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Basic Principles of Dynamical Systems

Frederick D. Abraham, Ralph H. Abraham, and Christopher D. Shaw

Introduction

There is a revolutionary new strategy of mathematical modeling of systems called dynamical systems theory. Although its roots reach back to Newton, Rayleigh, and Poincaré, the past two decades have witnessed a revolution in its language, concepts, and techniques for dealing with complex cooperative systems evolving through multiple modes of dynamical equilibrium (static, oscillatory, and chaotic). Hence, their applicability to biological and behavioral domains that have increasingly in the same period come to be understood in a similar light of synergistic cooperative systems. The mathematics provides models, simulation, cognitive strategies, and intuitively clear geometric representations for complex systems. It also serves as a unified philosophic view for understanding integrative, hierarchically organized, dissipative, irreversible, and evolutionary dynamical systems. In short, it is a world view as well as an elegantly simple modeling strategy.

The principle purpose of this chapter will be to present some of the

Frederick D. Abraham • Department of Psychology, University of Vermont, Burlington, Vermont 05405. **Ralph H. Abraham and Christopher D. Shaw** • Department of Mathematics, University of California, Santa Cruz, California 95064.

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basic concepts of mathematical dynamics, emphasizing the visual geometric approach of Abraham and Shaw (1982–1988) and is taken from Abraham, Abraham, and Shaw (1990). It progresses from elementary static and periodic attractors, through coupled periodic attractors and chaotic attractors, then to the concepts of stability and bifurcation, and finally, briefly, to complex dynamics. Each will be illustrated by classical systems. A secondary purpose will be to illustrate some applications of dynamical theory to psychology, proceeding from the more metaphorical, to simple, and then more complex dynamical models; from more contrived application of classic models to the development of more original models. The chapter is designed so that the reader may opt to skip to a section on psychological applications after an introduction to the relevant basic mathematical concepts. Some of these correspondences are as follows:

Basic Concepts	*Psychological Applications*
Interacting biological populations	Contingent operant behaviors
Damped oscillators	Alternative psychological states
Periodically driven self-sustaining oscillators	Coupled circadian oscillators
Sections, bifurcations, and complex systems	Consciousness
Bifurcations	Circadian and neuropsychobiological

There will be reminders when such jumps are appropriate and warnings if the jump may eventually result in excursions into some unfamiliar material, at which point one returns to the jumping-off point.

The chapter is intended both for those considering the use of this approach for their own research and those interested in becoming more casually fluent in dynamical theory. The emphasis on visual representation makes the technical ideas intuitively accessible and useable. Calculus is not assumed but will be helpful to those wishing to go deeper into the subject. The visual approach can be obtained more completely from Abraham, Abraham, and Shaw (1990) and Abraham and Shaw (1982–1988), and the mathematical–symbolic approach from works cited there. Equations in the "Basic Concepts" section, except for the very simplest and most obvious for the interacting biological population model, were deferred to Appendix B for the mathematically curious. Definitions of basic terms from the text were collect in a Glossary (Appendix A). We feel dynamical theory is not only important but also a joy. We urge readers to proceed only as long as they share this feeling.

Basic Concepts

Basic concepts considered here include not only concepts of state spaces, trajectories, phase portraits, attractors, characteristic exponents, information gain, stability, and many other features of static, oscillatory, and chaotic dynamical systems but also concepts relating to organizational changes in these systems, called bifurcations. Despite their esoteric sound, they provide some interesting perspectives into a variety of psychological systems.

Definitions of Elementary Terms

These should illustrate the ease of becoming familiar with the central concepts of this approach, and should, at the same time, hint at its potential richness.

State Spaces

A *system*, loosely speaking, is a set of interacting factors. Most important concepts in psychology, such as consciousness, learning, perception, maturity, sensation, meditation, communication, dopaminergic neurotransmission, sex, feeding, nigrostriatal system, sleep, hypothalamic–pituitary–gonadal hormone systems, attitude, and mental health are systems that we try to model. The process of modeling such systems is familiar enough to us. Some aspects are real (observable variables), whereas others may be imaginary (hypothetical or intervening variables), and we try to discover as many of these as possible and characterize the relationships between them (MacCorquodale & Meehl, 1948). A *system*, then, is a set of such variables whose values change over time.

Let's take the concept of maturity, a hypothetical personality construct, as an example of a system. Maturity is clearly a complex system composed of many variables. Assertiveness and the ability to plan for the future might be considered to be two of but many variables comprising the system of maturity. These may be hypothetical, but ultimately a variety of attempts are made to operationalize them, that is, measure observables behaviors assumed to represent them. For now, let's restrict our system of maturity to these two variables, assertiveness and planning for the future. For any given individual each of these variables change over time. The number of requests or demands made on others by an individual and the number of submissions by that individual to the requests or demands of others may vary from hour to hour, from day to day, week to week, year to year, and from one stage of life to another. The same is true for the amount of time spent planning for the future or the length of future being planned. Each of these variables may be described as a *time series*, a conventional representation (Figure 1). You may note that at any given point of time, we can characterize a set of measurements, one on each variable, which can be considered a vector. For example, for the individual's data shown in Figure 1, at 16 years of age, the vector consists of A = 61 on the assumed assertiveness variable measured on an interval scale of 0 to 100, and P = 8 on the assumed planning variable also measured on an interval scale of 0 to 100. Each such vector represents a *state* of the system. A *state* of a system is thus the vector of values, one for each of the variables of the system at a given moment. The state, or vector, is represented as a point in the graph. Table 1 summarizes the vectors, the observed states, the points from Figure 1.

An alternative way of graphing these values would be by using a *state space*. This would be a graph where the two axes would be the

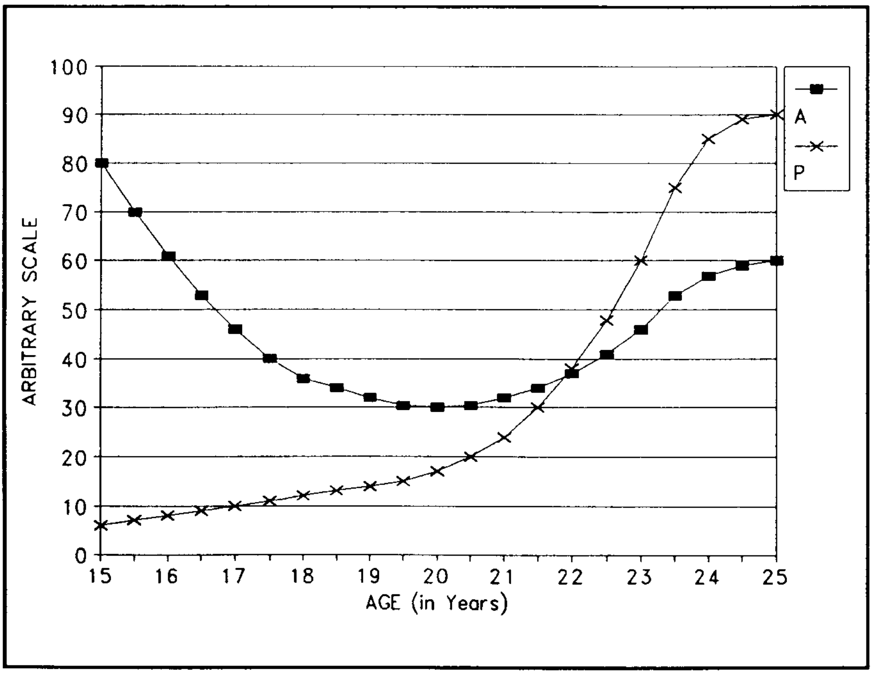


Figure 1. Maturity system: time series of assertiveness (A) and planning ability (P) as a function of age (from Abraham, Abraham, & Shaw, 1990, © Aerial).

Table 1. Vectors of Values for Assertiveness and Planning Ability for an Individual from 15 to 25 Years of Age (Taken from Figure 1)

Age	Assertiveness	Planning ability
15.0	80	6
15.5	70	7
16.0	61	8
16.5	53	9
17.0	46	10
17.5	40	11
18.0	36	12
18.5	34	13
19.0	39	14
19.5	31	16
20.0	30	18
20.5	31	20
21.0	33	24
21.5	35	35
22.0	38	38
22.5	41	48
23.0	47	60
23.5	53	75
24.0	57	85
24.5	59	88
25.0	60	90

state variables (the variables are also often referred to as system or state variables as well as simply variables), assertiveness and planning ability. The *state space* would be all the possible, states, points, or pairs of values (vectors) of assertiveness and planning ability. The state space showing the same points as in Figure 1 and Table 1 is shown in Figure 2. In general, such a plane displaying a dynamical system may not necessarily be filled (all pairs of values on the two variables might not be possible). Such geometric models are not restricted to flat cartesian coordinate systems but can be generalized to include curved spaces (called manifolds, Figures 6b and 7b).

Trajectories and Phase Portraits

Time and change in a system may be tracked in two different ways. The conventional form is the *time series* produced just shown (Figure 1). The other is to omit time as a graphic axis and use the phase space representation, keeping track of time by some labeling convention. For

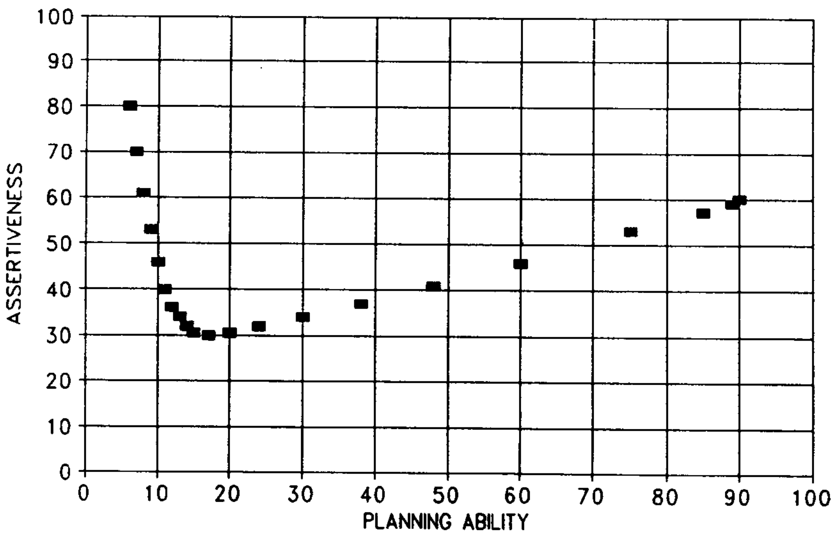


Figure 2. Maturity system: scatter plot of A vs. P for data of Table 1 and Figure 1 (from Abraham, Abraham, & Shaw, 1990, © Aerial).

example, we could connect the points of Figure 2 in the order of their occurrence, and the dots would represent not only the values in the state space, but the distance between them would represent the equal intervals of time between the observations (0.5 year) of the values of the state variables (Figure 3). Such a curve connecting temporally successive points in the state space is called a *trajectory*.

Time labeling of a trajectory may be performed with points, as here, or tick marks, color coding, or other representations, or omitted if rate of change is not considered of prime importance for the graph. Arrows indicating the direction of time are usually included. Notice that the equally temporally spaced points on the trajectory are further apart when one or both of the variables is changing rapidly as at the earlier and late middle parts of the trajectory, and closer together when both variables are changing slowly as at the early middle and very end of the trajectory.

The state space, filled with trajectories (only a few representative ones are usually drawn), is called the *phase portrait* of the system. A

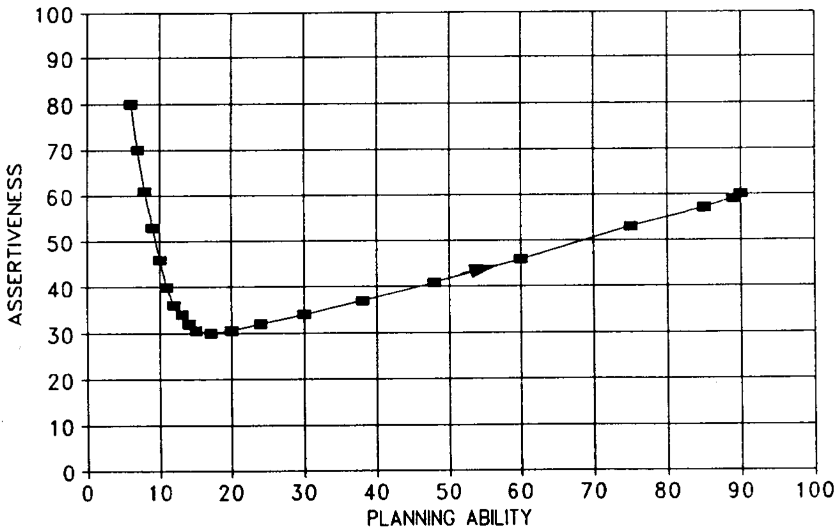


Figure 3. Maturity system: trajectory interpolated onto the scatter plot of Figure 2 with arrow showing time flow. (from Abraham, Abraham, & Shaw, 1990, © Aerial).

phase portrait for our system of maturity would be filled with trajectories representing different individuals from some population (Figure 4).

Vectorfields and Dynamical Systems

Dynamical systems are systems with special qualities. To describe these properties, it is necessary to introduce some special kinds of vectors. If we take any two points (vectors) on a trajectory, the difference between them is a new *bound vector*. Figure 5a shows such a bound vector for the trajectory from Figure 3 between the points representing the ages 16 and 22.5 with the arrow showing the direction of time. This bound vector has two values. One for the difference between the assertiveness values, $-20 = 41 - 61$, and one for the difference in planning, $40 = 48 - 8$ (values can be verified from Table 1 as well) showing a loss of 20 in assertiveness and gain of 40 in planning ability over this 6.5 year span for this individual. As mentioned before, the rate of change in each variable is reflected in the distance between

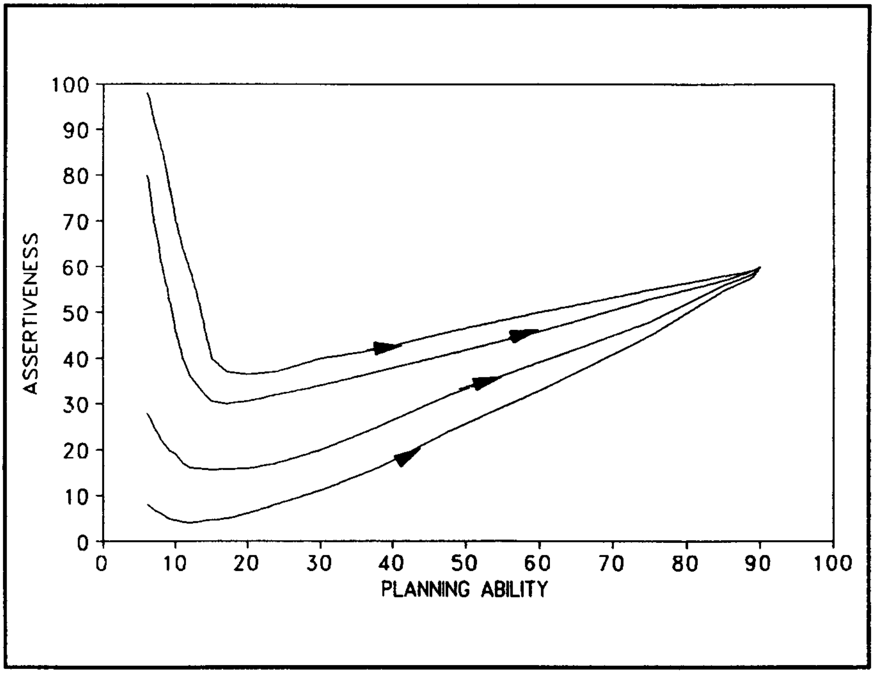


Figure 4. Maturity system: phase portrait with the trajectory of Figure 3 shown along with three of many others (not shown to keep the figure from turning black) (from Abraham, Abraham, & Shaw, 1990, © Aerial).

points. This rate, or *average velocity* of the change of state may be represented by a *velocity vector*, which is the bound vector divided by the interval of time to yield the rate of change in each variable per unit time. For this bound vector, that would be −20 assertiveness points/6.5 years or −3.08 points per year, and 40 planning points/6.5 years or 6.15 planning points per year. So now we have a new *average velocity* vector representing rate of change of the state variables, assertiveness and planning, which can be represented numerically, as just calculated, or visually in phase space (see Figure 5b). The *average velocity vector* then represents the average rate and direction of the change in the state of the system between two points in time. Now suppose we make our measurements continually in time or decide our trajectory may be represented by a continuous model. Further, suppose we allow the time for the second point in time to get closer and closer to our first point in time. As this time shrinks, the average velocity vector becomes a *tangent vector* at the first point in time (Figure 5c). It is also called the *instantaneous velocity* of the trajectory at that point in time, represent-

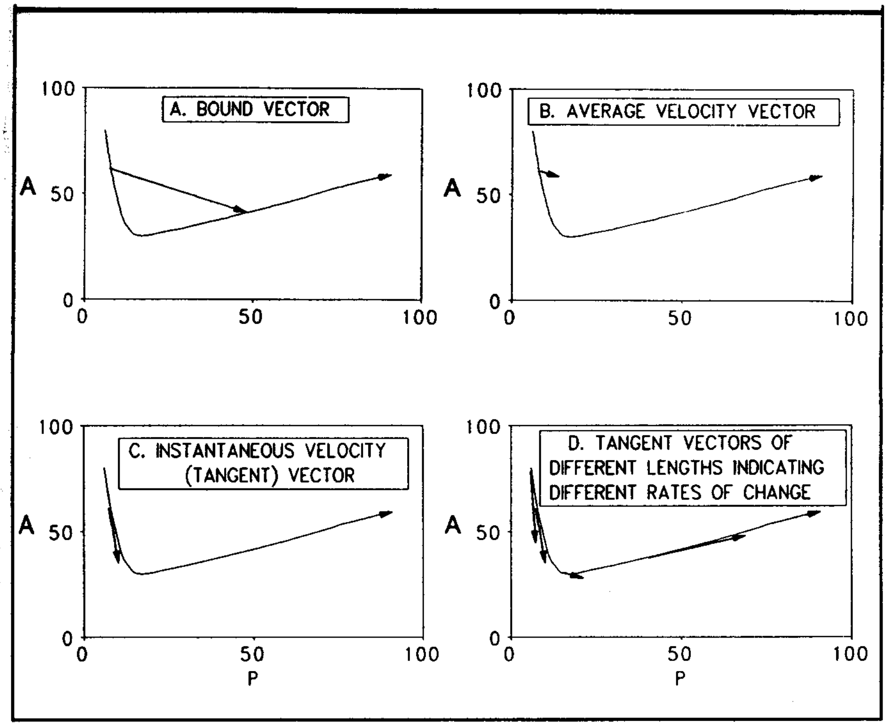


Figure 5 a–d. Maturity system: vectors (from Abraham, Abraham, & Shaw, 1990 © Aerial).

ing the instantaneous rate and direction of change in the state of the system at that point in time.

What does this instantaneous velocity vector tell us? Why have we derived it from the trajectory under the pretense of having a continuously instead of a discretely changing trajectory? What is the information contained in the instantaneous velocity vector? Simply this. It tells us the tendency of the system to change when in that state. It says in what direction and how fast the system should change on all variables simultaneously; how much assertiveness and planning ability are going to change. It is the thing that generates the trajectory. It moves the system to the next point on the trajectory where the next vector governs its next move. This process of deriving this instantaneous velocity vector is known as *differentiation* in vector calculus (the use of calculus will not be employed here; only the visual or geometric interpretations of dynamical systems are used that are hopefully intuitively clear without a knowledge of calculus). Several are shown on the same trajectory again (see Figure 5d) to emphasize that faster rates of change appear as longer vectors, slower rates of change as shorter vectors, just as distance between equally spaced time points on the trajectory did in our discussion of trajectories (see Figure 3) and bound vectors. Another example for a different, unspecified system shows a trajectory slowing (with shorter vectors) as it evolves and approaches its asymptote (Figure 6a). Asymptote here means a fixed state from which the system does not change; here the velocity vector is zero. Another example of a trajectory and some of its instantaneous velocity vectors for an unspecified system with a curved state space shows the tangent vectors projecting off the surface while the trajectory remains on it, with a roughly constant rate of change (see Figure 6b).

The state space is filled with these instantaneous velocity vectors, one at every point on every trajectory (assuming smooth trajectories). A *vectorfield* is the collection of all these instantaneous velocity vectors. Technically, a *dynamical system* is equivalent to this vectorfield. The vectorfield summarizes all the possible changes that can occur in the system. If you know the present state of the system, you know how it will change next. If you have an individual in a state of A = 40 and P = 11 you know that these velocity vectors will soon move the individual to A = 36 and P = 12 (Table 1, Figure 3). The term *dynamical system* need not be restricted to allude to the instantaneous velocity vectorfield but may be used more loosely to refer to a system that might be represented by such a vectorfield or to rules that represent sufficient information to produce a vectorfield. These rules usually take the form of descriptions of the rate of change of the variables; more specifically, usually equations. For example, such rules might be that assertiveness and planning always increase 5 points a year, or that assertiveness increases 3 points a year if assertiveness exceeds 20 and planning exceeds 35 points a year and is increasing (these are not the rules that generated the maturity system shown in these figures). A few of the instantaneous velocity vectors for our maturity system are shown in Figure 7 for points on the trajectories of Figure 4. Other examples of vectorfields are shown for unspecified systems using a flat (Figure 6a) and a curved manifold (Figure 6 and 7b's arrows indicate the direction of change in time, but tic marks on the curve representing the rate of change have not been shown).

