logistic period doubling sequence.

13.2.4 CD States

An *intantaneous state* of a CD consists of a map of the lattice in its physical space, L, into the state space of its standard cell, S, assigning an local state to each node. From this data, one may compute the control parameters at each node via the coupling functions, and thus obtain a derived map from L into the space, CxS, of the response diagram of the standard cell. We regard this map, or perhaps its image (a finite point set within CxS), as the *global state* of the CD.

13.2.5 CD Simulation

Simulations of a CD are done as follows. In case the local dynamics of the standard cell are specified by differential equations, a time interval and fixed time-step integration algorithm are chosen. Thus, the continuous dynamics are replaced by a discrete time scheme. Alternatively, a variable time-step method may be used, but forced to report the trajectory at fixed (larger) time steps. This is the method of *periodic reportage*, commonly used to provide the Poincaré section of a periodic attractor.

Our recipe applies to the discrete-time case only. We begin with an initial, global CD state, and fixed values of any free control parameters. Then, parallel computations are performed at each node, with the local values of initial state and control parameters, to obtain the next local state at that node. This determines the next global state. Iteration of this global step produces a sequence of global states, the global trajectory. This approaches a global attractor, in the space of all global states.

13.2.6 CD Visualization

The behavior of a cellular dynamical system under simulation may be visualized by various methods, here we mention three. In *Zeeman's method*, an image of the lattice in physical space is projected into the response diagram of the standard cell, where it moves about, close (more or less) to the locus of attraction. (It is closer if the coupling is weak.) Alternatively, the behavior may be visualized by the *graph method*, in which we attach a separate copy of the standard response diagram to each cell of the location space. Within this product space, the instantaneous state of the model may be represented by a graph, showing the local state occupied by each cell, within its own response diagram. A related method of particular importance in our current work with wo-dimensional CDs is the *isochron method*, in which isochrons (phase sins) are used to color the physical space. In any case, the behavior of the mplete CD system may be tracked, as the controls of each cell are sepatelly manipulated, through an understanding of the response diagram of the ndard cell provided by dynamical systems theory, in terms of attractors, basins, separatrices, and their bifurcations [10]. We now give some examples of these methods.

13.3 An Example of Zeeman's Method

This method appeared first in a model by Zeeman for the human heart. Organs typically contain many different types of cells. In the unusual case that there is only one type of cell, one could imagine a model for the organ consisting of a single CD system. This is the case with Zeeman's heart model. An explicit CD model for the organ would require an explicit model for the standard cell, which might be found in the specialized literature devoted to that cell. In this case, Zeeman uses a well-known qualitative model, with unspecified coupling.

13.3.1 Zeeman's Heart Model: Standard Cell

In Zeeman's heart model, the standard cell is the cusp catastrophe. Each node corresponds to a muscle fiber, and the standard cell is a model for the muscle fiber dynamics. The two control parameters are muscle tension and the concentration of some transmitter chemical. The state variable is the length of the fiber. The upper sheet of the locus of attraction corresponds to an elongated state, the lower sheet to a contracted state.

13.3.2 Zeeman's Heart Model: Physical Space

Let us imagine a heart-shaped region in the plane as the physical space of the heart model. The spatial lattice of our CD model resides in this heart-shaped region, and its standard cell is the cusp. Rather than map this lattice into the space of the cusp, we will map the whole region. Assuming adiabatic conditions (weak coupling), the image moves close to the locus of attraction.

13.3.3 Zeeman's Heart Model: Beating

In Zeeman's model, the beating of the heart consists of a sliding motion of the image of the physical space over the fold catastrophe of the cusp, falling from elongated to contracted states, as shown in Fig. 13.3. Presumably, this would be the result of a simulation of his model. In any case, this figure illustrates his novel method of visualization, which may be applied successfully in many other cases.

13.4 The Graph Method

We proceed with our examples of visualization techniques, using the one-dimensional logistic lattice, the object of much recent research. First, we must present this object as a CD.