The *phase portrait* of a dynamical system is the state space filled with trajectories. Drawing a few of these trajectories usually gives a good idea of the portrait and often the tick marks representing equal intervals of time, as in Figure 3, are considered adequate to represent rate of change in the state of the system, without resorting to the separate calculation or presentation of the vectorfield. Just as the in-

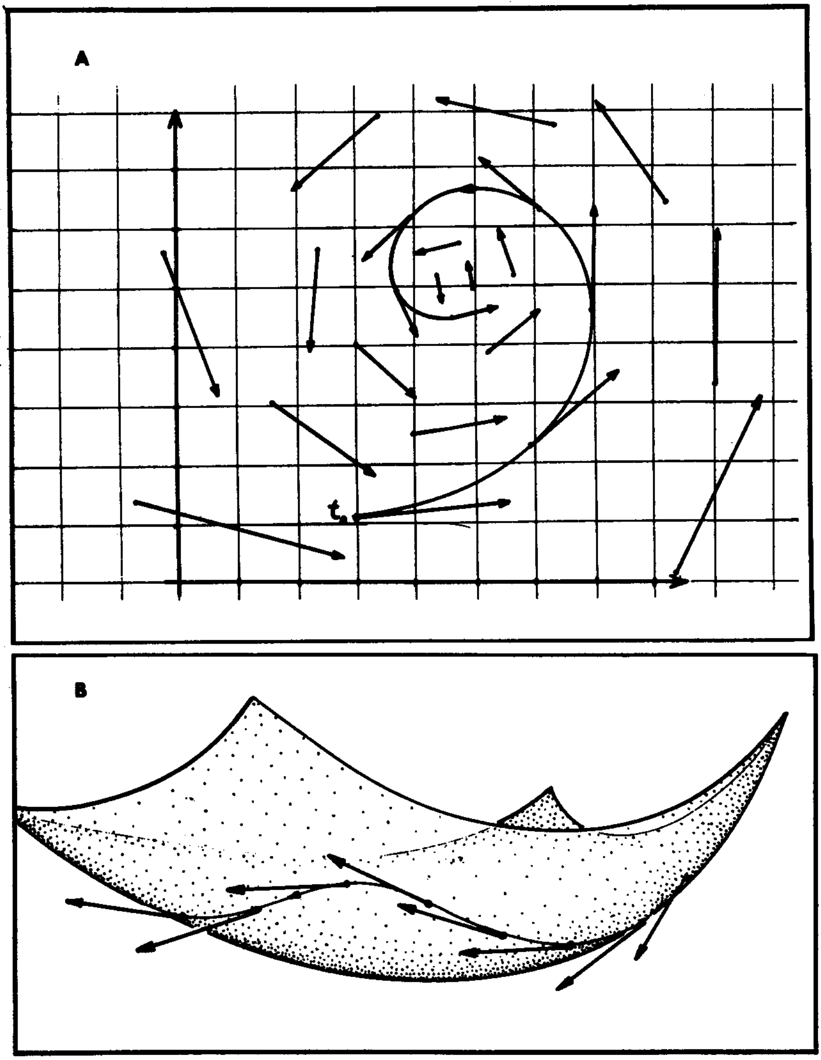
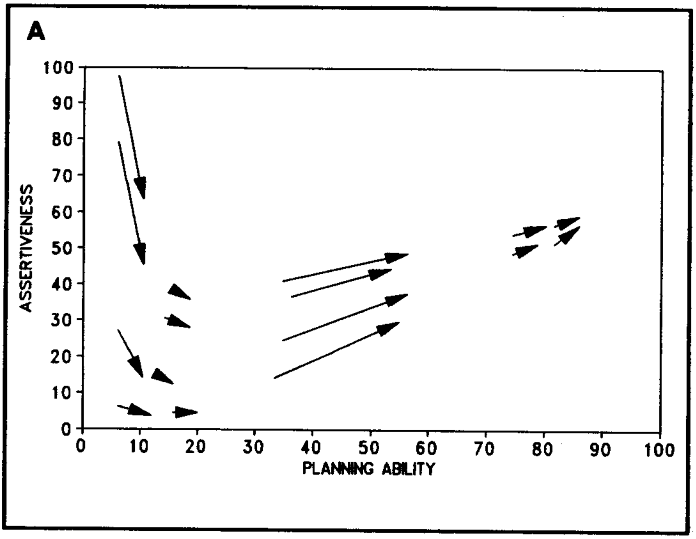


Figure 6. (a) Trajectory generated by tangent vectors at every point along it; (b) trajectory on a curved state space (manifold) with tangent vectors projecting off the surface (from Abraham & Shaw, 1982–88, © Aerial).

stantaneous velocity vectors may be derived from taking limits to differences on trajectories, so trajectories may be constructed by following a path of succeeding velocity vectors. Technically, this involves taking the limit of a summing process on the instantaneous velocity vectors, a process called *integration* in calculus. Again, we will not bother with



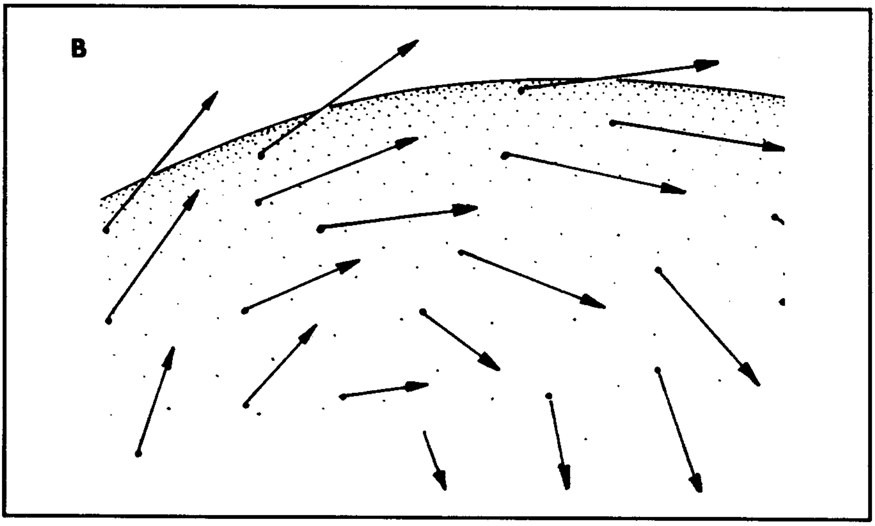


Figure 7. (a) Maturity system vector field: some instantaneous velocity vectors for trajectories of Fig. 4 (from Abraham, Abraham, & Shaw, 1990, © Aerial); (b) vectorfield: unspecified system assigns a tangent vector to every point of the manifold (from Abraham & Shaw, 1982–88, © Aerial).

the details of calculus; this point is only mentioned to emphasize that because there are mathematical rules for obtaining phrase portraits and vectorfields from each other, they contain essentially the same information. Each gives a different perspective on this information. One, the trajectory, gives the history of change of the system over time; the other, the vectorfield, gives the rules for the tendancy of change for each state in the system. Our use of the visual geometric approach is generally to emphasize the accessibility of that information, that is, its usefulness even to working scientists and psychologists without their having to resort to the less accessible mathematical tools of vector calculus. The integral/differential pair, the phase portrait and the vectorfield, containing all the information of the model system, may thus also be called a *dynamical system*. The vectorfield may be represented mathematically, as just mentioned, by a set of equations (in calculus, called first order, ordinary, autonomous, differential equations), which may also be referred to as the dynamical system. For a practicing scientist, presentations of real experimental data can directly produce an approximation to a phase portrait (Figures 1–4). To summarize, a dynamical system is a system that changes in time and may be characterized by phase portraits (a collection of possible trajectories) and vectorfields (a collection of instantaneous velocity vectors), which can be derived from each other and contain essentially the same information. The information may be the same as that conveyed in conventional time series graphs and in sets of equations, but the value of this approach, is to, hopefully, provide a quicker, visual, grasp of system change. Let's next take a look at some of those visual properties of phase portraits.

Attractors, Basins, Separatrices, Repellors, and Saddles

Experimental trajectories may present patterns if the systems have any regularity to their behavior. The task of the scientist is to discover those patterns; that of the modeler to discover reasonable models approximating the same patterns. A simple taxonomy of some common patterns can be given. Supposing our model of the maturity system assumed or found that the vector (A = 60, P = 90) (the most right-hand point in Figures 1–5; the last vector of Table 1) worked so well that any individual reaching that state remained in it. Some individuals might start at age 15 at these values and so would remain at this fixed state. Their trajectories would be a single point with an instantaneous velocity vector equal to zero (i.e., there is no tendency to change from that state). Such a point with a zero instantaneous velocity vector is a special kind of trajectory called a *constant, critical point, fixed point*, or *rest point*.

On the other hand, our model might assume fluctuations back and forth between different values. When you get too pushy, society pushes back; too shy and you get pushed around too much; too much planning yields too little fun; too little planning yields an empty wallet or something like that. Such individuals might cycle repetitively through the same set of states (Figure 8a). A nonconstant trajectory that closes upon itself is called a *closed trajectory*, or *closed orbit*. If the cycle time is constant and its instantaneous velocity vectors at each point remain identical each time through the cycle, then it may also be called a *periodic trajectory, cycle*, or *oscillation*. Maybe assertiveness and planning are up at the beginning of each month, low at midmonth, and pick up again toward the end of the month again as the paycheck runs out; a monthly cycle. New Year's resolutions suggest annual cycles.

Our maturity model might not be so exact as to create well-defined rest points or cycles. A trajectory might inexactly fluctuate around some constrained values for high-school years, around another set of states for some college years, and then near some point during early professional careers. Such trajectories that are neither fixed nor cyclic but that fill up a constrained region of the state space are called stochastic, strange, or *chaotic*.

Of course, most individuals will not start in these end trajectories of a rest point, periodic trajectory, or chaotic region. They may start in some distant state and gradually approach one of these trajectories as a limit (Figure 4 shows many individuals ending up at the same state). If many trajectories in the state space approach the special trajectories as asymptotic *limit sets*, then those special trajectories are called *limit points, limit cycles*, or *chaotic limit sets*. Such limit sets to which *all* nearby trajectories tend are called *attractors, fixed* or *static, periodic*, or *chaotic* (respectively). The end point of the trajectories shown in Figure

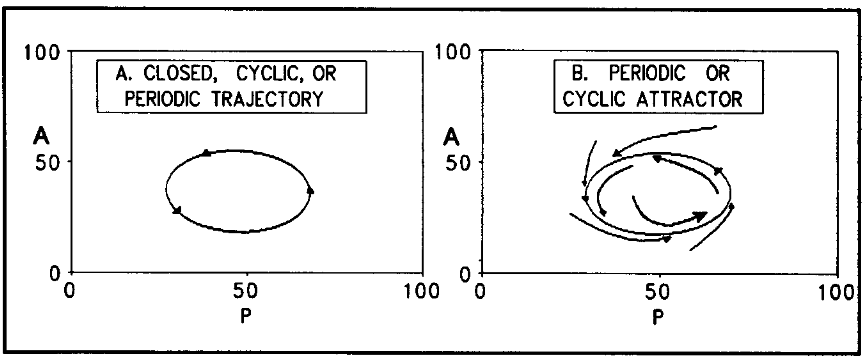


Figure 8. Cyclic trajectory and attractor (from Abraham, Abraham, & Shaw, 1990, © Aerial).

4 is a limit point. Trajectories approaching a limit cycle are also shown (see Figure 8b).

Now let's invent a new pretend experiment also based on the maturity system. Suppose we ask professionals in mental health to conceive an ideal state of adult maturity and then to draw trajectories representing how different individuals might develop as they approach that state as a limit point (static attractor). This imaginary experiment is very contrived and unlikely, invented for the purpose of illustrating a few properties of phase portraits. We repeat the experiment three times, the first as stated, with an ideal concept of maturity without specifying the gender of the individuals. The second, specifying an ideal concept of maturity for males, and the third, for females. So these trajectories are the result of informal models of the maturity system carried in the heads of the professionals. Suppose we got a complex phase portrait with trajectories approaching or departing a few different points (Figure 9). Here trajectories starting at high values of assertiveness and planning ability end up at a limit point high on both that we will say represents the professional's concept of male maturity. Those starting at low values on both variables might end up at a limit point lower on both variables representing the professional's concept of female maturity. Those trajectories representing individuals starting at values high on one variable, low on the other, might tend to the intermediate limit point, from which they tend to one of the two states representing maturity, male or female. A region of the state space containing all trajectories that tend to a given attractor comprise its *basin*. Here, there is one basis for the attractor representing the male state of maturity, one for the female (Figure 9). A *separatrix* consists of points and trajectories that are not in any basin, that is, do not tend toward any attractor. (Here they are all those points between the two basins, and they lie on the trajectories tending to the third, intermediate limit point, the shaded area; individuals arriving at this intermediate state may linger a while, but then internal and social pressures send them one way or the other.) A separatrix may form boundaries between basins (*actual separatrix* as shown here), or they may lie within a single basin (*virtual separatrix*, none shown). Points and periodic trajectories from which trajectories only leave are *repellors* (none shown here). Limit sets that some trajectories approach and others depart are *saddles* (the middle point of Figure 9). The arriving trajectories make up the *inset* of the saddle (here they are also the actual separatrices between the basins), whereas those departing comprise its *outset*. Saddles may be points (as shown here), cycles, or chaotic sets (examples to be shown later).

Now, where do you suppose the trajectories might be for the professional's models for maturity when sex is left unspecified? Suppose they

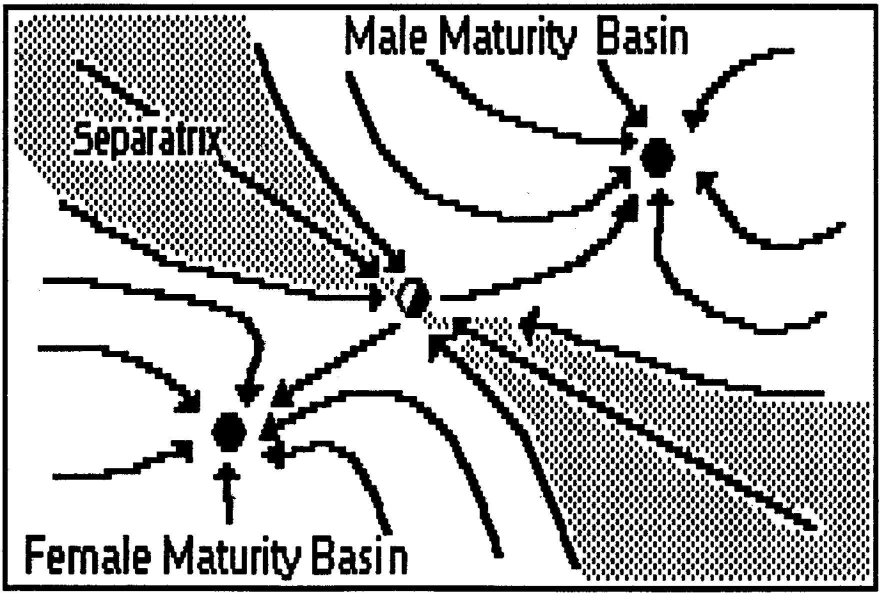


Figure 9. Mental health professionals, model of maturity system showing professional prejudice (from Abraham, Abraham, & Shaw, 1990, © Aerial).

were identical to those for their male model as shown. Would such a result represent their placing a higher value on male maturity than female? Would that represent an incorporation of societal prejudices by the mental health professions? Would that, in turn, have a feedback effect not only on their therapeutic practice, but on their influence on society, reinforcing its prejudices? This experiment has not been performed, and these results are very contrived, and such simple separation of trajectories and limit sets is highly unlikely for a maturity system. A similar experiment on just the limit points has been done and has revealed such prejudicial attitudes among professional's that underscores the importance that the biases of psychologists play not only in influencing the progress of science (the difficulty of creating truly objective experimental designs), and clinical practice, but in influencing social and legal systems as well (Broverman *et al.*, 1970).

Classic Examples

Compared to Freud, Jung "was more organic, expansive, and unfolding to purposive ends"; in short, dynamic, according to Hampden-Turner (1981), who continues:

> The psyche's structure is animated by the energy of the *libido*, a life force which in high intensities energizes will, affect and performance and in lower intensities energizes attitudes, interests and possibilities. Energy shifts ceaselessly to and fro along the structural axes in dialectical patterns, governed by the principle of *enantiodromia*, literally "the returning swing of the pendulum." The ego can intensify the energy flow towards one value polarity, but consciously or unconsciously it must flow back—tension–relaxation, openess–closure, evaluation–decision. In dynamic balance the psyche moves progressively (external adaptations) and regressively (internal adaptation). (p. 47)

The pendulum, historically, is one of the first systems to which dynamics was applied because it was clear enough to allow Galileo and Newton to discover the general principals of analyzing time and motion. Other historical, classic developments in dynamics also include the analyses of the gradient system by Newton, the buckling column by Euler, musical instruments by Rayleigh, electronic oscillators by Van der Pol, celestial mechanics by Poincaré, biological populations by Lotka and Volterra, and many others. We turn to some of then now, not for their historical significance, which is great, but because they continue to provide the mathematical foundations of the subject as well as to provide exemplary models for contemporary science.

Oscillations and the Pendulum

Galileo not only rolled polished brass balls down parchment-lined wooden grooves, but he swung a nasty pendulum. Fortunately for us, Newton considered a simple pendulum (Figure 10a). He showed that the pendulum is a system that could be described by two state variables. One is its position, measured by the angle. The other is its velocity, the rate of change in the position, measured in radians or degrees per unit time. The state space can thus be represented graphically by two axes, one for position, the other for velocity (Figure 10b). The state of the pendulum at any given moment has a position and a velocity that is the state of the system at that moment and is represented by a point in the state space. Its motion over a short period of time can be represented as a sequence of these states, which comprise a trajectory in the state phase space (Figure 11).

Imagine the pendulum is ideal, frictionless, continually oscillating back and forth between two extreme states. By ideal, we mean simply that it moves through the exact same sequence of states on each cycle. In other words, its trajectory repeats itself exactly on each successive cycle. This is an undamped oscillation. If we start the pendulum swinging with different forces and in different directions, each unique startup generates a different cyclic trajectory. The phase portrait contains all possible trajectories, but a practical sketch shows only a few of these

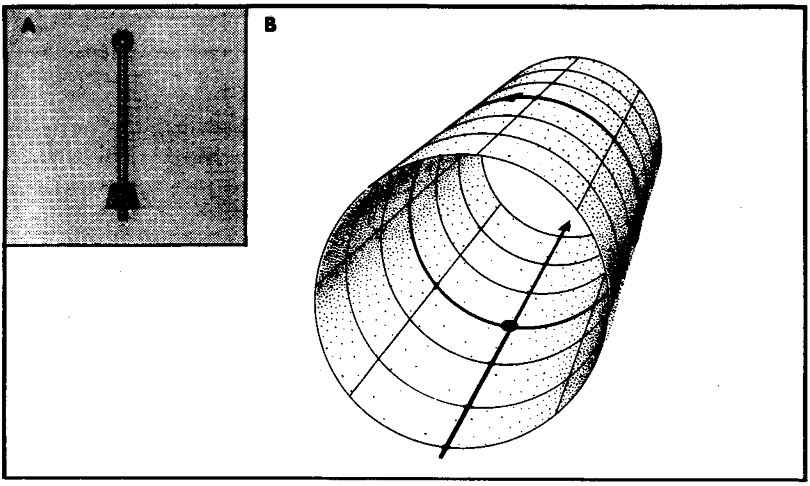


Figure 10. (a) Simple pendulum, (b) state space showing the rest point for the pendulum at rest. Circles represent angular position, straight lines represent velocity (from Abraham & Shaw, 1982–88, © Aerial).

(see Figure 11). If it is at rest, it stays that way; a critical point in the center of the portrait. Increasingly greater initial displacements and velocities are revealed as larger closed orbits (trajectories) on the portrait. As the initial force is made sufficiently great, the pendulum could swing to exactly the top position possible, the top of a circle of its motion above the pivot. It then could fall in either direction. The state representing the top of the motion is represented by a saddle point on the phase portrait; here that same point is shown twice emphasizing it can be approached from the right or the left depending on the initial condition. The portrait continues in both dimensions beyond the figure boundaries; just the central portion is shown. Trajectories both approach and depart from the saddles. The upper trajectory between the two saddles shows the pendulum approaching the top from the left. From there, the lower trajectory shows it falling back to the left. The two trajectories to the right of the right-hand saddle indicate the pendulum approaching from the right (lower of the two trajectories), and falling to the right (upper trajectory). Above and below those trajectories between the saddles are trajectories representing circular motions of the pendulum due to high initial velocities. Their velocities remain positive if the motion is always counterclockwise and negative if it is clockwise, if you choose that convention. The angle is always increasing or decreasing.

If the pendulum is real rather than ideal; there is friction in the system, and it eventually comes to rest in the center. This constitutes a damped oscillation. The angle and velocity decline. Or, more precisely,

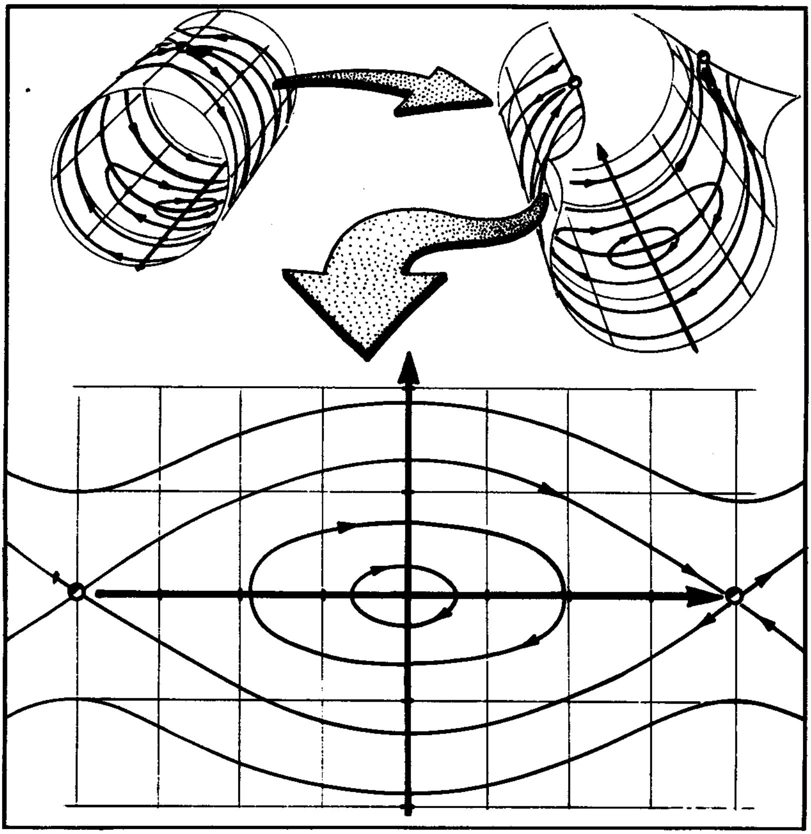


Figure 11. Simple conservative pendulum: phase portrait unrolled to show a few of the trajectories (from Abraham & Shaw, 1982–88, © Aerial).

the phase portrait reveals that the trajectories spiral into a focal point attractor (Figure 12).

We relabel the inner orbits of the ideal pendulum to depict some aspects of Jung's concept of *enantiodromia* (Figure 13). Many other psychological concepts could be modeled by the pendulum, but for many applications, we also find the buckling column suitable.

Damped Oscillations, Bifurcations, and Buckling Columns

Damped oscillations, like oscillations, are ubiquitous in psychological systems. Damped oscillatory effects of messages on attitude are used in the attitude models of Chapter 12. Damped oscillations were studied exhaustively by Euler using the buckling column (1778; Stoker, 1950).

If an elastic column has a weight placed on it, and then it is slightly

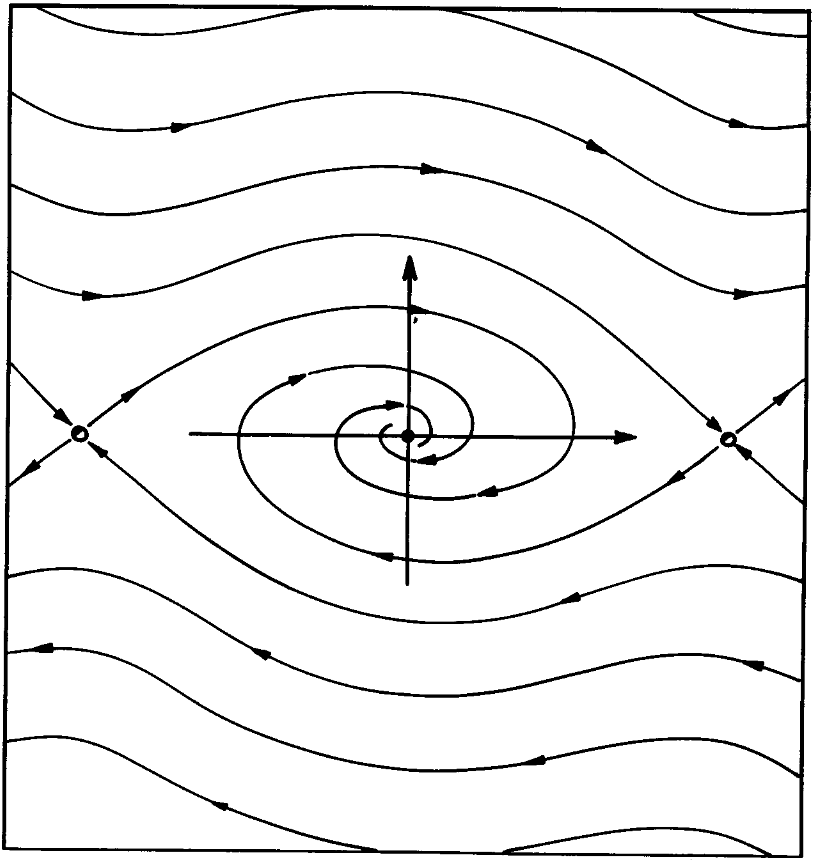


Figure 12. Dissipative pendulum: Phase portrait showing trajectories spiraling to a focal point attractor (from Abraham & Shaw, 1982–88, © Aerial).

displaced at its midheight, it will oscillate or buckle or both, depending on the magnitude of the weight and strength of the column (Figure 14). If the weight is below a critical value and the system is free of friction, then it acts similar to a frictionless pendulum in swaying undamped forever back and forth across the center of its normal upright position. If there is friction, then it will come to rest in the upright position, just as the pendulum with friction comes to rest. If the weight is greater and if the system is again frictionless, its swaying may or not be confined to one or the other side of vertical depending on the initial displacement. With friction, its swaying eventually becomes confined to one side, and then it comes to rest bent to that side. Thus the term *buckling*. If the weight is even greater, then the column may buckle immediately to one side without oscillating or even being pushed.

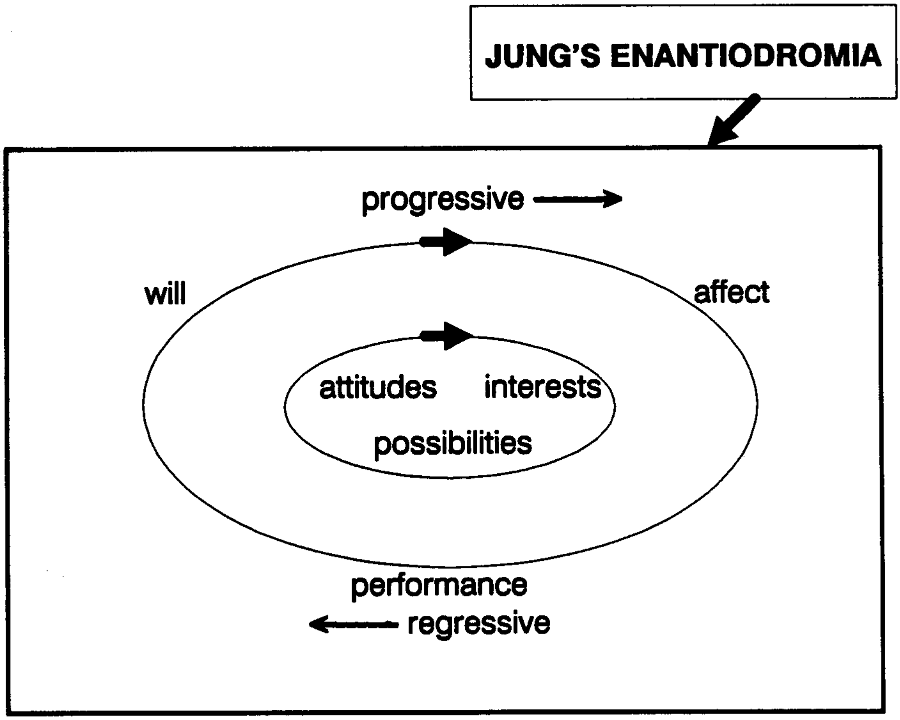


Figure 13. Pendulum model of Enantiodromia (from Abraham, Abraham, & Shaw, 1990, © Aerial).

The phase portrait is obtained by depicting velocity, v, as a function of the horizontal displacement, x, of any arbitrary point on the column, say its midpoint (Figure 15). In the frictionless system (Figure 15a), the trajectories, like the sustained oscillations they represent, continue forever. If the weight is sufficiently small, every oscillation crosses the center back and forth between two extremes. Each extreme is equidistant from the center, and the motion pauses at zero velocity momentarily at each extreme before changing direction for the return motion. With the addition of friction, this system with the light weight loses amplitude with each cycle, eventually coming to rest in the center, thus exhibiting damped oscillations as with the pendulum with friction (Figure 15b). With or without friction, each trajectory uniquely depends on the initial position and velocity imparted to the column.

What happens if the weight is increased past a critical value? Again, it depends upon the initial displacement and velocity imposed upon the column. If the initial velocity is small enough, the oscillations are confined to one side (Figure 15c). The velocity is again momentarily

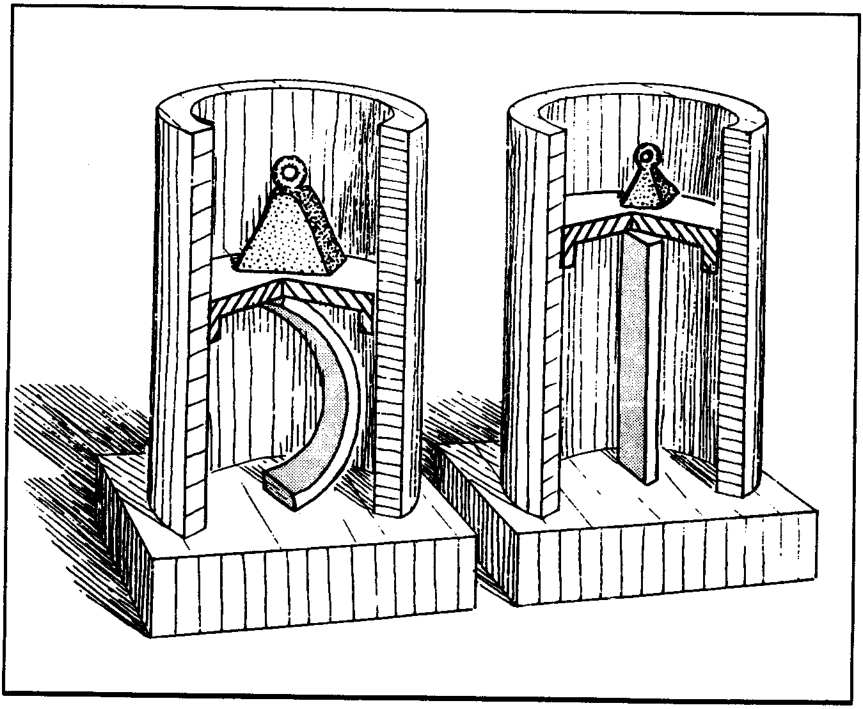


Figure 14. Euler's buckling column (from Abraham & Shaw, 1982–88, © Aerial).

zero at the extremes of the swaying, though both extremes may be on the same side of the center upright position. If there is no friction, the oscillations are undamped. If there is friction, the oscillations are damped, and the column comes to rest bent to one side. If the initial velocity is sufficiently great, then the oscillations cross the center, stopping momentarily at the displacement extremes and also slowing slightly as it crosses the center. There are two trajectories that show the motion of the column approaching the center and stopping there momentarily. From there it could continue its motion either to the right or the left, independently of the direction of approach. That point is a saddle, and the trajectories approaching it will occur if the initial condition imparted to the column is a combination of displacement and velocity existing on either of those trajectories. Again there is the case without friction, and the case with friction. Without friction, whether confined to one side or not, the osciallations are undamped, the attractors are cyclic, closed orbits. With friction, the trajectories spiral to fixed point attractors on one side or the other representing where the column comes to rest in a bent shape (Figure 15d). In this system with friction, there are two basins spiraling around each other, each representing spiral trajectories going to one of the two focal point attractors. The trajectories approaching the saddle point are separatrices between the two basins. The rate of spiraling and the location of the fixed point attractors is determined by the characteristics of the column and the magnitude of the weight; if the column is constant, then the phase portrait changes with the weight, the principal force parameter of the system.

Two of the most important properties of dynamical systems are demonstrated by the phase portraits of the buckling column. The first property is the importance of the initial conditions (the initial displacement and velocity of the column. This is most dramatic with the system with friction and the larger weights where the phase portrait shows the two spiraling basins. Initial displacements that are quite similar may initiate trajectories that are in different basins and thus end up at different attractors. This may be somewhat counterintuitive. For example, an initial displacement to the right with a particular velocity to the right may result in the column bent to the right; increasing the velocity to the right may result in the column ending up on the left. Increasing the velocity even more to the right may again result in its ending up on the right. All three of these cases are for the same initial displacement.

The second property is the importance of a critical threshold value of some control parameter of the system. Changes in the value of the control parameter results in changes in the phase portrait. These may be somewhat monotonic or linear for some range of the parameter, but passing the critical value may create a major change in the essential nature of the phase portrait. The change is called a bifurcation. The magnitude of the weight changes the phase portrait. Increasing the weight on the buckling column with friction changes the portrait from having the single focal point attractor of the system coming to rest at center (Figure 15b) to the portrait with two focal point attractors for the system coming to rest to the right or the left (Figure 15d).

Another common feature of visual dynamical models illustrated by the buckling column and the pendulum is that the state space often employs rate of change, the velocity, as a state variable, a dimension of the state space. And, as here, this rate of change or velocity is the rate of change of another state variable, in this instance, the position of the column or pendulum. The equations comprising the dynamical model may also be coupled by the sharing of other terms as well. The equations for the buckling column are in Appendix B 1. For didactic purposes, we have explored the buckling column in diverse areas of psychology. Tompkins's ideological theory (1963; this chapter, pp. 118–123), some of Jung's concepts of the psyche (Abraham, 1989; Abraham,

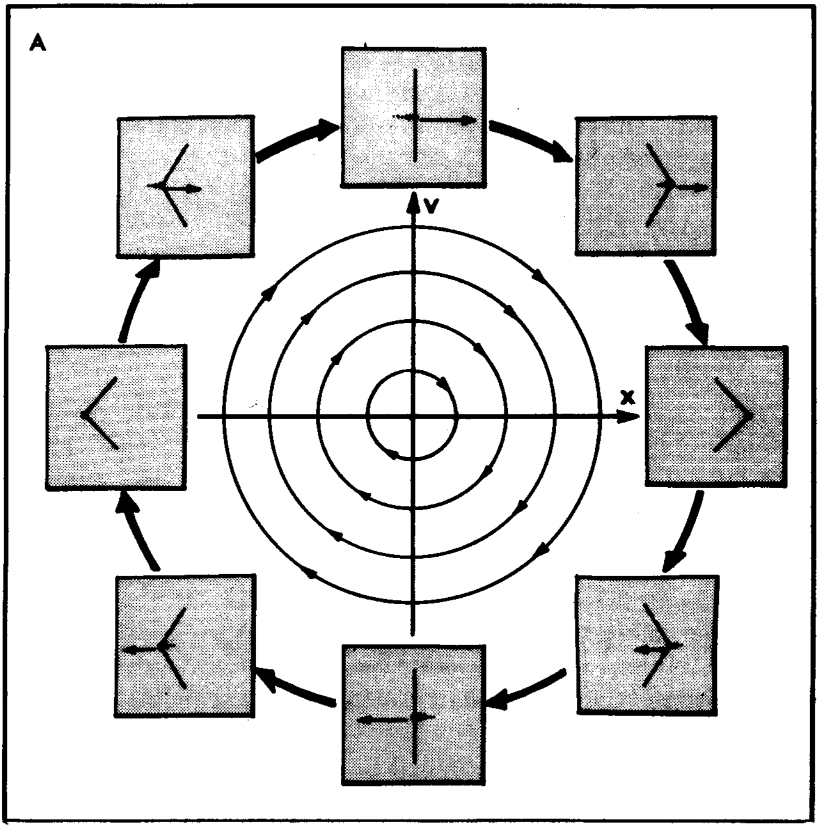


Figure 15. (a) Conservative buckling column: Phase portrait with the weight lighter than the critical value for buckling; (b) dissipative buckling column: Phase portrait showing trajectories spiraling to a focal point attractor; (c) conservative buckling column: Phase portrait with the heavier weight inducing buckled oscillations; (d) dissipative buckling column: Phase portrait with the heavier weight showing basins for two focal point attractors (from Abraham & Shaw, 1982–88, © Aerial).

Abraham, & Shaw, 1990), a psychoanalytic model of grief and depression with mood and self-image as state variables and dependency as a control parameter, and a cognitive model of attribution processes in motivation (Abraham, Abraham, & Shaw, 1990) illustrate but a few of the potential applications to Psychology.

Percussional musical instruments have also been modeled as damped oscillators (Rayleigh, 1877). Their consideration started the theory of nonlinear oscillators as their internal restorative and frictional forces were considered nonlinear functions of the velocity or displacement of the vibrational component of the musical instrument.

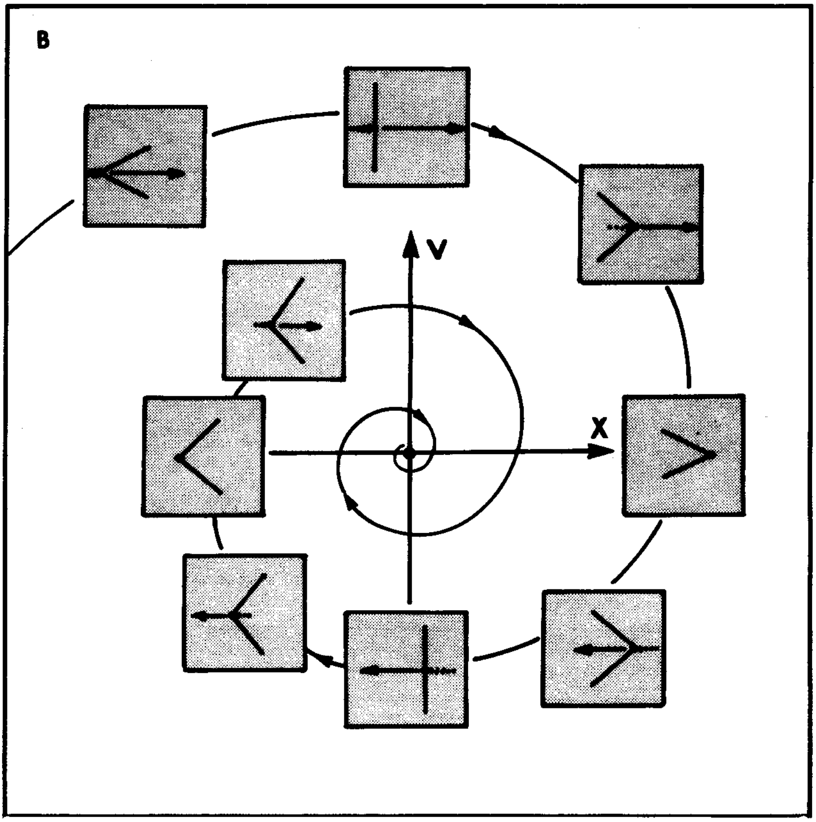


Figure 15. (continued)

Interacting Biological Populations

In behavioral psychology, competing responses and competing motives are often involved. In cognitive psychology, competing ideas are often considered. Some of these systems have been studied with variations of a dynamical model of competing biological species, one of the first in the life sciences.

The Lotka–Volterra model (1924, 1931) of prey–predator populations has generated a class of dynamical models in which the rate of change of the two populations is a function of both population densities. Figure 16 shows some of the vectors of the vectorfield. The state space for this prey–predator system has two dimensions; each dimension represents measurement of the population density, that is, the number of each species of fish for the part of the ocean or aquarium that

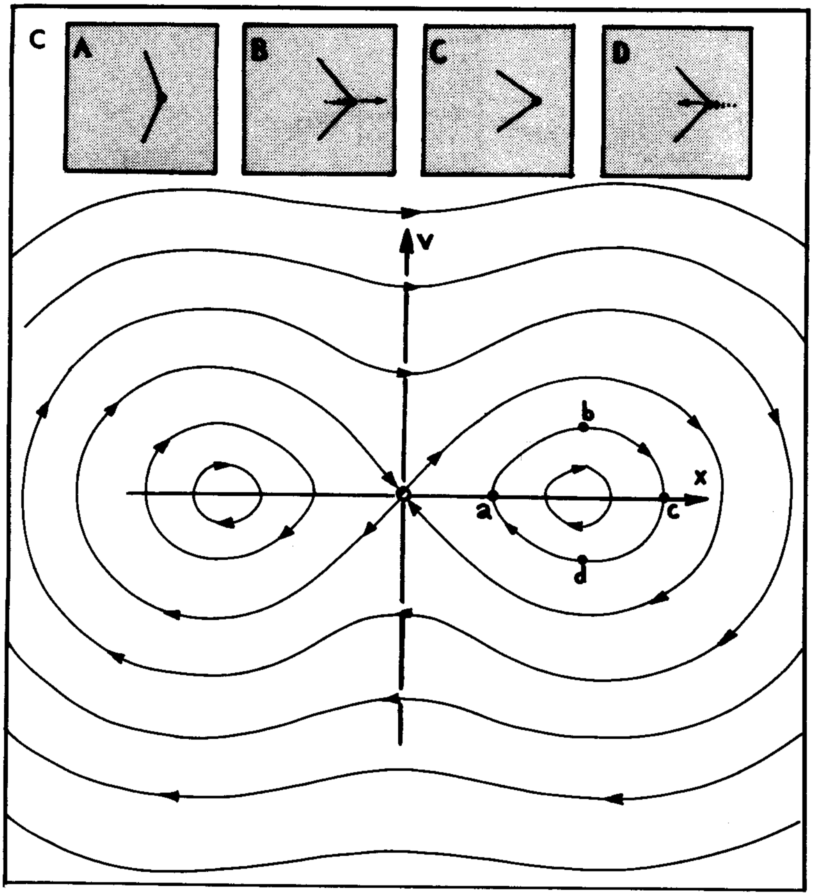


Figure 15. (*continued*)

might be considered a relatively closed environment. The arrows are velocity vectors representing the tendency of the system to change in each of the four main quadrants of this space. When both populations are relatively large, the big fish are well fed and tend to multiply, whereas the population of small fry tends to decline (Figure 1, instantaneous velocity vector C). Next, with many big fish and few small fry, both populations tend to decline as the fry population is too small to support the population of big fish (arrow D). When both populations are small, the population of big fish continue to tend to decline to a point where they fail to curb the population of small fry that now tends to increase (arrow A). When the small-fry population has recovered sufficiently, then they can support an increasing population of the big

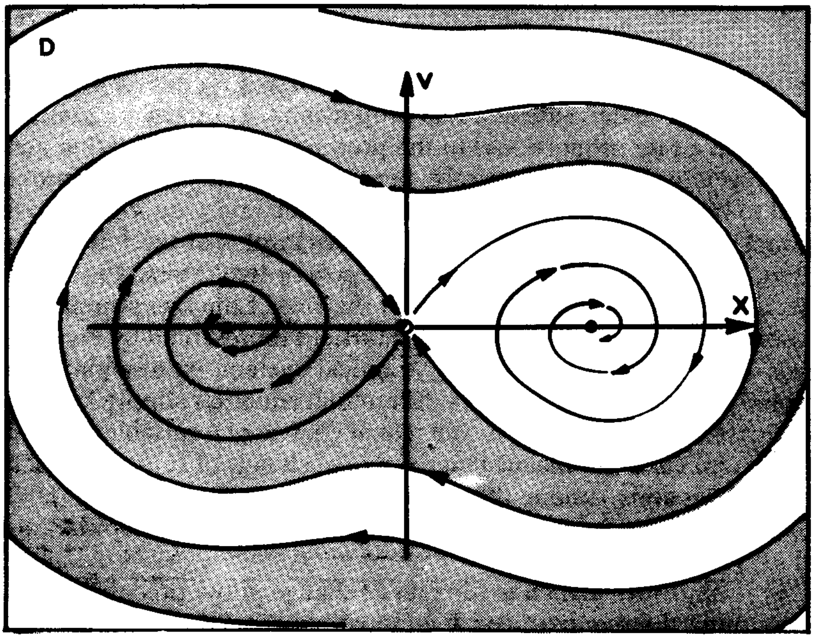


Figure 15. *(continued)*

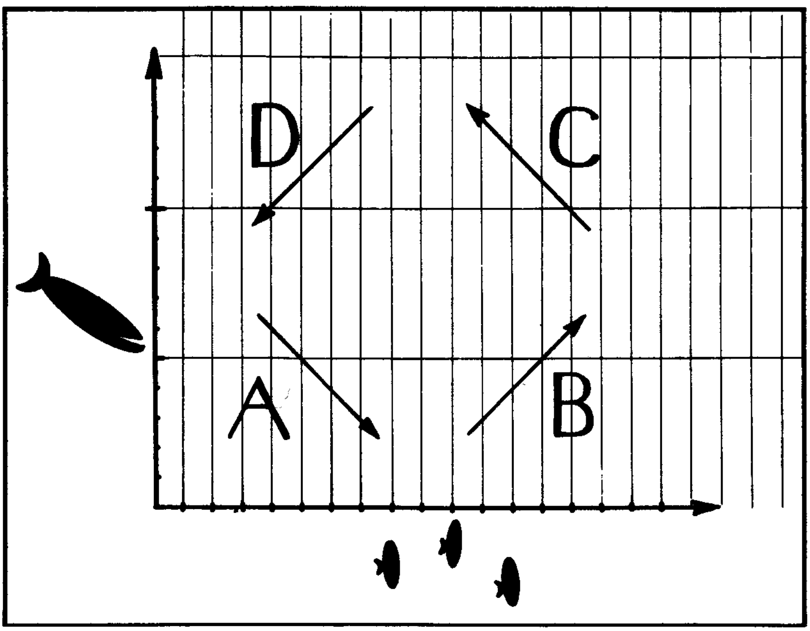


Figure 16. The vectorfield of the Lotka–Volterra model for two interacting populations of prey and predators (from Abraham & Shaw, 1982–88, © Aerial).

fish (arrow B). Thus the size of the predator population at any given moment tends to decrease at a rate proportional to its size and tends to increase at a rate proportional to the product of both population sizes. The prey population, conversely, tends to increase at a rate proportional to its own size and tends to decrease at a rate proportional to the product of the size of both populations (examples of the *law of mass action*, where the rate of change of the size of two interacting populations is proportional to the product of the size of the two populations). These assumptions can gerate the vectorfield represented by Figure 16. These assumptions comprise a dynamical system expressible as two coupled differential equations (a differential equation simply means an equation stating what governs the rate of change of a variable; coupling two or more variables means that the rate of change of at least one of the variables depends, among other things, on the other variable):

$x' = ax - bxy$, which is the rate of change of the prey population, x, and

$y' = cxy - dy$, which is the rate of change of the predator population, y, and where a, b, c, and d are positive constants.

The vectorfield, the verbal assumptions, and the differential equations have all been equivalent descriptions of the tendency of the system to change. What are the resulting changes in the populations, the trajectories generated by these tendencies? The phase portrait of this system (Figure 17a) is a nest of closed trajectories around a central rest point. The populations follow one and only one of these possible trajectories, depending on their initial sizes, which obviously comprise a point on that trajectory. That is, given an initial size of each population, the changes in their sizes keep cycling back through this point. This idealized model assumes that no factors other than the initial sizes and the tendancy to change that depends only upon population sizes influence the behavior of the system. Such a system is called a center. The rest point in the middle exists only when it represents the initial population sizes that then do not change.

When other factors are allowed, then many other types of portraits may be generated. For example, all trajectories could spiral in to a point, a focal attractor, illustrating the effect of ecological damping or friction, such as toxic wastes influencing reproductive rates or predation and escape behavioral capabilities (Figure 17b). In another example, all trajectories could tend to one of the closed periodic trajectories, a limit cycle (Figure 17c). This model is more realistic for a reasonably local or closed system. Other possibilities include point or cyclic repellors, perhaps due to improved fertility factors or alternative food supplies for both species. Even more complex systems could involve

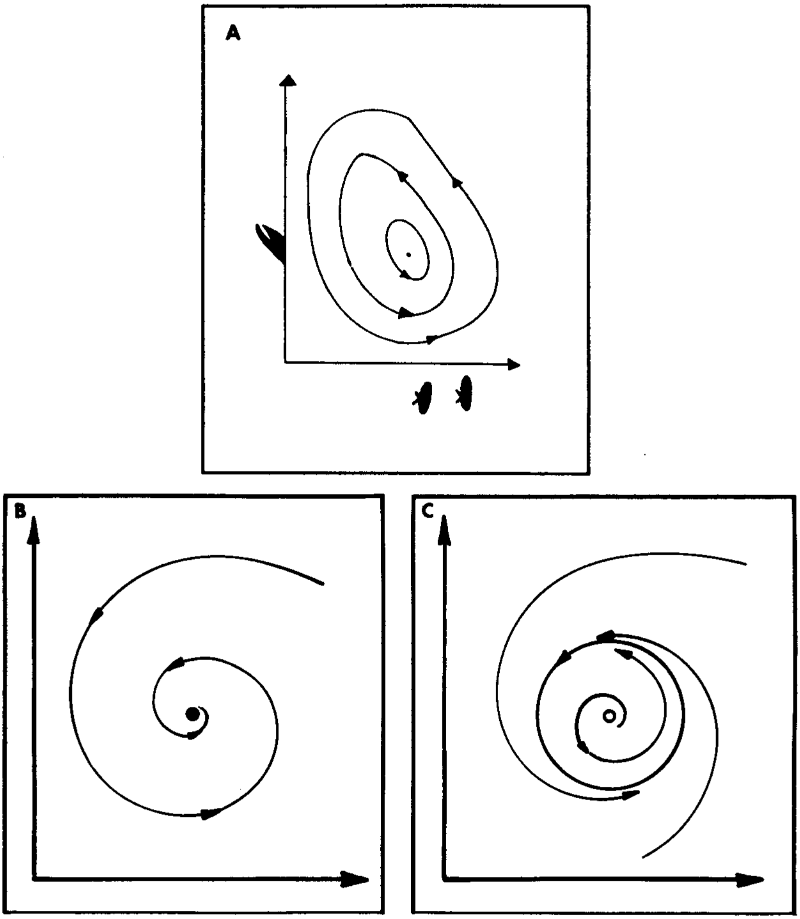


Figure 17. Phase portraits for the prey–predator model (a) basic Lotka–Volterra model: a center (from Abraham, Abraham, & Shaw, 1990, © Aerial); (b) ecological friction: focal point attractor; (c) relatively closed system: periodic attractor (from Abraham & Shaw, 1982, © Aerial).

multiple attractors and basins (see Abraham & Shaw, 1982; Holling, 1976; May, 1973; Rosen, 1970).