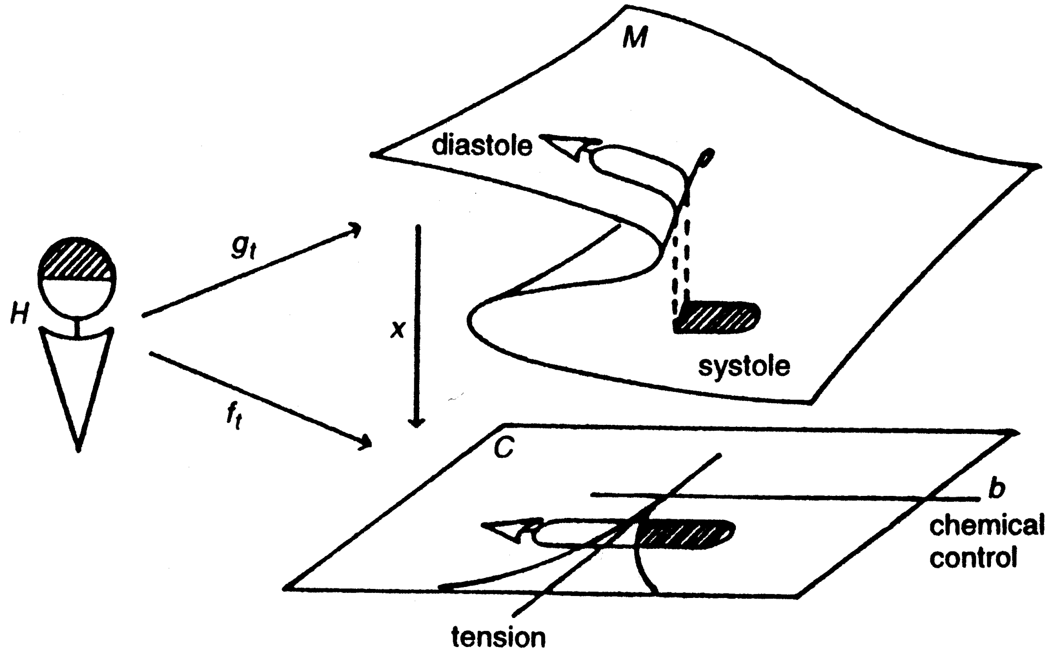

FIGURE 13.3. Zeeman's heart.

13.4.1 The Biased Logistic Scheme

The logistic map is the real-valued function of a single real variable,

$$f(x) = rx(1 - x). \tag{13.1}$$

Here, the control parameter, *gain*, r, should be in the range [1, 4], so f may be regarded as a map of the closed unit interval to itself, $f : I \mapsto I$. It serves our purpose to add to this function a constant called the *bias*, c, which is regarded as a second control parameter. Thus,

$$f(x) = rx(1 - x) + c \tag{13.2}$$

If this is to be regarded as a function from I to itself also, we must add clipping:

$$\text{if } f(x) > 1, \text{ replace it by one,}$$

$$\text{if } f(x) < 0, \text{ replace it by zero.}$$

Thus, we have an iterated function scheme with two control parameters. The response diagram of this scheme is a simple extension of the period doubling sequence of Fig. 13.2, as shown in Fig. 13.4, at least if the bias is small. (In our example and discussion below, following section 3 of Crutchfield and Kaneko [3], the range of the bias will be within ± 0.001.)