The law of mass action is similar to that found in dynamical chemical systems (Decroly & Goldbeter, 1982). It is not the same as Lashley's *law of mass action* that states that learning deficits are proportional to the amount of cortical damage (Lashley, 1929). But it is interesting to see the possibility of joining these two *laws of mass action*. Lashley's law on the effects of damage to the cortex on learning does not represent a dynamical system because it does not describe change over time. But when one considers recovery of behavioral function following neural damage, then the system could be a dynamical one. The two laws of mass action then meet only in the special situation where there is recovery of function of two competing learned responses. In an investigation of recovery of properties of electroencephalographic cospectra after hypothalamic damage, some features of competition between power and coherence suggested a competition between local and distant generation of some frequency components of the EEG. These were revealed in a canonical phase portrait from a discriminant analysis (Abraham *et al.*, 1973).

Historically, the prey–predator model is the most recent of our classical examples. It is especially relevant because it was developed for a system that involves the behavior of organisms and is thus psychological as well as biological. We apologize to the gentle reader for starting with an example that involves such violent behavior. The cooperative schooling behavior of fish or the cooperative breeding behavior of birds would have been gentler and most fascinating. The species competition model was used for its historical and didactic value. One of the fortés of dynamical systems theory is to explain complex cooperative systems as we shall see later. Even for the nature of evolution, Kropotkin emphasized cooperation. And, of course, competition is but a special case of negative cooperation. Because predatory and escape behavior involves evolution (behavioral change between generations) and learning (behavioral changes within a life span), it is not surprising that such models have been used to model evolution and learning. An example of such a learning model is given in the section called "contingent operant behavior," and readers, at their option, are invited to explore it now (p. 123). There is also a family model of cooperative/competitive dynamics (Elkaïm *et al.*, 1987). We also fabricated a small neural net based on the prey–predator model (Abraham, Abraham, & Shaw, 1990). Dynamical models of biological, psychological, and cultural evolutions are especially appealing because they can model saltatory as well as continuous change.

Sustained Oscillators

Oscillatory behaviors abound in biological and psychological systems, constrained by atomic, molecular, neural, and energy properties at one end, and by cyclic properties of the environment at the other. It is a tenent of the application of dynamical systems theory to dissipative cooperative biological systems to expect that there will be mutual and multiple interactions between variables that exist across many magnitudes of scale, from subatomic, molecular, biologic, behavioral, to environmental and even cosmic.

Lord Rayleigh (1877) used the damped oscillator to describe percussive musical instruments, as mentioned earlier. He then proceeded to generalize the analysis to sustained instruments where wind or bow makes a springlike object (reeds, lips, or strings) vibrate. As with the pendulum, buckling column, and other oscillators, the dimensions of the state space represent the displacement and velocity of the vibrating object, which also has internal friction (damping, resistive, or dissipative forces) and restoring forces opposing or assisting the externally applied force sustaining the oscillation. For the basic model, the restoring force is a negative linear function of the displacement, and the friction is a cubic function of the velocity. For small motions near the origin, the friction is a positive function of velocity assisting the motion (sometimes called negative friction; see the characteristic function in the insert, Figure 18a), and the resulting phase portrait has a point repellor at the origin from which trajectories spiral outward to a limit cycle. For larger motions, the friction is normal (a negative function of velocity resisting the motion), damping the trajectories in toward the limit cycle. A change of sign of the friction force creates a *sustained oscillation*.

The electronic oscillators invented by Helmholtz (and later the vacuum tube oscillators studied by Van der Pol) exhibited similar properties. The radio transmitter consisting of a power supply, triode, a tank circuit (a variable capacitor in parallel with an inductive coil and load resisters), and a negative feedback coil from the plate's tank circuit to the grid of the tube. The observed variables are average (rms) voltage and amperage. The phase portrait is very similar to the sustained musical instrument, having a point repellor at the origin and a periodic attractor around the origin (Figure 18b). A variant of this model yields a similar phase portrait with a periodic attractor that is a somewhat more flattened parallelogram, a portrait called a *relaxation oscillator* (Figure 18c). The speed along the attractor is relatively slow on the vertical segments, where the current is relatively constant and at its most extreme values, and relatively fast on the longer horizontal segments where voltage is constant at its extremes and the current is undergoing large and rapid change.

Van der Pol and Van der Mark (1928) employed this relaxation oscillator to model the heartbeat. The significance of this approach, the simulation of the heartbeat by the electronic relaxation oscillator, goes beyond providing an example of a dynamical system of great generality, that is, as acting as a model for the construction of similar models in

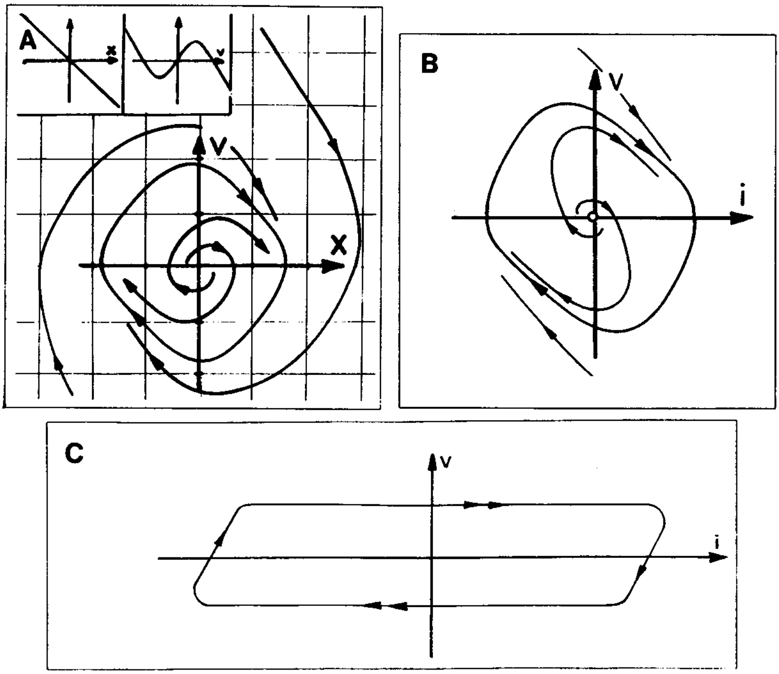


Figure 18. Self-sustained oscillators: phase portraits. (a) general model as for a reed woodwind instrument: limit cycle with a central repellor (reed displacement, X; reed velocity, V. Left insert: force is a negative linear function of reed displacement. Right insert: force is a cubic function of reed velocity); (b) Van der Pol's triode vacuum tube radio transmitter; and (c) relaxation oscillator (from Abraham & Shaw, 1982, © Aerial.

many fields of scientific investigation, a usefulness that certainly is vast and important in its own right. But, further, it ushered in the electronic era of experimental dynamics. Mathematical derivations of the models for musical instruments or electronic oscillators is arduous, whereas, on the other hand, using electronic, oscilloscopic, or computerized simulations is much easier and faster. Modern experimental dynamics is less concerned with the logico–deductive axiomatic theorem-proving approaches than with more direct comparison of theoretical phase portraits derived from the coupled differential equations representing the basic variables and parameters of the system, to empirically produced phase portraits derived from experimental data.

As with most modeling communities, this is a two-way or feedback process, sometimes with greater emphasis on the empirically derived portraits suggesting the conceptual mathematics by similarity to other better-known dynamical systems and sometimes with greater emphasis on the hypothetical conceptual system suggesting the empirical investigation needed to reveal or confirm the phase portrait. This reciprocal interaction between modeling and observation is a well-known fundamental aspect of science.

To emphasize this flexible and empirical nature, we have demonstrated the sustained oscillator as a dynamical system by using the visual approach showing the phase portraits.

Forced Coupled Oscillators

Rayleigh went on to establish yet another new, extremely significant branch of dynamics, forced vibrations, involving attractors in three dimensions. We consider two types. In both, there is a periodic driving force or device, and a driven device. In one type, the driven device is a damped oscillator tending to rest. In the other, it is a self-sustained oscillator. In psychological systems, most oscillators, from neurons, to neuroendocrines, to sexual, appetitive, sleep, and personality changes, are coupled to related biological and environmental oscillators.

Periodically Driven Damped Oscillators. The classical example is the effect of a mechanical vibration on a pendulum or a spring (figure 19) studied by Duffing (1918). A biological example would be the effect of climatic seasons on ecologically interacting populations such as in the prey–predator model.

Forced Linear Spring and the Response Diagram. Duffing's model can be approximated by a motor driving a spring attached to a sliding weight (Figure 19b). This figure shows a strobe light illuminating the driven oscillator and triggered at a fixed phase of the driving oscillator (shown for phase zero; the motor arm at full right). If not driven, the weight would behave like the damped pendulum or lightweight buckling column coming to rest at center. Its phase portrait would be a spiral attractor at the origin in a planar state space of velocity as a function of displacement of the weight (spiral as in Figure 17b and dimensions as Figure 18b). The state space for the driving motor is not a plane, but a ring (forgetting starting up transients, the limit cycle is the entire state space) consisting of the phases, from 0 to 2π. The frequency and amplitude are considered constant.

We may combine these into a three-dimensional state space (Figure 20a), which we show, for clarity, in the isochronous harmonic case (one cycle of the weight is completed during one cycle of the motor), with vertical planes representing the weight's state space, and the horizontal ring representing the motor's state space, shown being bent from a straight time line to the ring more representative of its cyclic nature.

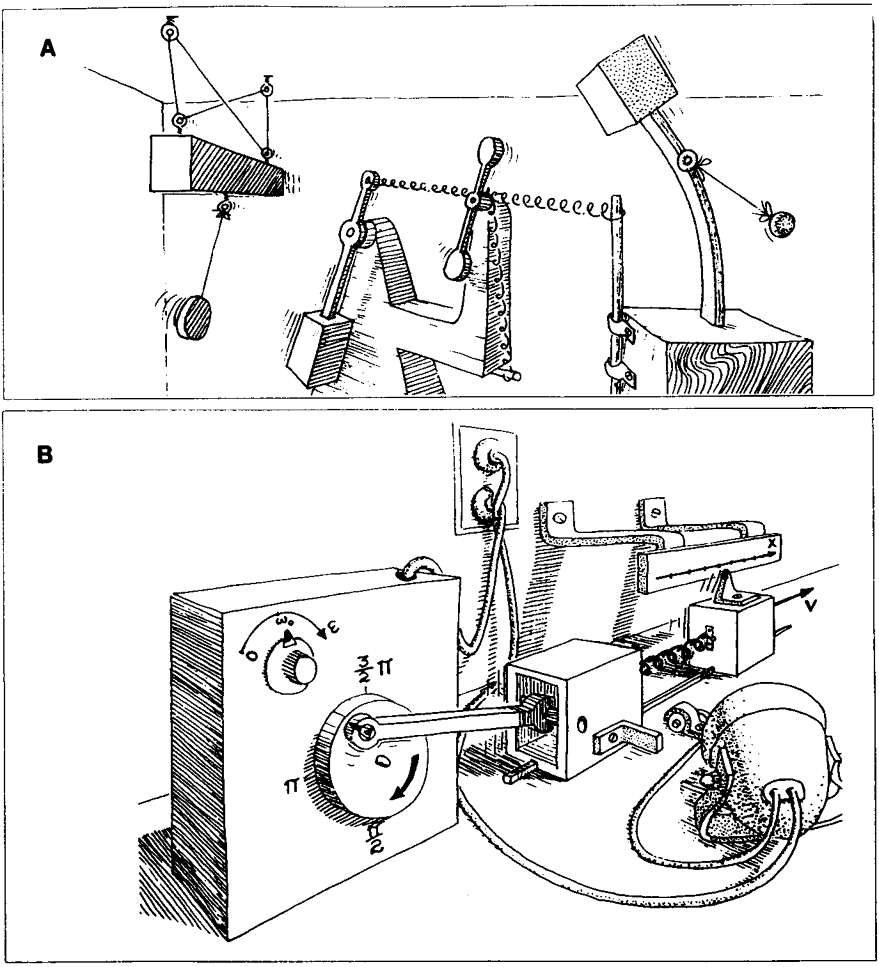


Figure 19. Forced damped oscillators. (a) devices used by Rayleigh, Duffing, and Ludeke (the driving oscillator is only approximately sustained; the drive oscillator is damped). (b) model equivalent to Duffing's (from Abraham & Shaw, 1982–88, © Aerial).

The cylinder doesn't exist, it is not an attractor, that is, not an invariant manifold (collection of trajectories) or phase portrait, but merely a visual aid to help show the location of the trajectory for the isochronous harmonic (the undulating solid and dotted line) that is the only trajectory on it.

A trajectory can be visualized spiraling in toward this isochronous harmonic periodic attractor, shown as it passes through the phase zero plane (Figure 20b). The strobe plane for phase zero can be shown with

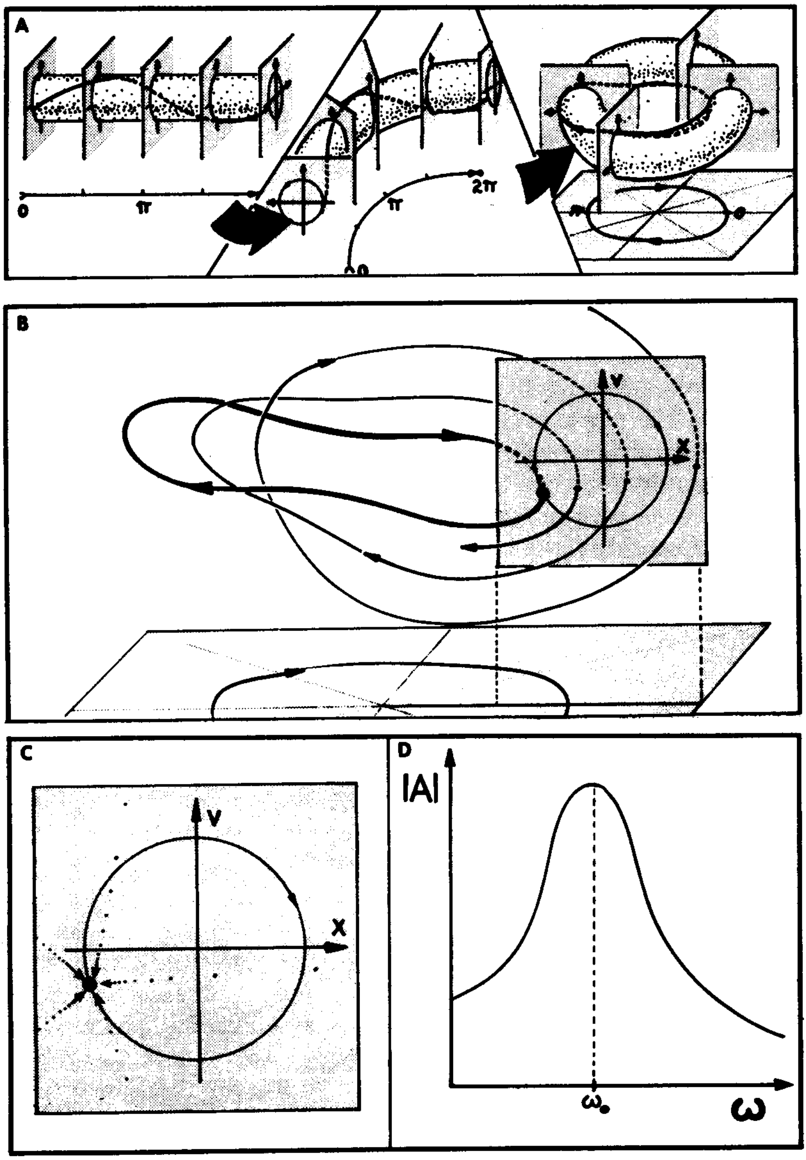


Figure 20. Ring model for forced damped oscillator (linear spring). (a) driving cycle bent into ring, with isochronous harmonic attractor; (b) trajectory approaching the attractor; strobe plane at phase zero; (c) Strobe (Poincaré) section, phase zero, showing several trajectories approaching the attractor; and (d) response diagram: response amplitude as a function of the driving frequency (from Abraham & Shaw, 1982–88, © Aerial).

several trajectories approaching the attractor's point of intersection in this plane (Figure 20c & d). If the motor is slow, the trajectories for the motor and the weight are in phase; both reach phase zero at the same time, the attractor's location in the strobe plane at phase zero of the ring for the motor (Figure 20a), representing maximum displacement and zero velocity of the weight in the representations.

Increasing the driving frequency closer to the natural frequency of the spring weight still produces an attractive limit cycle of the isochronous harmonic, but the phase has slipped. In this case, the maximum amplitude of the weight is also greater but does not coincide with phase 0 of the driver, so the imaginary torus (Figure 20a) is larger in diameter. This amplitude is greatest when the driving frequency is the same as the free frequency of the driven weight (resonance; ω_0 in Figure 20d). The phase of the weight in this case is $\pi/2$ (a quarter cycle) behind the driving motor. As the driving frequency is increased even more, the maximum amplitude gets smaller (Figure 20d), and the phase lag increases even more. This has been for the case of the linear spring where the restoring force is a negative linear function of the displacement. Now let's look at replacing the linear spring with a hard spring where the force is greater than a linear increase as displacement (amplitude) of the driving motor is increased.

Forced Hard Springs and the Cusp Catastrophe. The restoring force of the hard spring is a cubic inverse function of the displacement amplitude (Figure 21a). One consequence is that a clarinet with such a reed would blow flat on the softer Mozart, sharp on the louder Clarinet Polka (dotted line, Figure 22a). Another consequence is a hysteresis effect on the amplitude–frequency response diagram when the forcing frequency is systematically raised and lowered (Figure 21b). In the region between the large jumps up or down, there are two amplitudes for each frequency, depending on whether the frequency is being increased or decreased. Thus, in the region of the phase portrait of the ring model corresponding to the effects of that range of frequencies, there will be two periodic attractors representing isochronous harmonics (Figure 21c). Because there are two attractors, some trajectories go to one, some to the other. And thus there are two basins, with a third limit cycle (a saddle) between them (shown in Poincaré strobe section in Figure 21d, three dimensionally in Figure 21e). The inset of this saddle cycle comprises the separatrix of the two basins. If the amplitude of the forcing oscillator is varied, a family of response curves, with increasing maximum amplitudes of increasing frequency are generated (remember the clarinet with the hard reed we just mentioned, and Figure 22a). This response diagram can be shown in three dimensions with the hysteresis loop (Figure 22b). Such curves, where there are jumps from one value to another, are called *folds* (2D as in Figure 21b) and *cusps* (3D as in Figure 22b) in catastrophe theory.

The ratio of the driven frequency to the driving frequency, the *rotation number*, for the isochronous harmonic attractor is 1. This ratio need not be 1. If it is rational, and an integer greater than 1, ultraharmonics are generated in the driven oscillator, and the attractor rotates an integer number of times in the velocity-displacement plane during a single phase cycle of the driving oscillator (Figure 22c). If the ratio is rational and smaller than 1, subharmonics are generated; for example, it takes several cycles of the driver to each cycle (or cycles) of the driven oscillator (Figure 22d). In both cases, the attractors are limit cycles passing through the same points in any given phase plane. If the ratio is irrational, the attractor is a solenoid, that is, wraps densely around an imaginary torus, progressing around this torus, never going through the same point more than once (Figure 22f). Irrational and exotic rational ratios are rare, occurring with small amplitudes, exemplified by the timbre they give to various musical instruments.

Pythagoras (*ca.* 550 B.C.) discovered the world of perfect ratios and harmonies with his monochords using implication and deimplication from but the first four ultraharmonics (and an inversion on the interval of the fifth) as the basis for the generation of most of the scale (Zarlino, in the sixteenth century, extended this to the fifth harmonic and generated the remaining note, the seventh degree of the scale, *b* of the *c* scale). For example, the 3 : 2 ratio (Figure 22e; from classic 12 and 8 lb and the 9 and 6 lb (ocades?) hammers Pythagoras is said to have heard at the blacksmiths) yields a 12th above the fundamental (third harmonic by implication), and an octave below that (half that frequency by deimplication), which is the 5th (g on the *c* scale) (Apel, 1972). The Beaker people (*ca.* 4500 B.C., Stonehenge), the Chinese, and the Babylonians, and possibly even the Cro-Magnons, also knew of the harmonic foundations of scales.