13.4.2 The Logistic/Diffusion Lattice

In our application, the simple logistic/diffusion lattice, we will build a CD by coupling the bias parameter of each node to the states of the closest neigh-

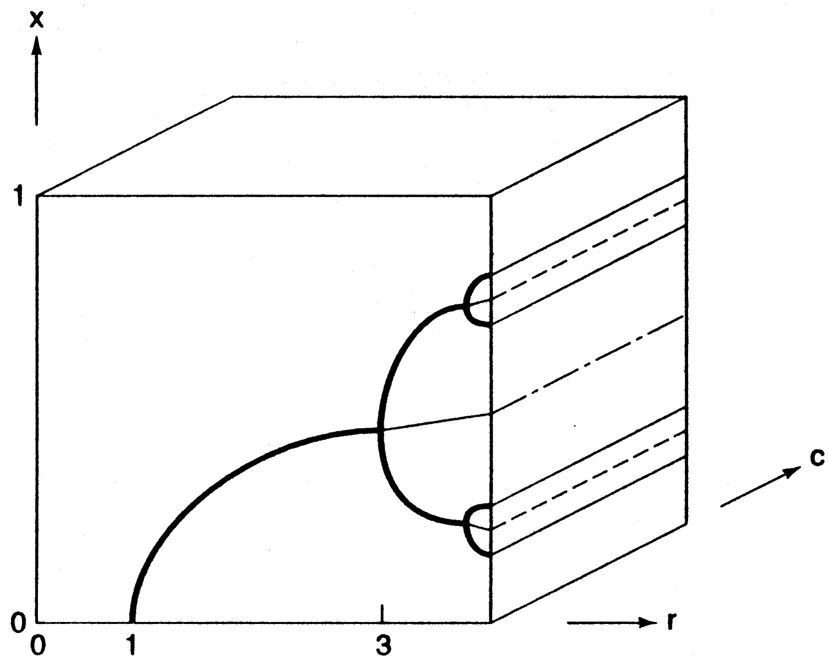

FIGURE 13.4. Response diagram of biased logistic scheme.

boring nodes by Laplacian coupling, leaving the gain free. The physical space will be the closed real interval, $I = [0, N-1]$, where N is the number of nodes, equally spaced in the physical space. We will set N and r (at a common value for each node) at the start of any simulation. Below, we will use $N = 128$ and $r = 3.5$.

At the ith node, $i = 1 \cdots N-2$, the next state, x_i^+, is given by

$$x_i^+ = r x_i^0 (1 - x_i^0) + c(i), \tag{13.3}$$

where x_i^0 denotes the current local state, and the the local bias is determined by

$$c(i) = \frac{\gamma}{2N^2}(x_{i+1}^0 + x_{i-1}^0 - 2x_i^0), \tag{13.4}$$

and γ is set to a constant value at the start of a simulation. In our example, $\gamma = 4.8\ k$ or about 5000.

These formulas are applied at the endpoints, $i = 0, N-1$, by special rules, called the *boundary conditions*. In our example, we will use the *toral* boundary conditions, in which the 0th and $N-1$st nodes are identified. Thus $c(-1)$ is replaced by $c(N-2)$ in Eq. (13.4) when $i = 0$, and so on.

13.4.3 The Global State Graph

Choosing the exemplary values mentioned above, we may now visualize the global states of this 1D/CD, the logistic/diffusion lattice, by the graph method.