Periodically Driven Self-Sustaining Oscillators. *Entrainment and Braids.* Here we consider forcing a periodically attractive system rather than a system with a point attractor. If the two self-sustained oscillators are uncoupled, their state space may be taken to be a toroidal surface. The cross-section of this torus is a circle representing the phases of one of the oscillators (ignoring startup transients). The cycle running lengthwise around the torus represents the phases of the other oscillator (Figure 23a). Note that this state space (2D) differs from that of the ring model (3D) for the forced damped oscillator. Here, both cycles represent phases, and the torus represents the state space, not just a convenient visualization for the limit cycle. When the oscillators are

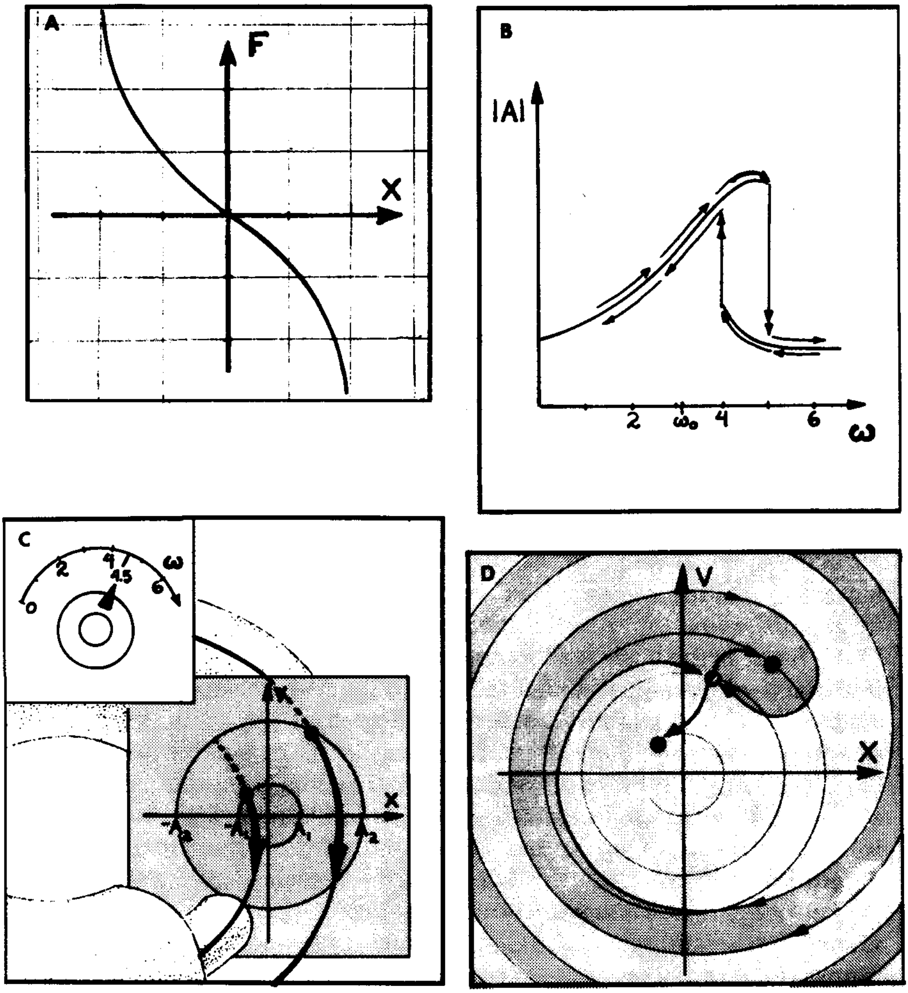


Figure 21. Forced hard spring (a) force as an inverse cubic function of displacement; (b) response diagram: amplitude of the attractor as the driving frequency is changed, showing the hysteresis loop of Duffing (double fold catastrophe); (c) large and small attractors at an intrahysteresis frequency; (d) strobe plane showing the basins for each of the attractors and the saddle cycle and separatrix; and (e) the completed ring model (from Abraham & Shaw, 1982–88, © Aerial).

coupled, they normally become *entrained*. This could be considered as the phase portrait being perturbed by the addition of a small (unspecified) vectorfield to the state space. Peixoto's (1961) classical theorem bringing together differential topology and classical dynamics describes a generic situation in which a finite, even number of closed

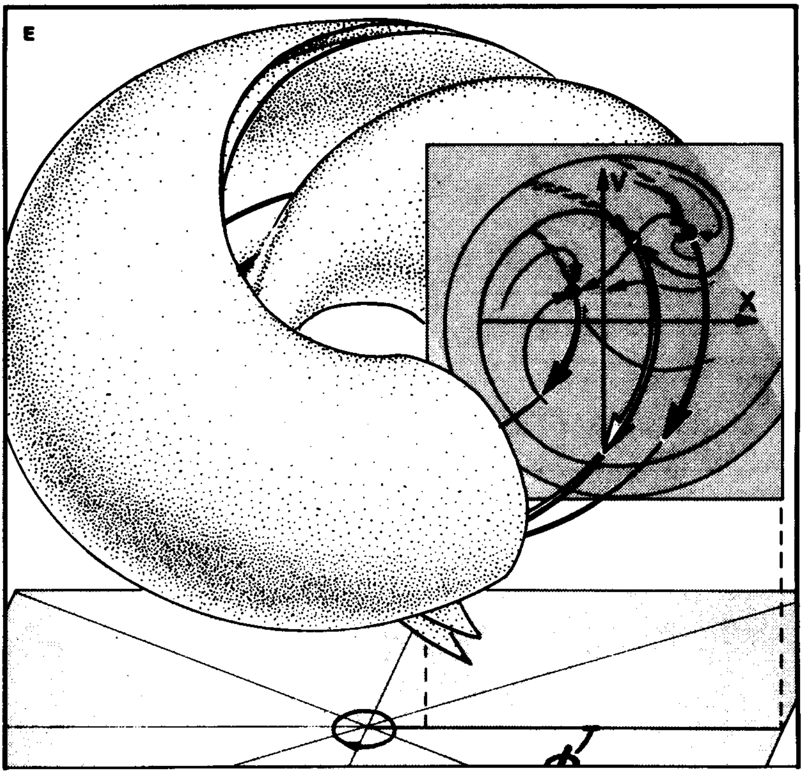


Figure 21. *(continued)*

trajectories all wind around the torus the same number of times (two are shown in Figure 23b). Every other one is an attractor; the alternate ones are repellors. Each clock has to decide which way to go to fall in line with the other. This is *frequency entrainment* (not necessarily phase entrainment). This *braid* is structurally stable; further small perturbations of the system make no significant change in the phase portrait. Huyghens's seventeenth century observation of frequency and phase entrainment of the pendula of clocks on a wall provides an example of coupling self-sustained oscillators with mutual or reciprocal influence.

Returning to the more asymmetrical case having a dominant enforcer (such as a motor) coupled weakly (as by a light spring) to a submissive enforcee (maybe some type of mechanical clockwork), it is useful to enlarge the 2D torus model of the state space to a 3D ring model (Figure 24a). First shown for the uncoupled situation (Figure

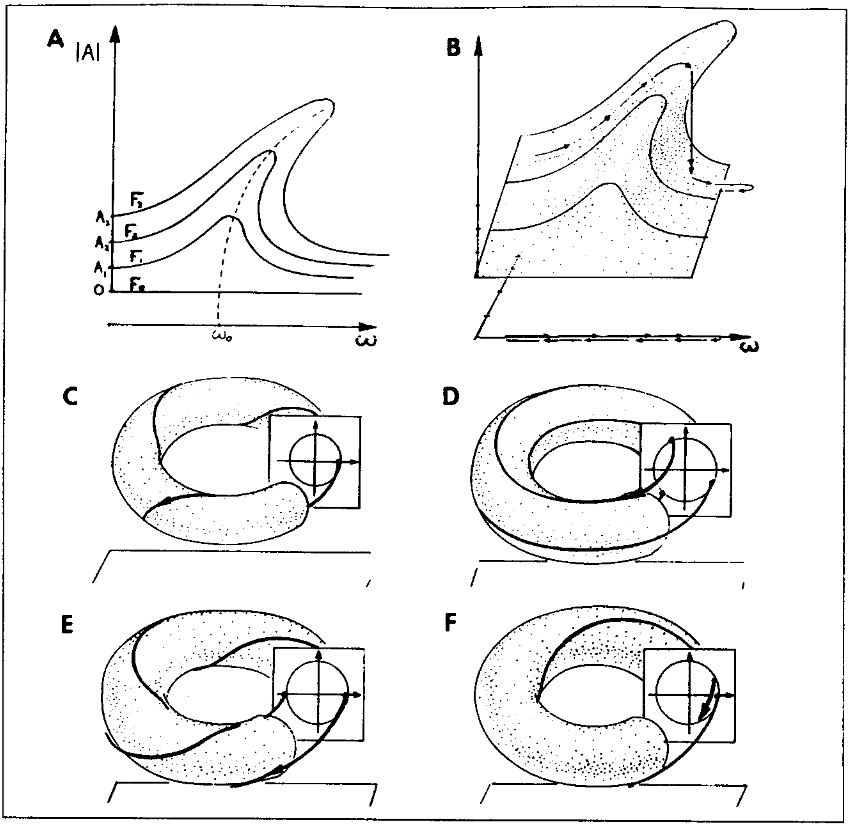


Figure 22. Forced hard spring. (a) response diagram: response curves for different forcing amplitudes; (b) cusp catastrophe: the forcing amplitude is given its own axis yielding a 3D response diagram; (c) third ultraharmonic: winding ratio is 3 : 1; (d) third subharmonic: winding ratio is 1 : 3; (e) winding ratio is 3 : 2. Rotation number and harmonic ratio are synonyms; and (f) toroidal winding never repeats; the winding ratio is irrational (from Abraham & Shaw, 1982–88, © Aerial).

24b), the 2D phase portrait for the driven clockwork is now filled with trajectories instead of consisting of the 1D single-limit cycle alone. The inner part of the basin contains a repellor in the center and trajectories spiraling outward to the limit cycle. The outer part of the basin contains trajectories spiraling inward to the limit cycle. The state space for the driving motor is the 1D horizontal ring representing the cycle of phases again (longitudinal axis of the torus).

In the Poincaré section or strobe plane, we would see a trajectory (discrete or strobed) approaching a point (which could be considered the section of the limit cycle) as a series of points as the trajectory

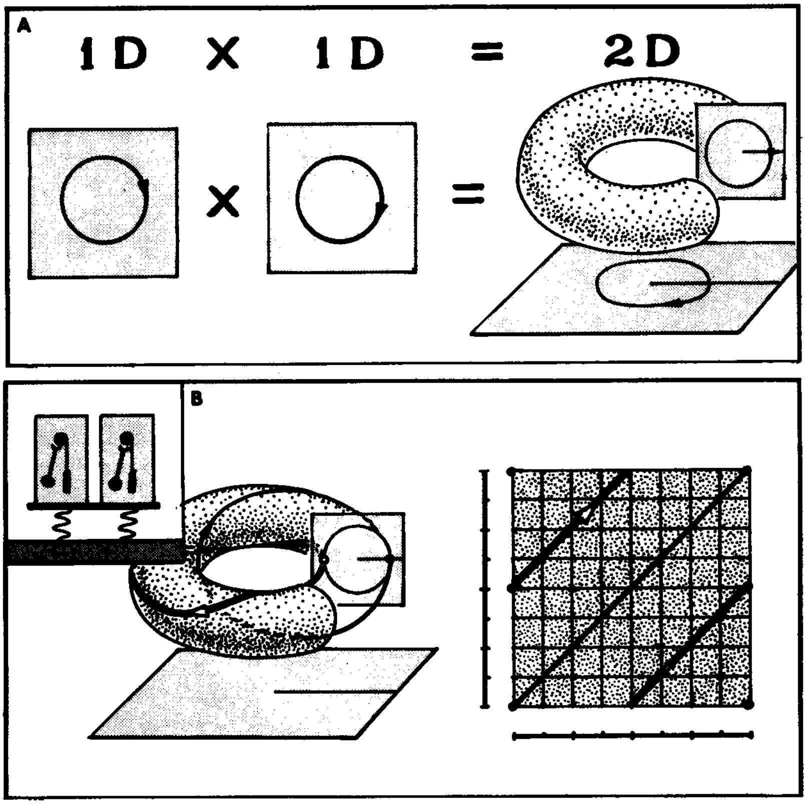


Figure 23. Forced self-sustained oscillators: 2D torus model: (a) the horizontal ring represents the driving oscillator; the vertical ring represents the driven oscillator; and (b) the coupled system considered as perturbation of uncoupled system; alternate closed trajectories are cyclic attractors and repellors; there are no other limit sets; this kind of phase portrait is called a braid; and the axes at the right are the phases of the two clocks (from Abraham and Shaw, 1982–88, © Aerial).

successively crossed the plane while spiraling in or out toward the limit cycle. This is shown for the weakly coupled, in-phase, isochronous case in the three-dimensional model (Figure 25a). If we set the clock with the motor out-of-phase, at clock phase π, trajectories will drift forward or backward into phase, on the torus, toward the in-phase limit cycle just described. Other nearby trajectories from both the inner and outer regions spirally approach this out-of-phase limit cycle. So this is a saddle-type limit cycle, with nearby trajectories first approaching the periodic saddle, before their departure (Figure 25b) along the attractive torus to the entrained in-phase cycle. The torus is shown

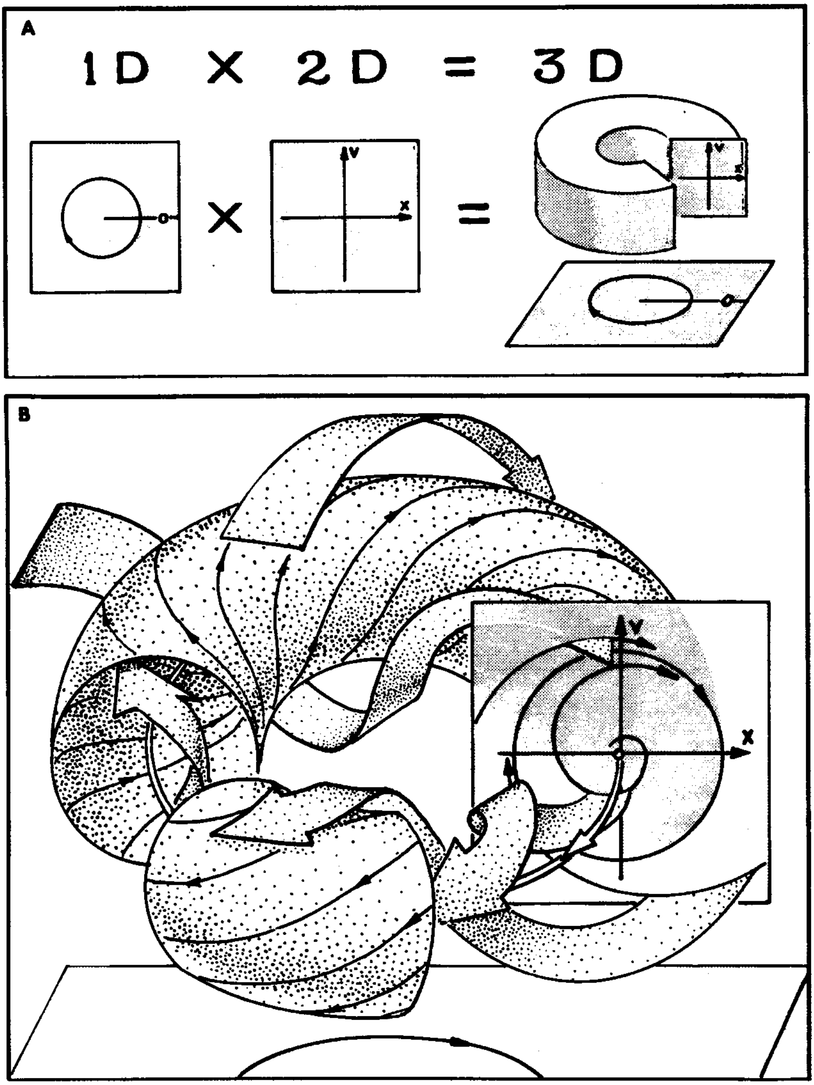


Figure 24. Forced self-sustained oscillators: 3D ring model. (a) the 1D vertical ring of the driven oscillator is replaced by a 2D V vs. X plane; and (b) oscillators before being coupled; the torus is an invariant manifold; it is attractive, but not an attractor; every trajectory starting on it stays there; others approach it (from Abraham & Shaw, 1982–88).

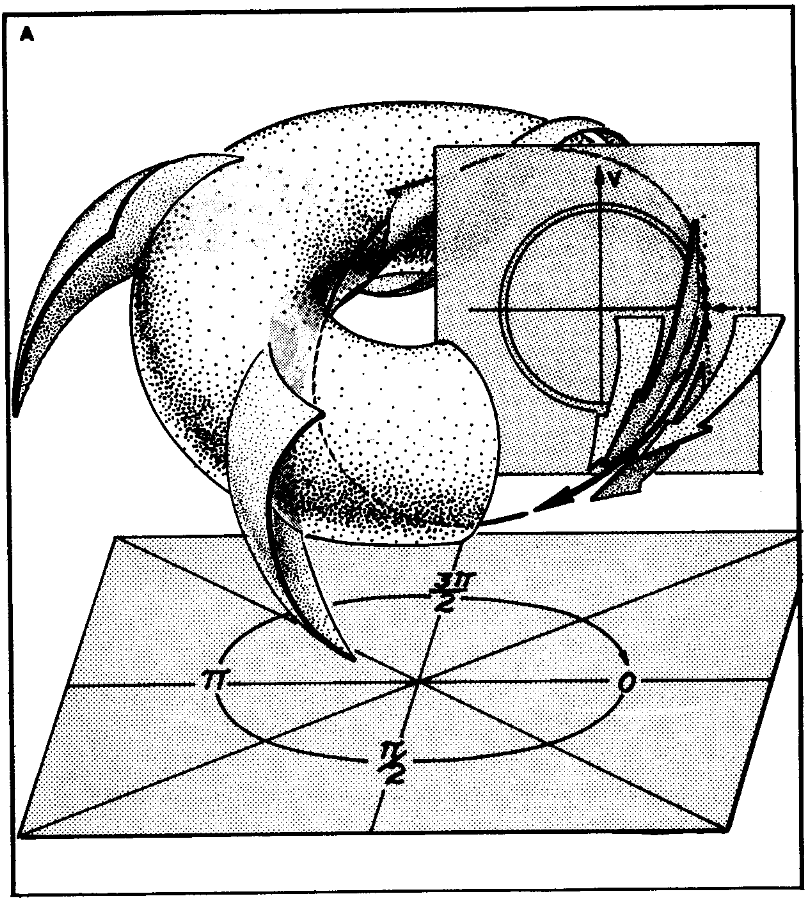


Figure 25. Forced self-sustained oscillators: 3D ring model (continued). (a) oscillators coupled showing isochronous trajectory; in-phase case; the periodic trajectory is an attractor; and (b) periodic out-of-phase saddle attracts amplitudes but repels phases as ribbon arrows show (from Abraham & Shaw, 1982–88, © Aerial).

again not as an invariant manifold but just as a construction to assist visualization of the phase portrait.

Somewhere in between the two limit cycles, the in-phase periodic attractor and the out-of-phase periodic saddle, there will be a central repellor, a periodic trajectory winding around its own noninvariant locating torus (Figure 26). Its outset is the central portion of the basin of the periodic attractor. So where is the invariant manifold for the attractive limit cycles of this dynamical system? It is the torus described by the outset of the periodic saddle as the saddle and periodic attractor

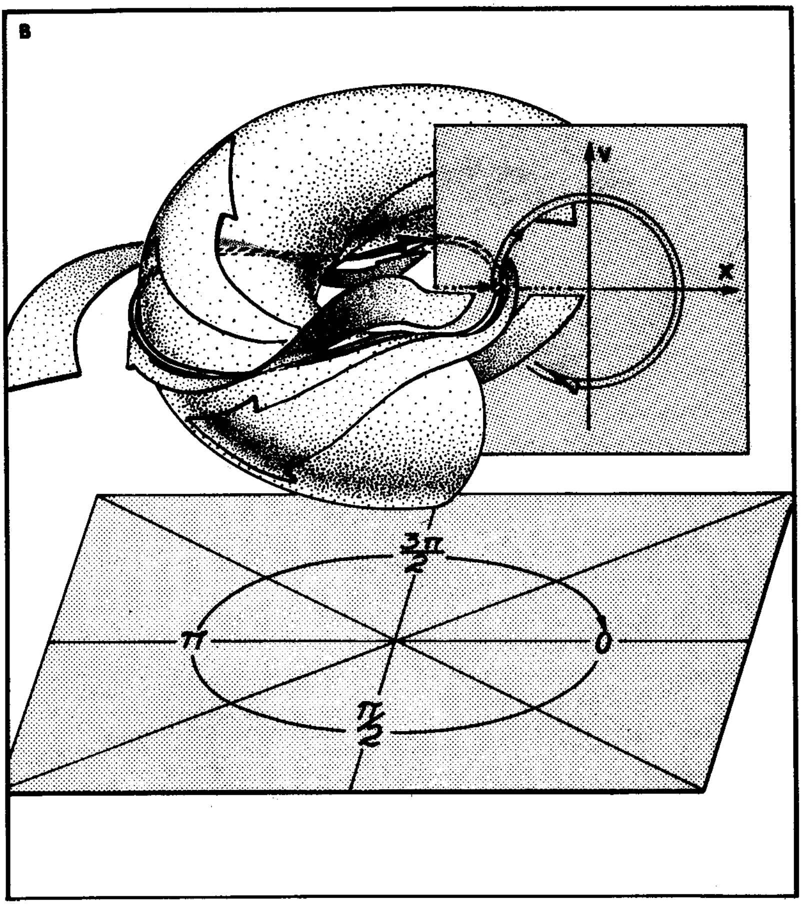


Figure 25. *(continued)*

wind (braid) around each other on that torus (Figure 27). The insert shows this torus in cross-section as a dotted circle and as the shaded torus in the main figure. It is therefore established that this phase portrait has at least three isochronous periodic trajectories: the braided saddle and attractor and the central repellor. This compound oscillator has been presented so far for equal or nearly equal frequencies in the two oscillators.

The Response Diagram for Frequency Changes. We may vary the frequency of the driving oscillator above and below the isosynchronous frequency and plot the maximum amplitudes of the three periodic trajectories as a function of the driving frequency (Figure 28). This may be done with coupling springs of different strengths as a second control

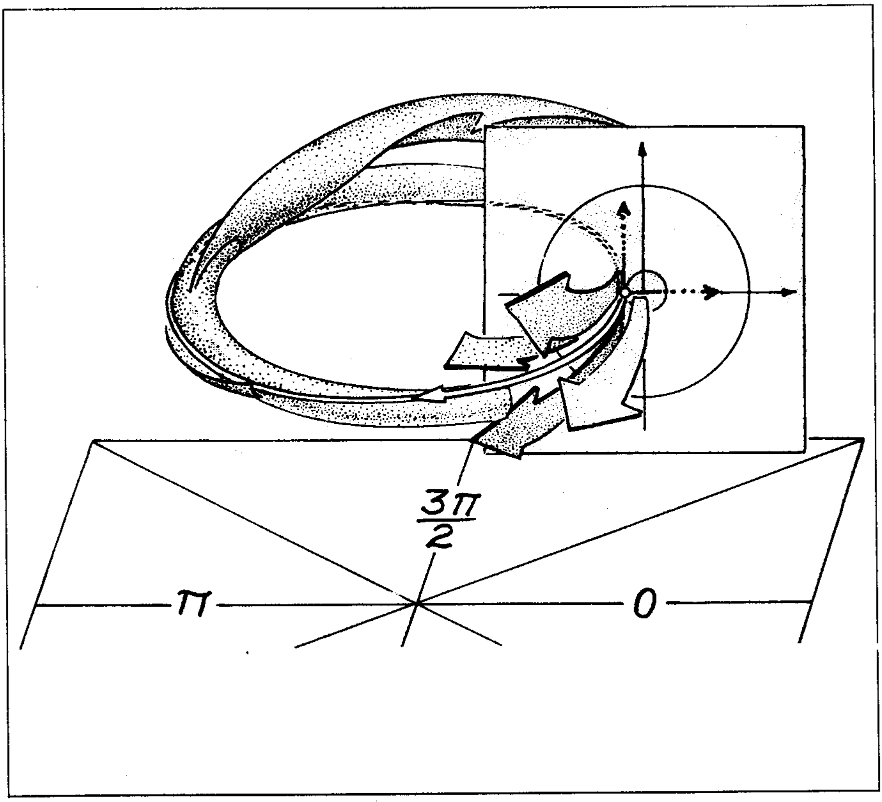


Figure 26. Forced self-sustained oscillators: 3D ring model (continued): coupled oscillators showing isochronous trajectory; out-of-phase case; this periodic trajectory is a repellor shown winding around its locating torus; its outset (ribbon arrows) comprises the central portion of the basin of the periodic attractor; this central repellor lies located somewhere in between the two limit cycles, the in-phase periodic attractor, and the out-of-phase periodic saddle (from Abraham & Shaw, 1982–88, © Aerial).

parameter. Consider first the results for the weakest spring represented by the innermost circle and lowest dotted line near the bottom of the figure. The solid line of the top of the circle shows the maximum amplitude of the periodic attractor that diminished as the driving frequency deviates more from the isochronous frequency. The dotted line of the bottom of that same circle shows the simultaneous variation of the maximum amplitude of the periodic saddle that is increasing as the driving frequency deviates more from the isochronous frequency. When the two amplitudes become equal, for that and greater deviations of frequency, they cease to exist! This is the *periodic annihilation catastrophe*, and this *response diagram* shows an example of a cata-

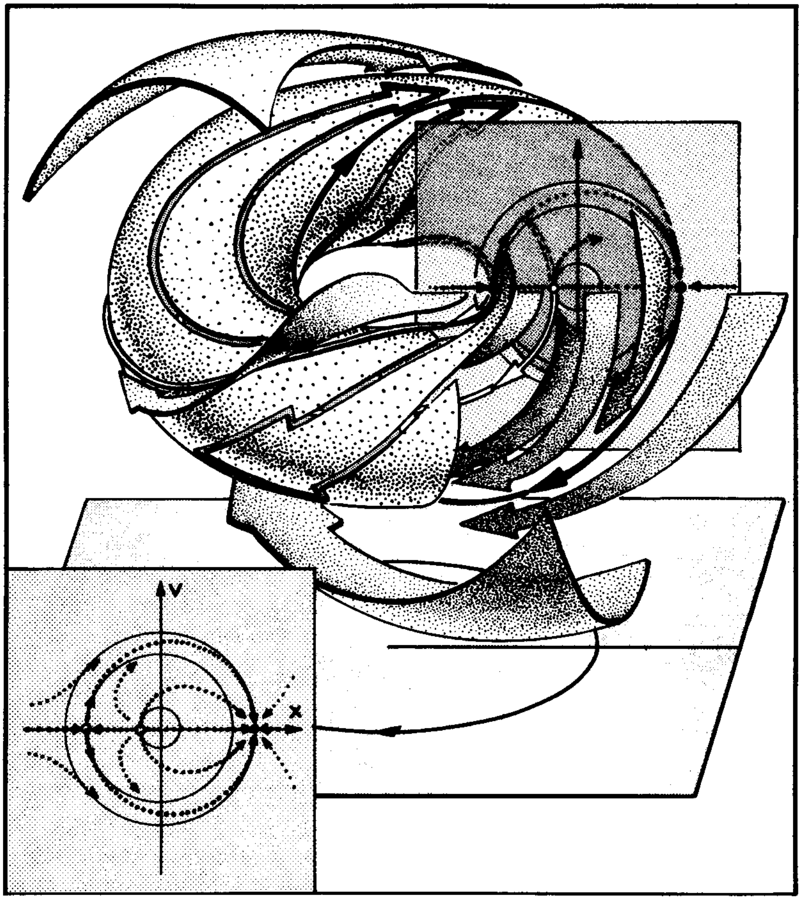


Figure 27. Completed phase portrait of forced self-sustained oscillators. Composite of braided periodic attractor and saddle on invariant torus with central repellor; solid circles in strobe plane are on locating tori; dotted circle represents invariant torus (from Abraham & Shaw, 1982–88, © Aerial).

strophic bifurcation. The bottom dotted line represents the diminishing amplitude of the period repellor that does not annihilate at these frequencies. For the strongest spring (the Ω-like response diagram), the saddle may cancel either the attractor or the repellor; for smaller deviations in frequency, the saddle and the repellor disappear, but the periodic attractor remains; when the deviations are great, the attractor and saddle disappear.