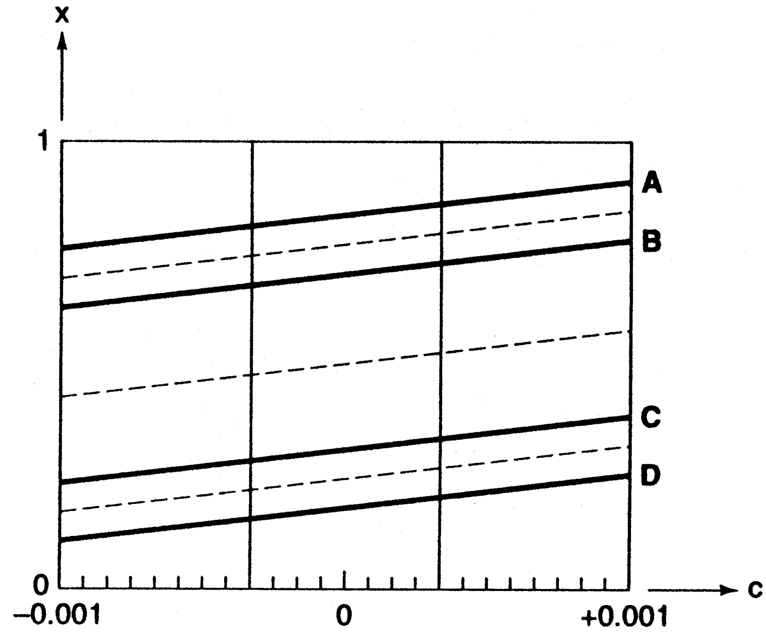

FIGURE 13.5. Response diagram, gain fixed at 3.5.

Fixing $r = 3.5$ at each node, we may reduce the response diagram shown in Fig. 13.4 to two dimensions, as shown in Fig. 13.5. The locus of attraction comprises four nearly straight lines, slightly rising to the right, corresponding to the four points of the periodic attractor. These are visited in the order: A, C, B, D. In between are the two curves of the Period two repellor, and the curve of the fixed repellor. All three curves are shown dashed. Attaching this 2D response diagram (visualized as a vertical plane) to each node in the 1D physical space (shown as a horizontal line) creates the 3D space in which the graph of a global state, $i \mapsto (x(i), c(i))$, may be plotted.

In Fig. 13.6 we show the graph of a state of the exemplary CD, taken from Crutchfield and Kaneko, [3, Fig. 2 (top)]. The projection into the plane of (i, x) is shown in Fig. 13.6a. The initial state leading to this global attractor is the sinewave (shown dashed). The global attractor, in this case, is a periodic trajectory of Period four. The periodic trajectory (superimposition of four successive global states) is shown by the lighter solid curves. One of the four instantaneous global states is shown as a heavier, solid curve. Note that although the coupling is weak, nodes in large regions of physical space are pulled far from their attractors by the combined influence of their nearest neighbors. In Fig. 13.6b is shown the projection of the heavy graph onto the (i, c) plane. In Fig. 13.6c is shown the projection onto the (c, x) plane. Note that this plane contains the reduced response diagram of the standard cell, as seen previously in Fig. 13.5. The graph of the instantaneous state is a curve, in the full 3D space of (i, c, x).

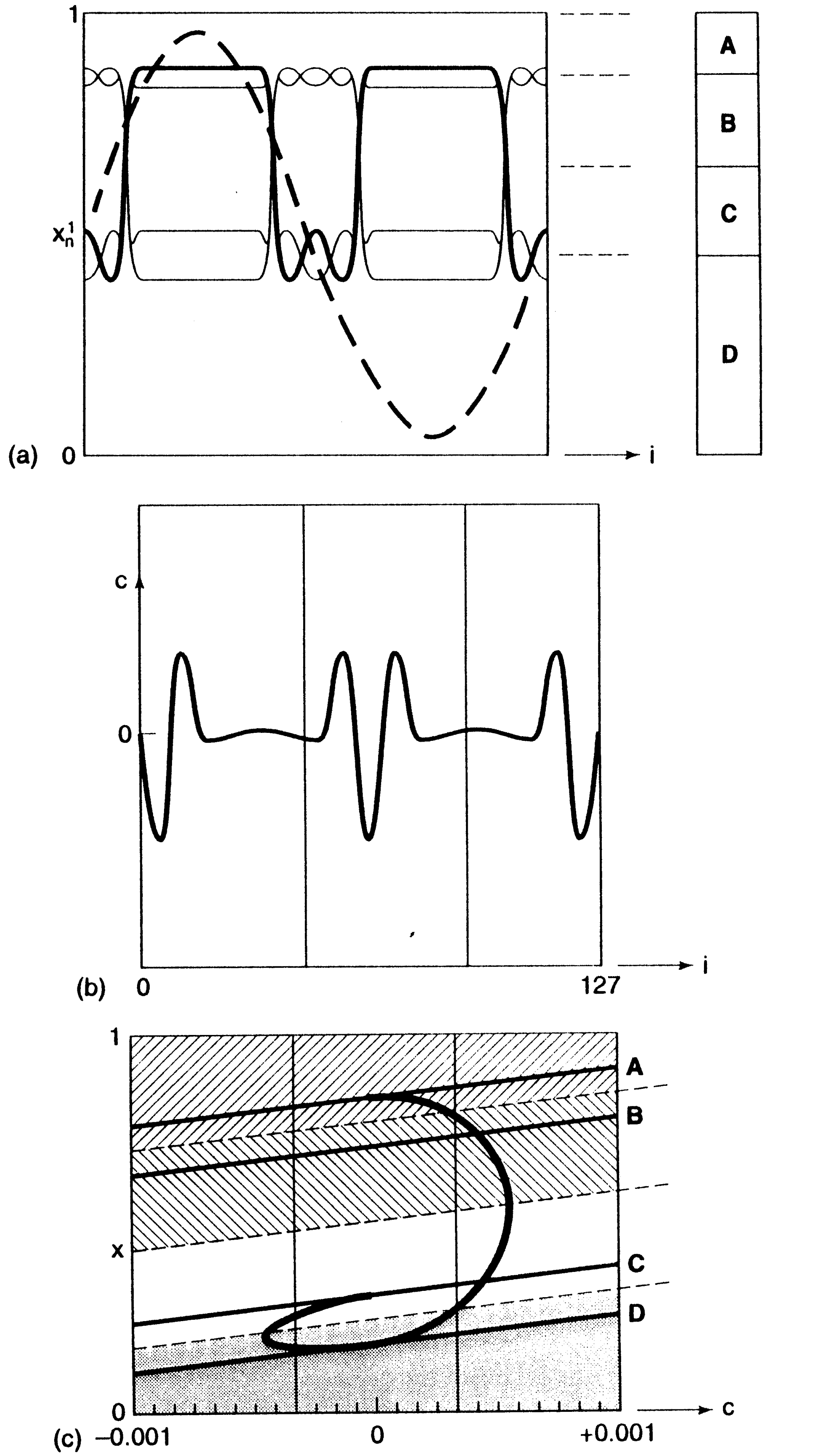

FIGURE 13.6. (a) Graph projected on (i, x) plane. (b) Graph projected on (i, c) plane. (c) Graph projected on (c, x) plane.

13.5 The Isochron Coloring Method

The graph method, for even slightly more complicated CDs, will require too many dimensions for direct visualization. Thus, we seek algorithms for pulling back the essential information from the graph to some kind of a map in the physical space. In our current example taken from Crutchfield and Kaneko [3], this will be a division of the physical space into interval regions. These could be colored for graphical presentation. The algorithm we are using currently in our research with 2D CDs is based on the projection of global states into the response diagram of the standard cell, as shown in Fig. 13.6c. It may be explained as follows.

13.5.1 Isochrons of a Periodic Attractor

Note that in the response plane, Fig. 13.5, the dashed curves (loci of the repellors, which are virtual separators) separate the plane into four strips. Each contains one of the four solid curves of the locus of attraction. These represent its phases: *A*, *B*, *C*, and *D*. Given an initial state of the biased logistic scheme in one of these phases, say *A*, the eventual state after many iterations (a multiple of four) will be phase *A*. These strips, which are basins of attraction of the four-fold iteration of the logistic map, are called *isochrons* of the response diagram. Only periodic attractors have isochrons, which constitute a decomposition of the basin of attraction of the periodic attractor into disjoint pieces. Periodic trajectories of continuous dynamical systems have isochrons also, but we will not discuss these here.