Bifurcation is one of the most important topics in the theory of dynamical systems and the one that makes it so relevant to complex

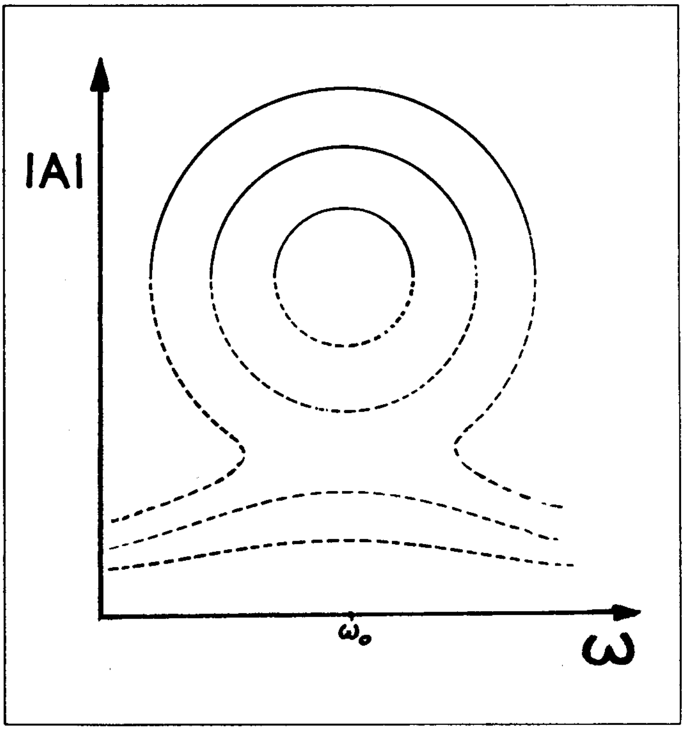


Figure 28. Response diagram of the Van der Pol system: frequency response diagrams for three different strength springs (from Abraham & Shaw, 1982–88, © Aerial).

cooperative systems in biology and psychology. Phase portraits that describe the behavior of a system for a given set of parametric values is valuable enough, but the ability of the dynamical system to generate pictures of how the attractors undergo sudden reorganization through bifurcations, as control parameters make small transitions across critical threshold values, gives visual dynamics its great usefulness as a scientific modeling strategy. A *dynamical scheme* is the dynamical system as a function of a control parameter. The response diagram is the picture of the scheme, just as the phase portrait is the picture of the dynamical system. The response diagram is the primary map in the applications of nonlinear dynamics. The topics of chaos and global behavior further extend this analysis from the classical systems considered so far, into the modern era of applications to biological and psychological systems.

Before proceeding to these topics, we'll conclude with final classical examples of the Van der Pol model for the case of the forced electrical oscillators. This model has a periodic attractor, a periodic repellor, and a periodic saddle. It has a periodic attractor even if the coupling strength between the oscillators is zero. If either the capacitance or inductance in the circuit is negative, the periodic attractor shrinks to a point in a classical bifurcation. With capacitance, inductance, and the coupling strength all positive, many important bifurcations have been observed. The equations are shown in Appendix B3.

It may now prove interesting to explore the sleep system involving forced circadian oscillators (p. 127).

Characteristics of Limit Points and Cycles

Liapounov's *characteristic exponents* and Poincaré's *characteristic multipliers* characterize some important geometric properties of limit sets and the behavior of their nearby trajectories. These and the related concept of the hyperbolic nature of limit sets are important for the understanding of generic and stable features of limit sets features that relate to their general rather than their idiosyncratic interest.

Index, Characteristic Exponents, Hyperbolic Limit Point, and the Spectrum

The *index* of a critical point of a dynamical system is the dimension of its outset. A *characteristic exponent* (CE) of a critical point of a dynamical system is a complex number that measures the rate and character of approach and departure of nearby trajectories with respect to the critical point. It measures the strength of attraction or repulsion in a given direction and the rate of spiraling. (A complex number N takes the form $N = a + bi$ where a is the real part and bi is the imaginary part. They are often plotted on the complex plane with the vertical axis being the imaginary axis and the horizontal being the real. In trigonometry the vertical component of an angle is the sin and the horizontal is the cosine, so a complex number is strongly related to trigonometry as $N = \cos \phi + i \sin \phi$, and thus their usefulness for describing cyclic activity. Algorithms of linear algebra are required to assess CEs.) In general, the number of CEs of a critical point will be the same as the dimension of the state space.

A limit point is *hyperbolic* if none of its CEs has a zero real component. Otherwise it is *nonhyperbolic*. The set of CEs pictured in the complex plane comprises the *spectrum* of the critical point. The phase portraits of some exemplary attractive, repulsive, and saddle-type hy-

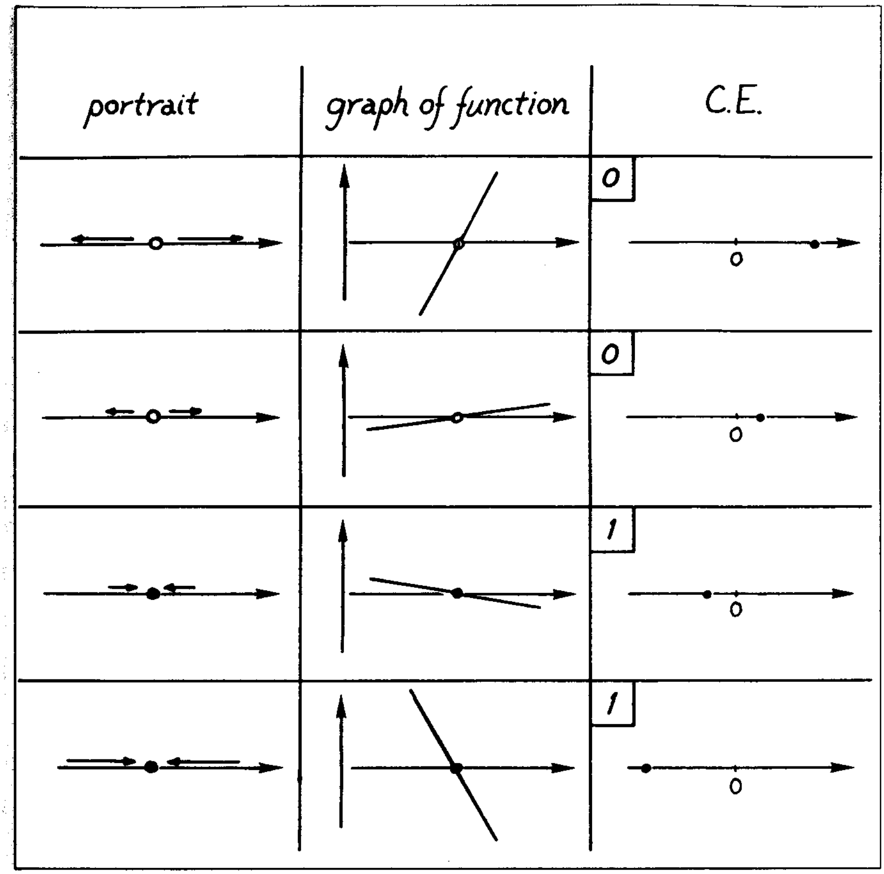


Figure 29. Characteristic exponents for hyperbolic limit points in 1D phase portraits. 1st row: strong repulsion; 2nd row: weak repulsion; 3rd row: weak attraction; and 4th row: strong attraction (from Abraham & Shaw, 1982–88, © Aerial).

perbolic limit points (with index and CEs) are shown for the one-, two-, and three-dimensional cases (Figures 29, 30, 31).

The index for the one-dimensional case is 1 for repellors and 0 for attractors. In this context there is only one CE for a critical point, a real number. The graphs of the vectors as a function of displacement are shown also. Note the positive slope in the case of repellors and the negative slope in the case of attractors (Figure 29). The CE is this slope. The CEs are greater for higher rates of repulsion and attraction.

The indexes for the two-dimensional case again follow the dimensionality of the outset, for example, unity in the case of the saddle

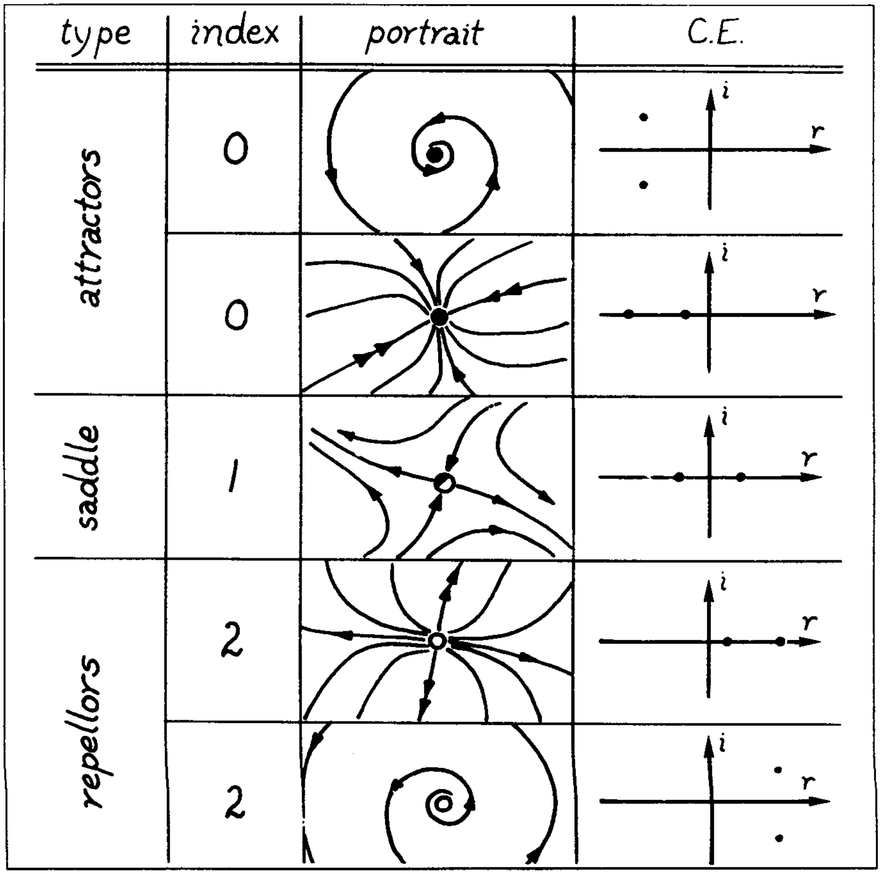


Figure 30. Characteristic exponents for hyperbolic limit points in 2D phase portraits (from Abraham & Shaw, 1982–88, © Aerial).

(Figure 30). The CEs are in the right half-plane for the repellors, the negative half-plane for the attractors, and there is one of each for the saddle (positive for the outset; negative for the inset). For the nodal points, the CEs are real numbers, but for the spiral limit sets, they are conjugate complex numbers (symmetrical about the horizontal real axis). The size of the real component indicates the strength of attraction (left, negative) or repulsion (right, positive), whereas the imaginary component indicates the rate of spiraling. The three-dimensional case can be seen to be composites of these one- and two-dimensional conditions with three CEs (Figure 31).

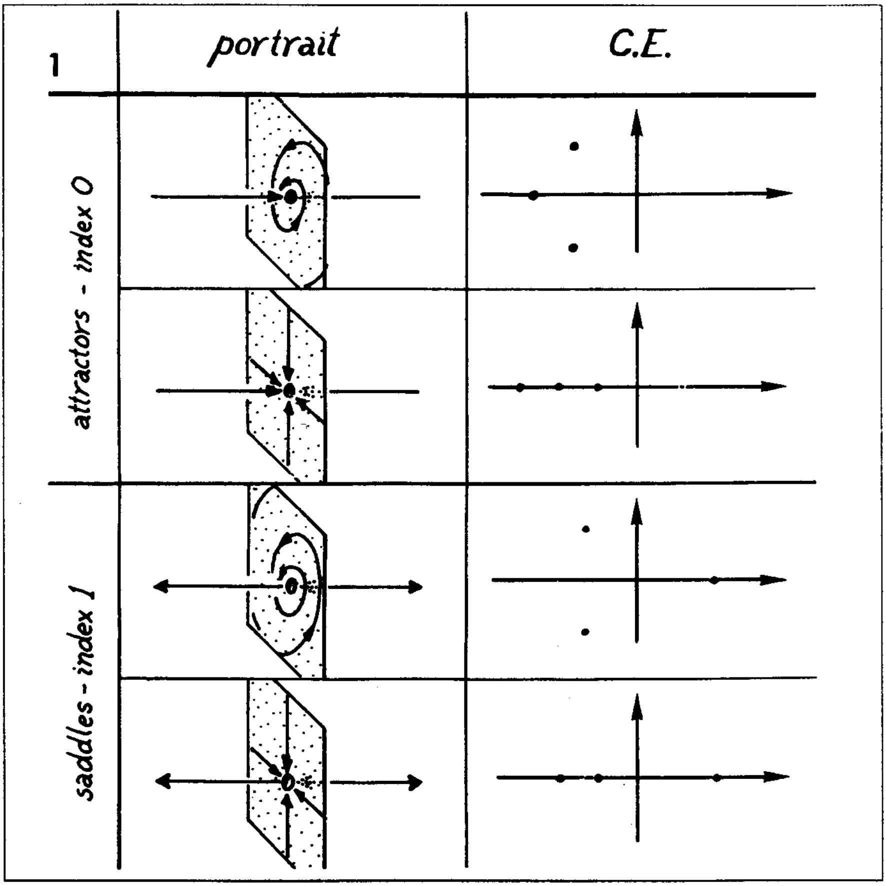


Figure 31. Characteristic exponents for elementary hyperbolic limit points in 3D phase portraits. Part 1. attractors of indices 0 and 1; and part 2. attractors of indices 2 and 3 (from Abraham & Shaw, 1982–88, © Aerial).

Poincaré Sections, Index, Characteristic Multipliers, and Hyperbolic Limit Cycles

The *characteristic multiplier* (CM) of a limit cycle is an extension of the idea of the characteristic exponent (CE) for a critical point. It is a measure of the rate at which a trajectory approaches or departs the limit cycle. Remember that the trajectories approaching or departing a limit cycle (e.g., Figure 18) can be strobed in a hyperplane perpendicular to the trajectory (Figure 20–26), called the strobe plane, or *Poincaré sec-*

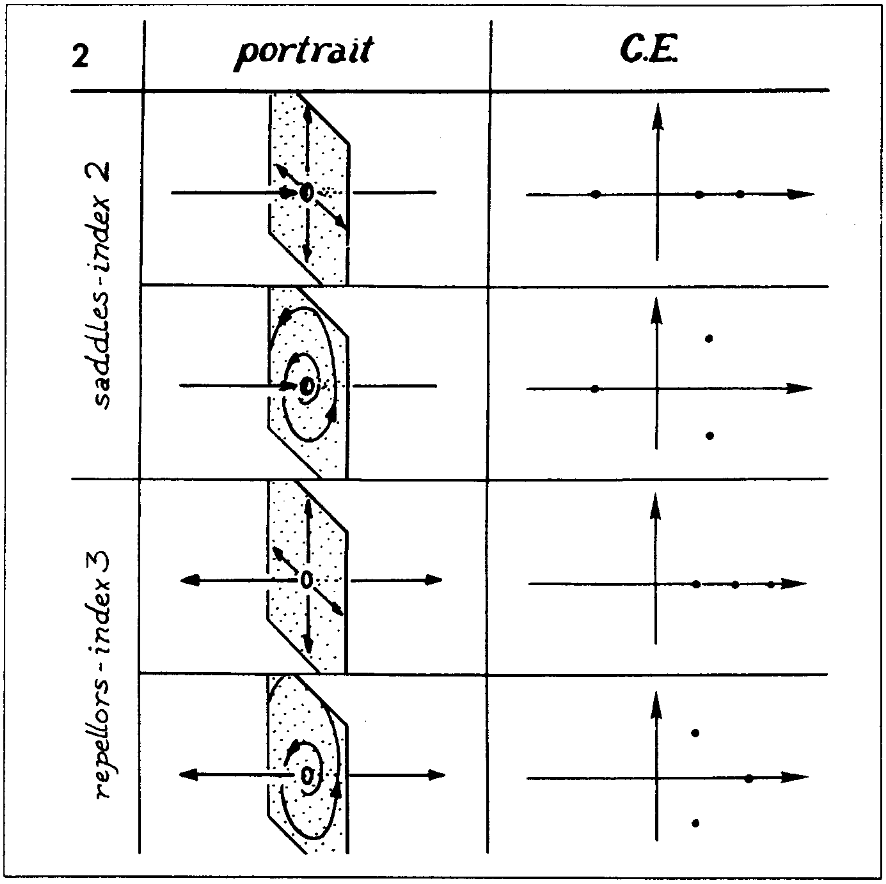


Figure 31. (continued)

tion (Rayleigh attributes such strobing originally to Plateau, 1836). For those examples, the state space was three dimensional; the strobe plane was two dimensional. For now, consider the state space two dimensional and the strobe plane one dimensional. In this context there is one CM, a real number. If you were to plot the closeness of the trajectory to the limit cycle as a function of the closeness on the previous pass of the trajectory through the section, then the *characteristic multiplier* (CM) would be the slope of this function at the limit cycle (Figure 32).

CMs can be classified (Figure 33). The CM is between ±1 if the limit cycle were attractive; greater than +1 or less than −1 if it were repulsive. It is positive when the trajectory remains on one side of the attractor in the Poincaré section; negative if it alternates between op-

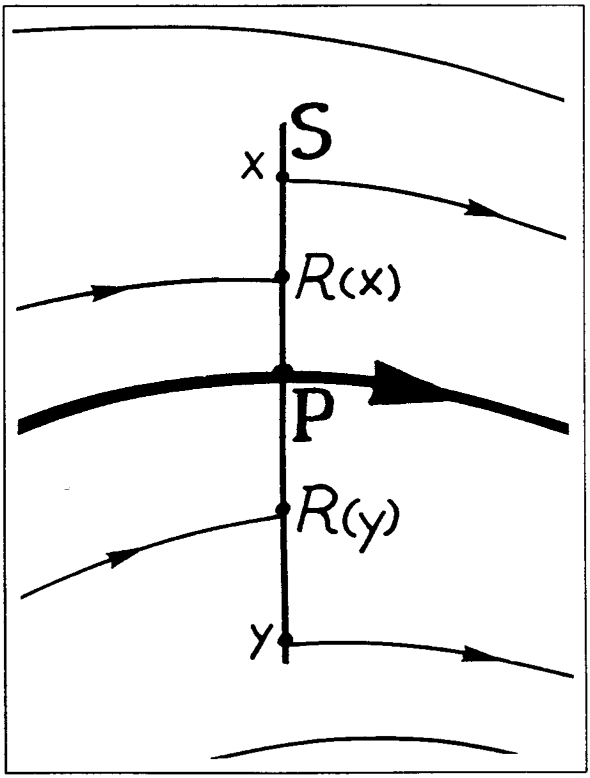


Figure 32. The first return map: start at a point x in the strobe section S above P and follow its trajectory as for the Van der Pol system; this trajectory follows around near the limit cycle; eventually, it passes through the section S again at R(x), the first return, closer to P as the limit cycle is attractive (from Abraham & Shaw, 1982–88, © Aerial).

posite sides of the attractor in the Poincaré section as if on a Mobius strip. If the CM is equal to +1, then the limit cycle is nonhyperbolic, a center where all trajectories are closed orbits cycling on themselves such as with the frictionless pendulum, buckling column, and competitive species (Figures 11, 15a, 17a). The CM cannot be 0 or infinite. The index is the dimension of the outset, seen within the strobe plane. Here, it is 1 for repellor and 0 for an attractive cycle.

3-D Limit Cycles

In general, there is one less dimension for the Poincaré section than the state space, and one CM for each dimension of the Poincaré section. So for the three-dimensional case, such as Duffing's model that included

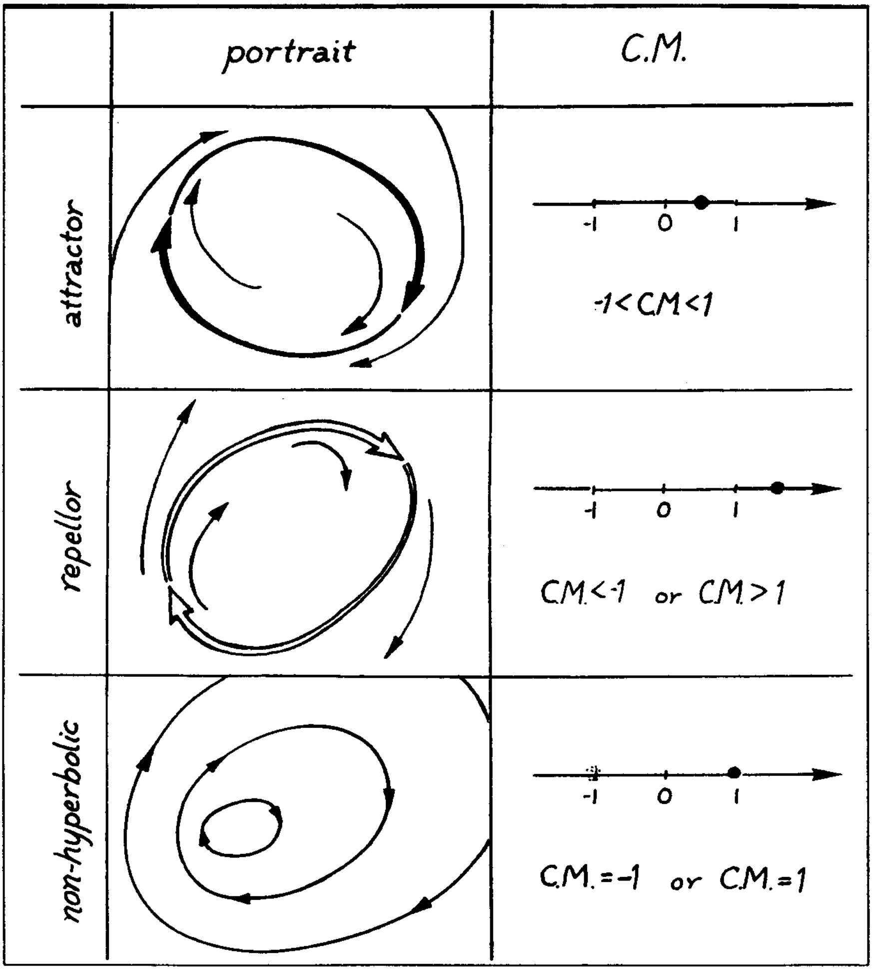


Figure 33. Characteristic multipliers for limit cycles in 2D state spaces (from Abraham & Shaw, 1982–88, © Aerial).

repellant, attractive, and saddle-type limit cycles (Figure 27) there is a two-dimensional strobe plane and two CMs. For the saddle-type limit cycle (or periodic saddle), one CM, greater than 1 (or less than −1) represents the outset (Figure 34, middle row), and other CM, between −1 and +1, represents the inset. The CMs are represented on the complex CM plane on which the unit circle is shown. CMs inside the circle represent attractors; those outside represent repellors. The nodal limit

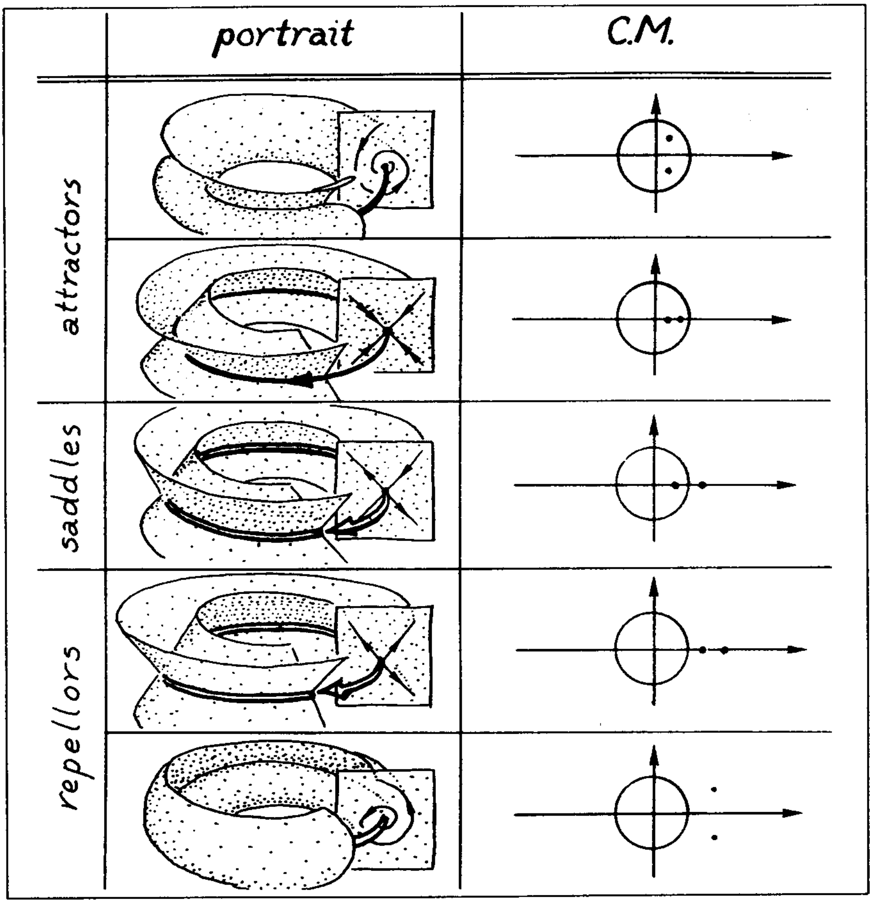
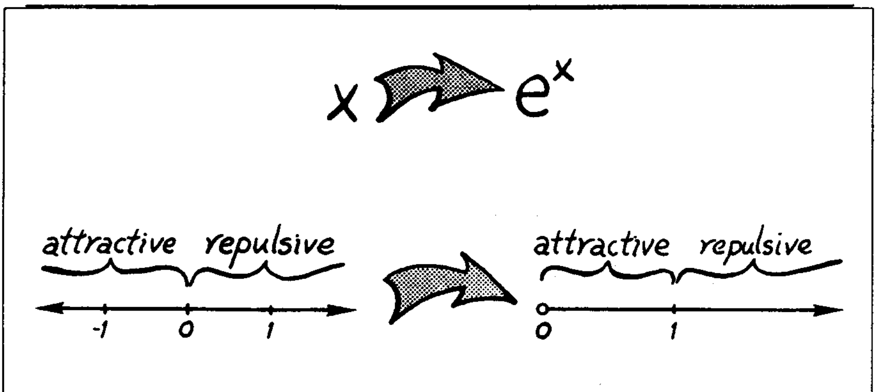


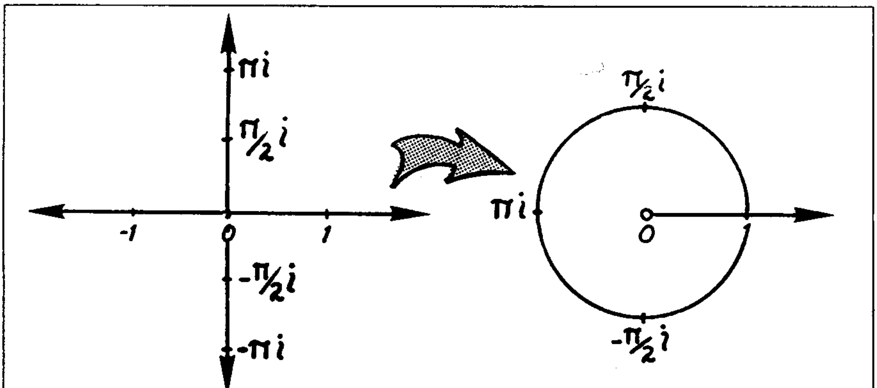
Figure 34. Characteristic multipliers for elementary limit cycles in 3d state space (from Abraham & Shaw, 1982–88, © Aerial).

cycles (Figure 34 middle three rows) have real CMs (on the horizontal axis); the spiral limit cycles (top and bottom rows) have complex components.

You may have noted a relationship between the CE plane of the 3-D point limit sets and the CM plane of the 3-D periodic limit sets. The CM plane is polar, with the magnitude of the CM representing the strength of attraction or repulsion, and the angle characterizing the degree of spiraling. The CM plane corresponds to the exponential of the CE plane (Figure 35).

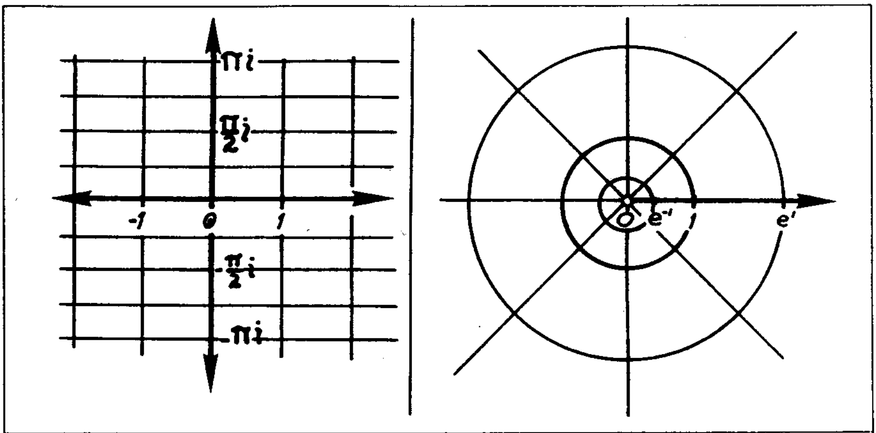


Step 1. Map the horizontal axis of the CE plane onto the positive half of the horizontal axis of the CM plane, using the natural exponential function. Note that ***zero is mapped to one***.

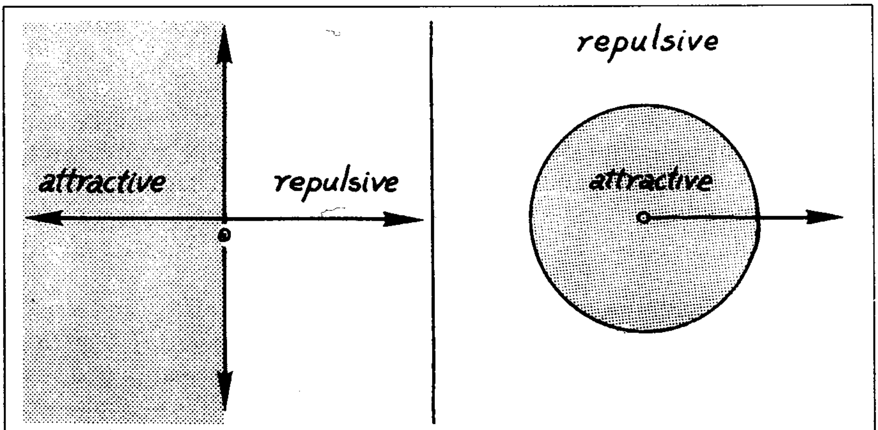


Step 2. Wrap the vertical real axis of the CE plane around the unit circle of the CM plane. ***Linear measure becomes angular measure*** on the unit circle.

Figure 35. Correspondence of the characteristic multiplier plane to the characteristic exponent plane (from Abraham & Shaw, 1982–88, © Aerial).



Step 3. Horizontal lines in the CE plane are mapped into rays in the CM plane, radiating from the origin. Vertical lines in the CE plane are wrapped around concentric circles in the CM plane. No point in the CE plane goes to the origin of the CM plane.



Note. The attractive and repulsive regions of the CE plane are mapped onto corresponding regions of the CM plane.

Figure 35. (*continued*)

Chaotic Attractors

Periodic attractors, like musical instruments, may have harmonics. Their power spectra are discrete, with positive values at frequencies that are integral multiples of the fundamental frequency, corresponding to shorter periods. Real dynamical systems are more likely not to be so perfect. The attractors may be noisy, involving orbits confined to regions in the state space that are not exact points or cycles but that nevertheless have definite periodicities and geometric structure. Their power spectra may be continuous, for example, 1/F noise or white noise. The discovery of chaotic attractors gave impetus to the move away from linear univariate experimental designs, within which one is destined to throw away most of the variability encountered in experimental measurement as random noise and to explore the use of dynamical systems as a way of reducing many-variate systems to elegant fewer-variate models that would represent the structure inherent in such variability. Thus we turn now to seeing how some historical chaotic attractors characterized such complexity.

Some Classic Examples

Homoclinic Tangles: Poincaré's Solenoid. These are chaotic limit sets of saddle type and are not always attractors. Although they are too complex to depict adequately in this brief presentation, a couple of their features help to introduce this topic. This solenoid could be considered to be like an infinite coil of wire (Figure 36a). The cutaway shows its Poincaré section as intersecting planes of insets and outsets, which are thickened curves with an infinite number of pieces. The intersections of inset planes and outset planes form trajectories of saddle type.

If you consider a pair of these trajectories and their inset and outset planes (Figure 36b that shows these planes as twisting strips), note that the inset of one intersects with the outset of the other, creating two new saddles and departing trajectories that return to the point of departure. Thus the term *homoclinic*. Poincaré despaired this complication. An infinite set of these saddle-type trajectories, some closing upon themselves around the solenoid and some never closing, comprise the complicated chaotic limit set of saddle type known as *Poincaré's solenoid*. It serves not only as an introduction to chaotic attractors but because it occurs in the forced Van der Pol system that is commonly used as a model for many forced coupled oscillators in nature (e.g., circadian), it is thus very relevant for psychological and biological modeling.

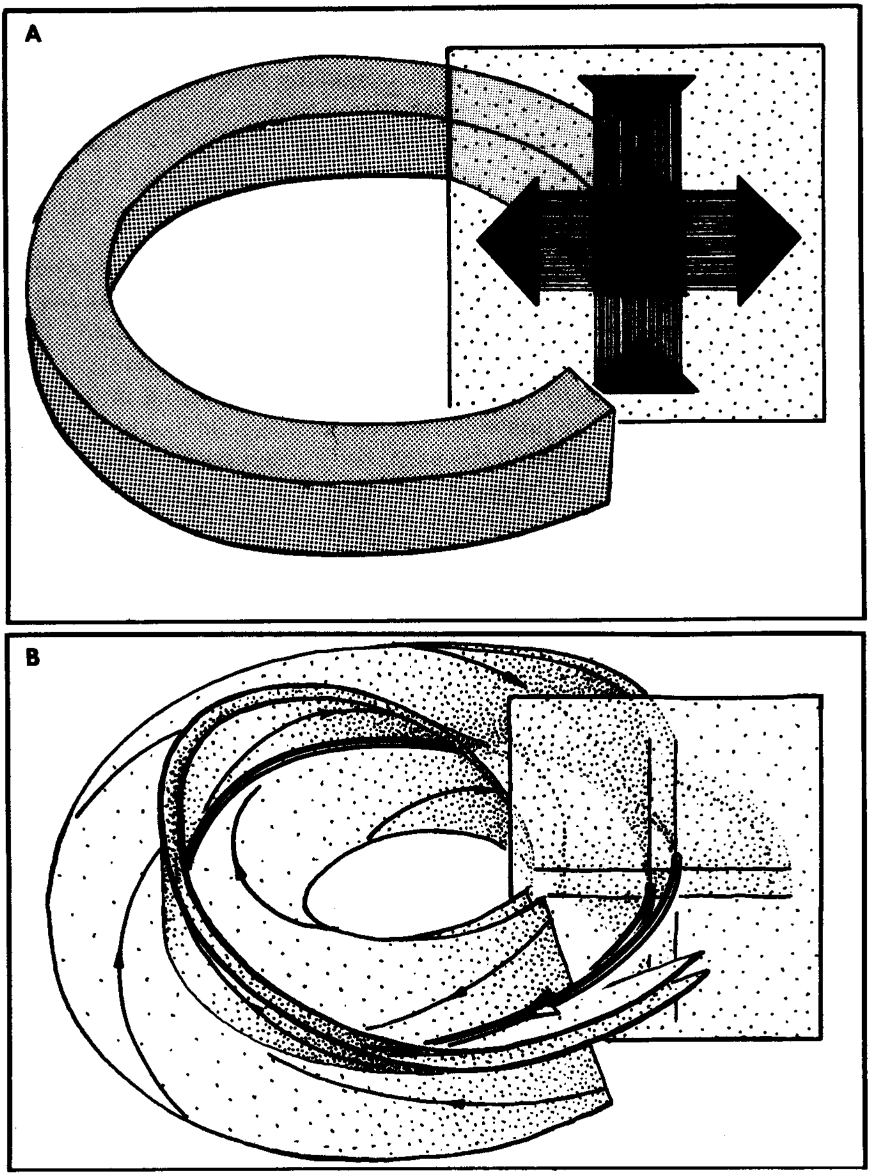


Figure 36. Poincaré's solenoid. (a) Poincaré section of Poincaré's solenoid; and (b) homoclinic saddle intersection of inset and outset planes; the saddle cycle completes two revolutions before closing; likewise the inset and outset planes, each of which twists like a Mobiüs strip (from Abraham & Shaw, 1982–88, © Aerial).

Birkhoff's Bagel. This was discovered studying the classic forced Van der Pol oscillator (Birkhoff, 1932; Cartwright & Littlewood, 1945; Charpentier, 1939, 1946; Holmes, 1977; Levinson, 1944; Shaw, 1980). A driving oscillator operates at a high frequency relative to the driven oscillator (e.g., 2 : 1). In the Poincaré strobe section, it will take many passes of the trajectory through the plane to characterize this strobe section of the attractor that appears like a continuous closed curve *of Birkhoff* instead of the single point of an isochronous system or a few points of an harmonic (Figure 37).

The most interesting feature of this curve is that you can't predict exactly the long-run future of a trajectory from knowing about a trajectory starting from a nearby initial point. Errors are amplified. This apparently random nature, along with the relative complexity of the attractor, has given rise to descriptive terms such as *strange* and *chaotic*. Changing the strobing phase completes the ring model, and the bagel emerges. As the phase around the bagel progresses, beaks appear, elongate, and become pleats pressed flat against the surface. The bagel consists of an infinite number of pleats pressed flat against the thickened toroid, giving rise to its *fractal* microstructure, characteristic of known chaotic attractors, and associated with its unpredictable behavior.

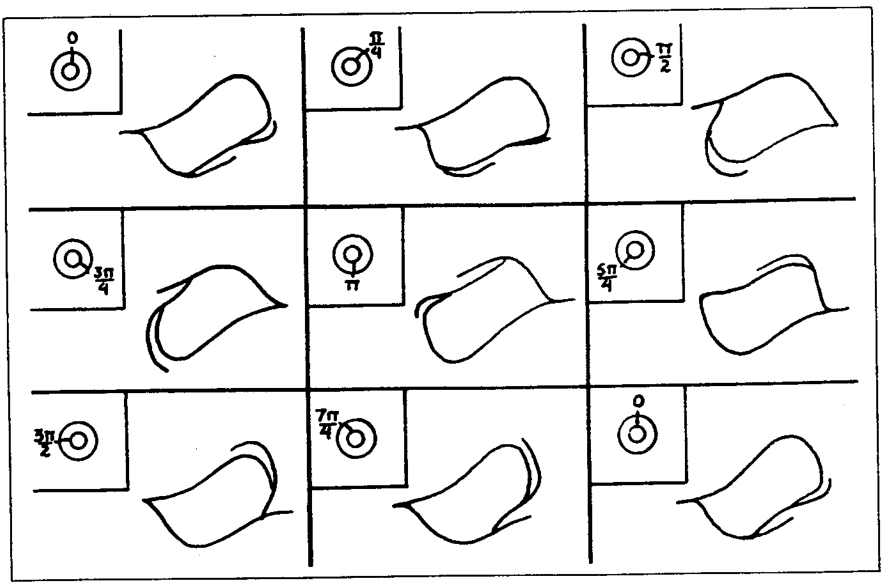


Figure 37. Birkhoff's bagel: Poincaré sections at different phases of the driving oscillator (from Abraham & Shaw, 1982–88, © Aerial).

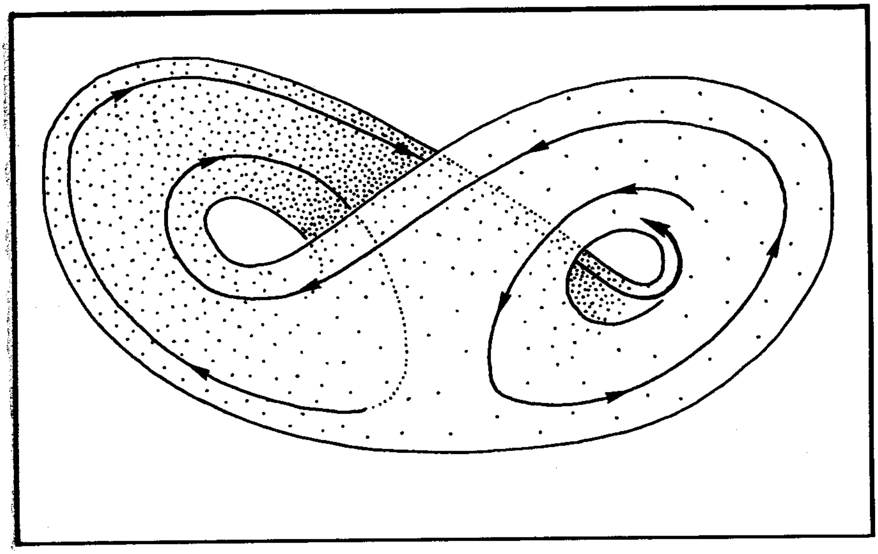


Figure 38. Lorenz's mask (from Abraham & Shaw, 1982–88, © Aerial).

Lorenz's Mask. Lorenz attempted to model air currents in the atmosphere, as represented by a sea of Bénard cells packed in a hexagonal lattice, using a dynamical system derived from fluid dynamics. Using computer simulation, he found an interesting attractive object (Figure 38) in this 3D system. The object is strictly determined, but its trajectories are highly erratic, orbiting one hole for a while, then jumping to the other. Its chaotic behavior dismayed Lorenz who had hoped to use it to predict the weather.

Rössler's Band. Inspired by Lorenz, Rössler created a simpler system in 3D. The successive crossings of a trajectory through the strobe plane seem to fuse in the Poincaré section as an arc. Changing the phase of the strobe plane to view progressively around the ring model reveals *Rössler's band* (Figure 39a). Like the bagel and the mask, it, too, has a thickened surface, a fractal microstructure, and limited predictability of the attracted trajectory. These attractors have highly diverse applications, and many others may yet be found. Unpredictability means that small differences between trajectories at one point in time may be amplified to very large differences significantly later, as we now describe.

Characteristics of Chaos

Unpredictability and Sensitivity to Initial Conditions. A theoretical system, such as the Duffing oscillator or the Rössler band, is

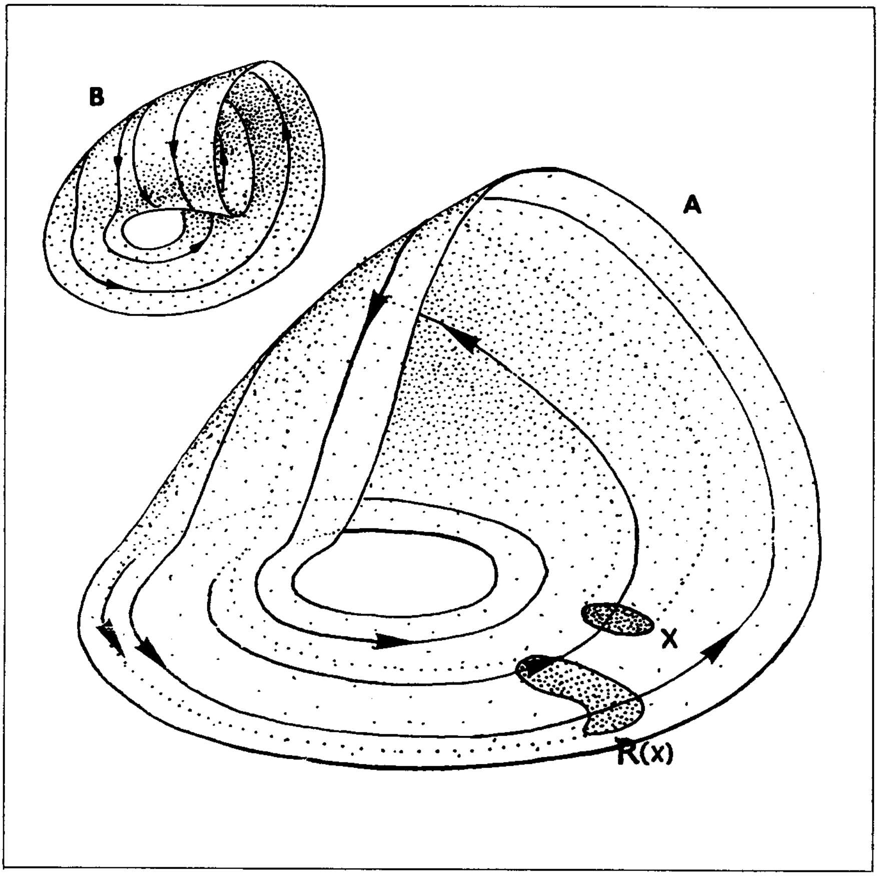


Figure 39. Rössler's attractors: (a) Rössler's band: an area of trajectories in a strobe plane, x, has expanded after the first return of those trajectories to area R(x); they continue to expand; and (b) Rössler's funnel (from Abraham & Shaw, 1982–88, © Aerial).

strictly deterministic. If you know its position in the state space precisely at any given time, you can predict exactly its position at any other time. However, running simulations on mechanical or electronic devices or measuring properties of real systems in nature involves problems of resolution and uncertainty. Thus the exact position at any given moment can never be known exactly. Furthermore, the divergence property of chaotic attractors ensures that a small difference at any given moment in the position of two trajectories will be amplified by exponential growth and will become a large difference at a later time. This is called *sensitive dependence on initial conditions* (Ruelle, 1980).