13.5.2 Coloring Strategies

A simple coloring strategy is used in Zeeman's method, Fig. 13.3. There, the lower sheet of the locus of attraction is colored black, the upper sheet, white. Note that nodes of the physical space, which are far from their attractors, are not colored by this rule. Thus, we would like to extend the rule outward from the loci of attraction, into the entire 3D response diagram of the standard cell. For control points inside the bifurcation set, the cusp curve in the horizontal control plane, this is easy, We use the locus of separation (the intermediate sheet of the cusp surface) to divide the wedge-shaped column into white (upper) and black (lower) regions. For points outside the bifurcation set, however, there is only one attractor, or sheet of the locus of attraction. We may use gray for this entire region, which is neither black nor white.

In the case of the logistic/diffusion lattice, we may color the four isochrons in different colors. Additional information may be encoded by varying the hue or saturation of a color according to the distance of the point from the attractor. This method is easily extended to the case of a CD with 2D physical space. We color the isochrons of the response diagram of the standard cell. Then, we map the physical space into the standard response diagram by the graph method, projected, and imagine the physical space moving around in the colored diagram as the simulation progresses. At each step, the colors encountered by the moving image are pulled back to the physical space, creating an animated color movie within it, *the isochron movie*. This is a substantially different movie than the one usually seen, in which the color indicates the values of the state variable, x, which we call *the state movie*. Another useful visualization is the *control movie*, in which the control space is colored, and these colors are pulled back to the physical space. A useful still picture view (for periodic global attractors) is the *period view*, in which each point of the physical lattice is colored according to the period of that node. This may be obtained from the isochron movie, and is analogous to the Fourier transform of a real function.

13.6 Conclusions

Here we have described three methods for obtaining animated color movies from a cellular dynamaton simulation; the familiar state movie method, and two new ones: the isochron and control movie methods. Suitable for periodic states primarily, the extension to some chaotic attractors having approximate isochrons, such as the Rössler attractor, may prove useful. Other color movie methods, based on entropy, Lyapunov exponents, spectral shape, symbolic dynamics, and so on, may be more appropriate for chaotic CDs [13]. The research results on experimental dynamics of physically 2D CDs will require video publications, such as the new aperiodical from Aerial Press, *Dynamics Showcase: a Hypermedia Journal.*

References

[1] R.H. Abraham, in *Mathematical Modelling in Science and Technology*, edited by X.J.R. Avula, R.E. Kalman, A.I. Leapis, and E.Y. Rodin, (Pergamon, Oxford, 1984).

[2] R.H. Abraham, in *Mathematics and Computers in Biomedical Applications, Proceedings IMACS World Congress, Oslo, 1985*, edited by J. Eisenfeld and C. DeLisi (North-Holland, Amsterdam, 1986).

[3] J.P. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin, (World Scientific, Singapore, 1987).

[4] R.H. Abraham, J.B. Corliss, and J.E. Dorband, Int. J. Chaos and Bifurcations **1**, 227 (1991).

[5] R.A. Fisher, *The Genetical Theory of Natural Selection*, (Dover, New York, 1930/1958).

[6] N. Rashevsky, *Mathematical Biophysics* (University of Chicago, Chicago, 1938).

[7] E.C. Zeeman, in *Catastrophe Theory*, edited by E.C. Zeeman (Addison-Wesley, Reading, 1977).

[8] E.C. Zeeman, in *Catastrophe Theory*, edited by E.C. Zeeman (Addison-Wesley, Reading, 1977).

[9] E.C. Zeeman, *Catastrophe Theory* (Addison-Wesley, Reading, 1977).

[10] R.H. Abraham and C.D. Shaw, *Dynamics, the Geometry of Behavior* four vols. (Aerial, Santa Cruz, 1982–1988).

[11] R.L. Devaney, *An Introduction to Chaotic Dynamic Systems*, 2nd ed. (Addison-Wesley, Reading, 1989).

[12] R.H. Abraham and A. Garfinkel, in *Ultradian Rhythms in Life Processes*, edited by D. Lloyd and E.L. Rossé (Springer-Verlag, New York, 1992).

[13] K. Kaneko, Physica D 37, 60 (1989).