If you are told only which half of the state space the system is currently in, you have precious little information (1 bit in information theory, which measures in base 2 logarithms). But if you are told exactly the point the system occupies, you have infinite (all the) information about its current condition. In experimental practice, you normally know that the system is within a small region, such as the small area in a strobe plane (x shown in Figure 39). That region contains many trajectories, and if they are followed approximately one cycle, when they next pass through the strobe plane, they describe a larger region (R(x) in Figure 39). Thus there is a *loss of information* about the future. Proceeding iteratively, there is continued information loss; continued spreading of the recurrence map, R. This is the meaning of unpredictability in the context of chaotic attractors: Any small error in the measurement of the current state eventually leads to ignorance of the position of the trajectory within the attractor in the future. You know it's in the attractive region and can characterize its motion, but you can't predict exactly where it will be at any given moment.

Divergence and Information Gain (Shaw, 1984). Similarly, the divergence of trajectories leaving a repellor implies a loss of information at a later time compared to measurement at an earlier time. A new measurement at the later time would then imply a gain of information because if we now extrapolate backwards in time from the new measurement, we get convergence or contraction, a smaller area than yielded by the initial measurement made at that time, using the same measuring instruments. We gain information about the initial state. Thus *diverging flows* provide increasing information about initial states in the past. *Information is gained* about the past.

Conversely, near a point attractor, the flow of trajectories converge. Here earlier distinct points eventually become indistinct experimentally, and extrapolation backwards tells us nothing about initial states. Thus *converging* flows provide decreasing information about past initial states. Information is *lost* about the past.

Expansion, Compression, and Characteristic Exponents. Near chaotic attractors, divergence and convergence (expansion and compression), occur simultaneously. How can this apparent paradox of simultaneous compression to the attractor and expansion along the attractor be the case? How can there be continued expansion within a bounded region? Remember the periodic saddle in 3D (Figures 21, 34, and 36). It attracts in some directions (near trajectories approach) and repels in others (near trajectories depart). Also remember that the chaotic attractors consist of thick surfaces (from an infinite number of surfaces packed closely together). Each trajectory is saddlelike, with the attractive direction crossing the thick surfaces, and with the repelling direction tangent to the surface. Thus there is convergence toward the attractor and divergence along it, saddlelike. If you try following separate trajectories around the Rossler band, for example, you can follow this divergence for a while (Figure 39), but, of course, they stay in the bounded attractor, as repeated folding prevents global expansion. The outsets of the repelling trajectory remain within the attractor itself. This leads us to a closer examination of the microstructure of this thick surface.

Fractal Microstructure. To view the microstructure, a Poincaré section is made, revealing repetitive folding of a surface into infinitely many layers of the thick surface of the attractor (Figure 40a). But it's not quite the multiple U-folding of a line either. To get a better view, cut the Poincaré section with another plane perpendicular to it (Figure 40b), the *Lorenz section*, and examine the trajectories at the line of intersection (Figure 40c).

The number of dots representing the trajectories crossing is infinite but reveal a definite pattern. This pattern is not known exactly but appears to be as if created by a *Cantor process*, which could be imagined as the successive decimation of a line, as by iteratively removing middle thirds or middle fifths (Figure 40d). The remaining segments are examples of a *Cantor set*. Expanding these dots into the layered planes represents the thick surfaces of the attractors, and the Cantor nature of the layering, which is called the *fractal microstructure* of the thick surface. There are many useful measures of the complexity of this fractal microstructure, such as the *fractal dimension*. The study of the relationship between these measures of fractal dimension and the CEs measuring divergence of trajectories along the attractor is an active area of investigation. But this is where the diverging trajectories go, that is, into the infinite complexity of the microstructure within the thick surface of the attractor.

Noisy Power Spectra. Chaotic attractors yield noisy power spectra with varying degrees of periodicity evident depending on the attractor. The spectrum contains less information than the phase portrait or the time series. Nonetheless, it may provide clues to assist in identifying appropriate dynamical models. A summary of exemplary attractors, their phase portraits, time series, and power spectra are shown in Figure 41).

Global Features

Global generic features of dynamical systems are important as they relate to the stability of the system. We discuss but a few of them here.

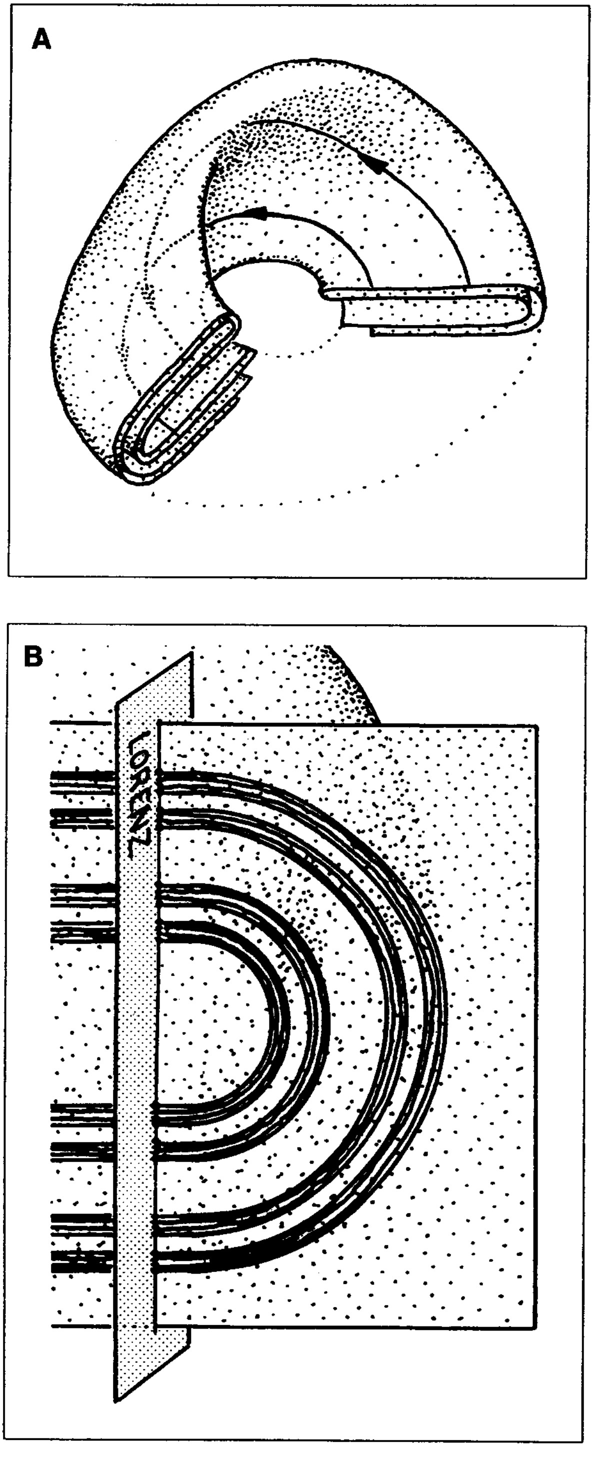
Generic Local Properties

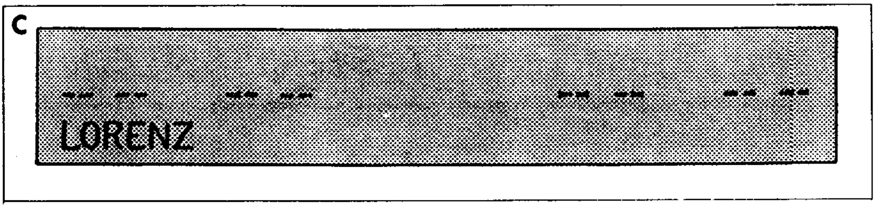
Typical, garden-variety, dynamical systems possess *generic* properties. For example, here are a couple of these generic properties: A dynamical system has property G1 if all of its critical points are hyperbolic. (Recall Figures 29–31 for definition and examples). These are points that have no CEs on the imaginary axis. Several types of elementary critical points can coexist in a given system (Figure 42a). A dynamical system has property G2 if all of its limit cycles are elementary (hyperbolic). These are cycles that have no CMs on the unit circle. (see Figures 33 and 34 for definition and examples.) A braid with several limit cycles on the 2D torus is an example (Figure 42b).

Transversality and Tangles

Transversality is a property of a *saddle connection* existing at the intersection of the outset of a saddle (*donor*) and the inset of a saddle (*receptor*). The trajectory comprising this connection may be *heteroclinic* (the donor and receptor are different) or *homoclinic* (from a saddle to itself). Transversality means that the intersection must be between two planes that cross each other as cleanly as possible (e.g., the intersection may not be tangential). In the heteroclinic case in 3D, these conditions can be met only from saddle points of index 2 to saddle points of index 1 or to saddle cycles, or from saddle cycles to saddle points of index 1 or saddle cycles. These are the four heteroclinic generic saddle connections in 3D (Figure 43). In the homoclinic case, the only generic (transverse) connection is with a saddle cycle. Homoclinic connections are very important features of a phase portrait. No transverse connections are possible in 2D. Heteroclinic intersections are always quite complex and are often referred to as *tangles*. Some chaotic attractors may be dissected into tangled outsets. For example, the Lorenz mask is a chaotic attractor in 3D with three saddle points: two donors of index 2 with spiral outsets and a receptor of index 1 with a nodal inset.

Consideration of heteroclinic intersections leads to a third generic property, G3, which requires that all such connections be transverse. Note that in 2D, G3 can be satisfied only if there are no saddle-to-saddle connections. (The model of the frictionless buckling column had ho-


A
B
LORENZ



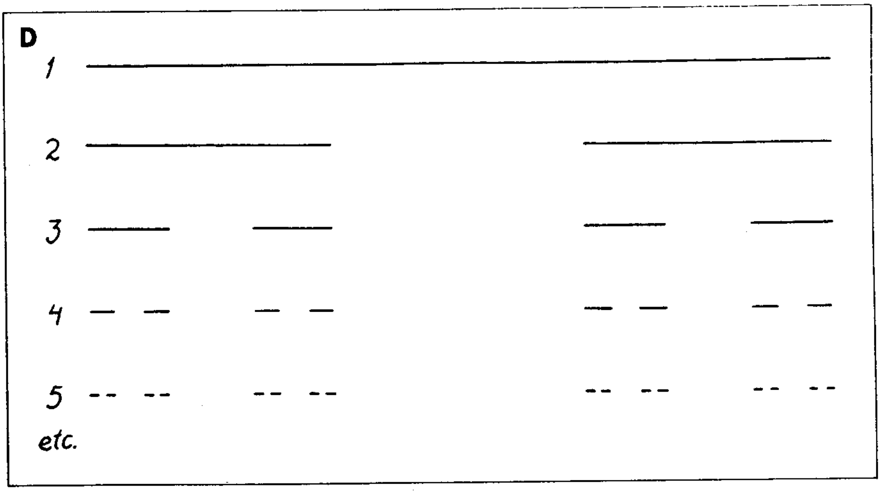


Figure 40. *(continued)*

moclinic trajectories violating property G3, whereas the model of the dissipative buckling column satisfied property G3, see Figures 15c and 15d). In 3D, G3 is satisfied only in the four heteroclinic cases (Figure 43) and the one homoclinic case. Note that the connections of the generic saddle cycles, like the generic saddle points, are hyperbolic.

Structural Stability and Peixoto's Theorem

Topological equivalence of two phase portraits means there is a *homeomorphism* (continuous "rubber sheet" deformation) of the state space

---

Figure 40. Fractal microstructure: (a) Poincaré sections of Rössler's attractor showing the apparent folding and layering of the thick attractive surface; (b) Lorenz section cutting through the Poincaré section; (c) Lorenz section rotated showing dots as cross-sections of trajectories within a layer of Rössler's band; the number of dots is infinite but follows a cantor or fractal process of iterative decimation of filling (infra); and (d) Cantor's process of iterative decimation, removing the middle third of the line segments progressing downward (rules other than that of the middle third process may be used) (from Abraham & Shaw, 1982–88, © Aerial).

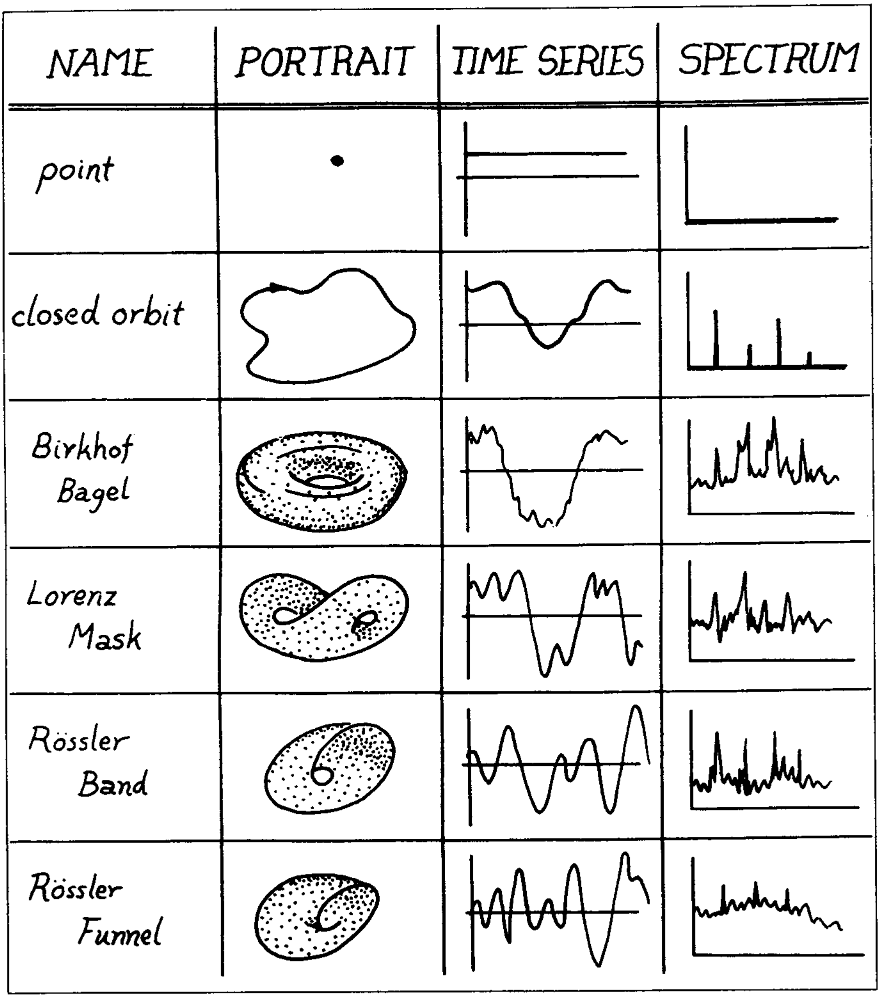


Figure 41. Summary (from Abraham & Shaw, 1982–88, © Aerial).

that maps one of the portraits to the other, preserving the arrow of time on each trajectory. A *deformation* of a dynamical system is simply the adding of a weak vector field (a delta perturbation) to the original one; if the result is topologically equivalent within a specified deviation, epsilon, then there is *epsilon equivalence*.

A dynamical system (vectorfield) has the property S of *structural stability* if all delta perturbations of it have epsilon-equivalent phase portraits. That is, the addition of some other vectorfield does not affect the system in any significant way (Figure 44a). Dynamical systems in

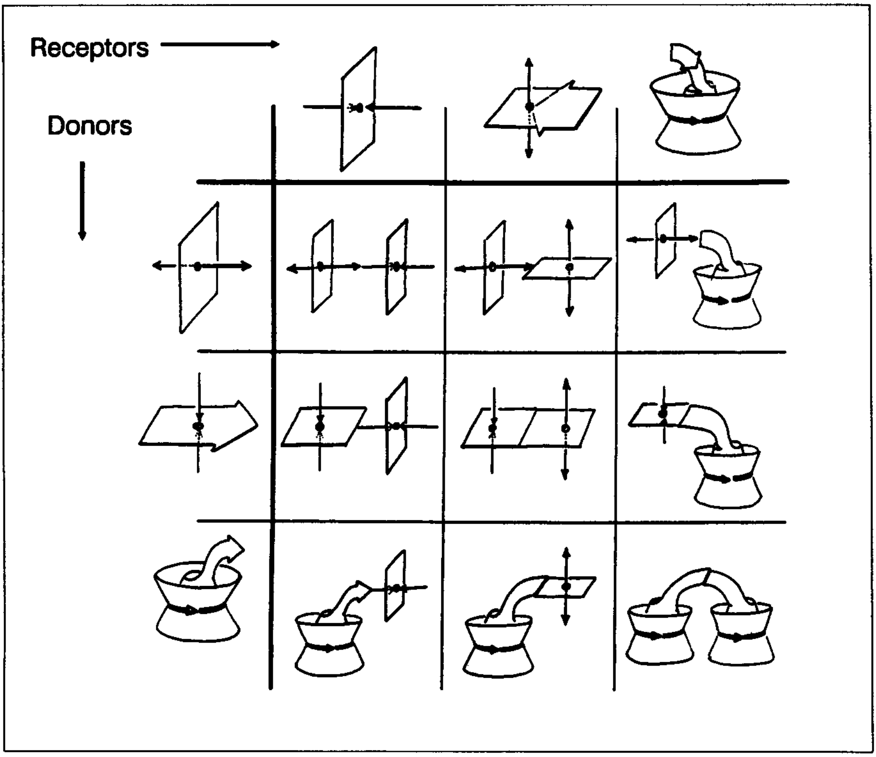


Figure 43. Heteroclinic saddle connection in 3D; possible donors include hyperbolic saddle points and cycles; of the nine possibilities, the four in the lower right are the generic ones (from Abraham & Shaw, 1982–88, © Aerial).

2D are structurally stable because they possess generic properties (G1-3 and some others), as stated by a famous theorem of Piexoto (1961). Systems violating these properties are unstable, and the addition of a small perturbation changes the system significantly. For example, a center is notoriously unstable. Only very weak friction or drag is required to change the models of the buckling column, the fish populations, or the rings of Venus (Figures 15, 17, 44b).

Bifurcations

As we have seen, a dynamical system is a vectorfield representation of the habitual tendencies of some variables to change. It may be represented visually or by differential equations. In either case, its *phase portrait* may be constructed by laborious integration. Such systems

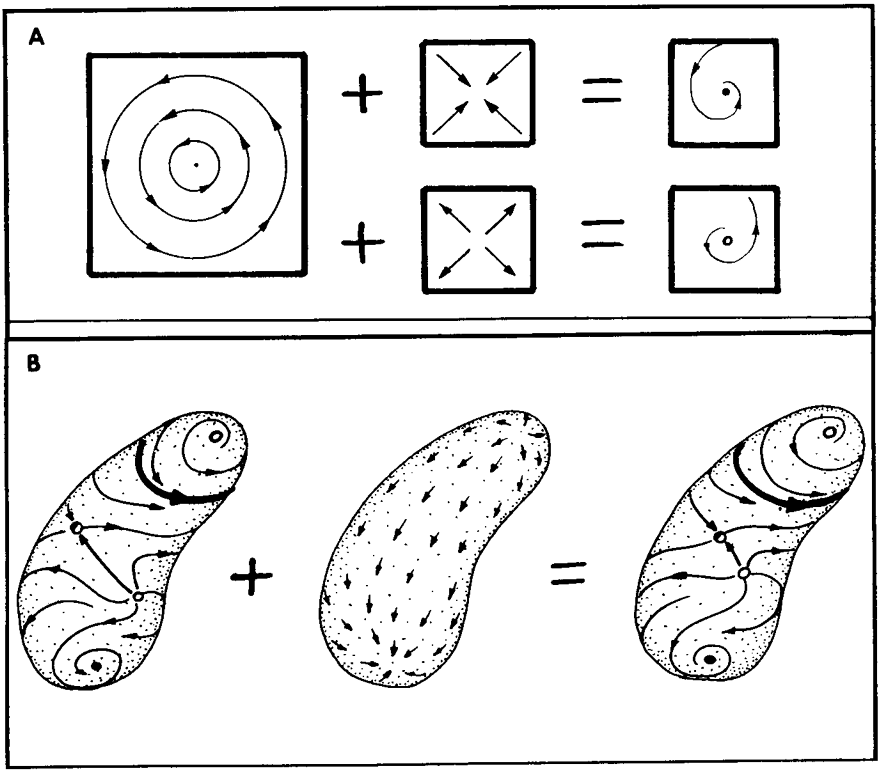


Figure 44. Structural stability: (a) a center may be perturbed into either a point repellor or a point attractor, depending on the inclination of the perturbation; (b) generic properties ensure structural stability (from Abraham & Shaw, 1982–88, © Aerial).

usually contain some parameters, and the phase portrait maintains a particular structure as long as the parameters remain fixed. Some of these parameters may be subject to control in an experimental situation and hence are called *control parameters*. A dynamical system with control parameters is called a *dynamical scheme*. Often the phase portrait changes in an insignificant fashion with changes in the control parameter over some small range of their values, but major changes may suddenly occur. The phase portraits become topologically nonconjugate, nonequivalent. These significant changes are called *bifurcations*. The configurations of the attractors as a function of the control parameters are summarized in a *response diagram*. The response diagram is the important map of a dynamical scheme, just as the phase portrait is the useful map of a single dynamical system. Generally, a response diagram may be built by piecing together maps of atomic events (generic bifurcations) provided by mathematical theory. Mathe-

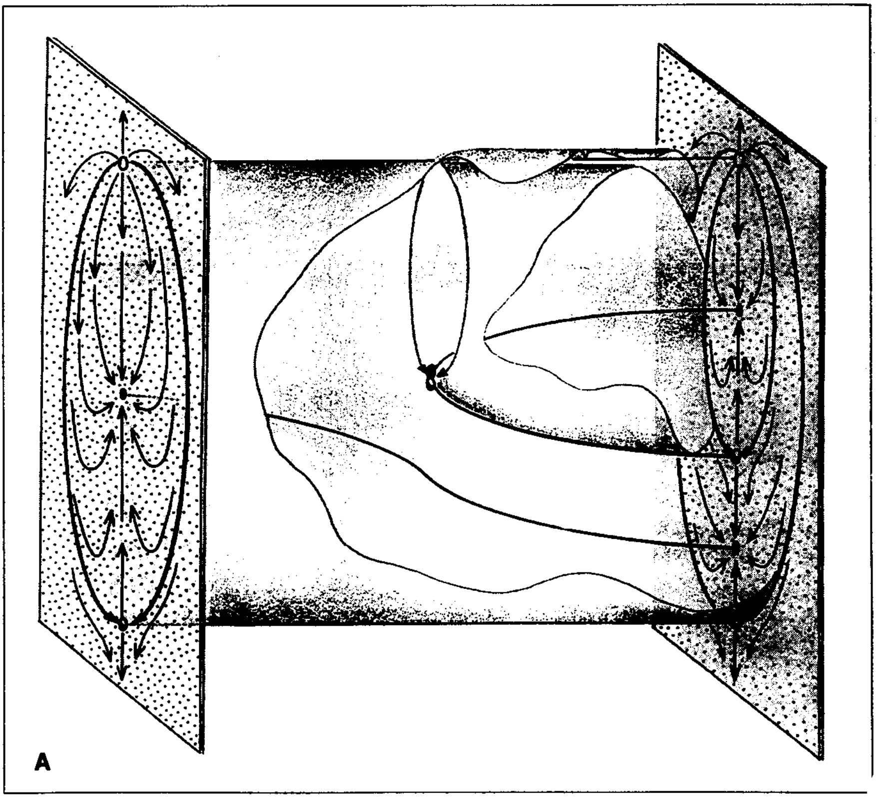


Figure 45. Response diagrams of creation and annihilation: (a) creation of a static attractor, its basin, and a cyclic saddle which is a separatrix between the two basins (going from left to right on the horizontal control parameter), or their annihilation (going from right to left); and (b) creation of a periodic attractor and repellor outside of original cyclic attractor within a repellor (left to right along control parameter; annihilation going from right to left; the cyclic repellors are actual separatrices; the central point repellor is a virtual separatrix) (from Abraham & Shaw, 1987, © Plenum Press).

maticians are now trying to classify the generic bifurcations that may occur in response diagrams. Here are some known generic bifurcations with one control parameter. There are three types: catastrophic, subtle, and explosive.

Catastrophic Bifurcations

Catastrophic bifurcations occur when a limit set appears or disappears, out of or into the blue, so to speak. There are three types, depending on the type of attractor created: static, periodic, and chaotic.

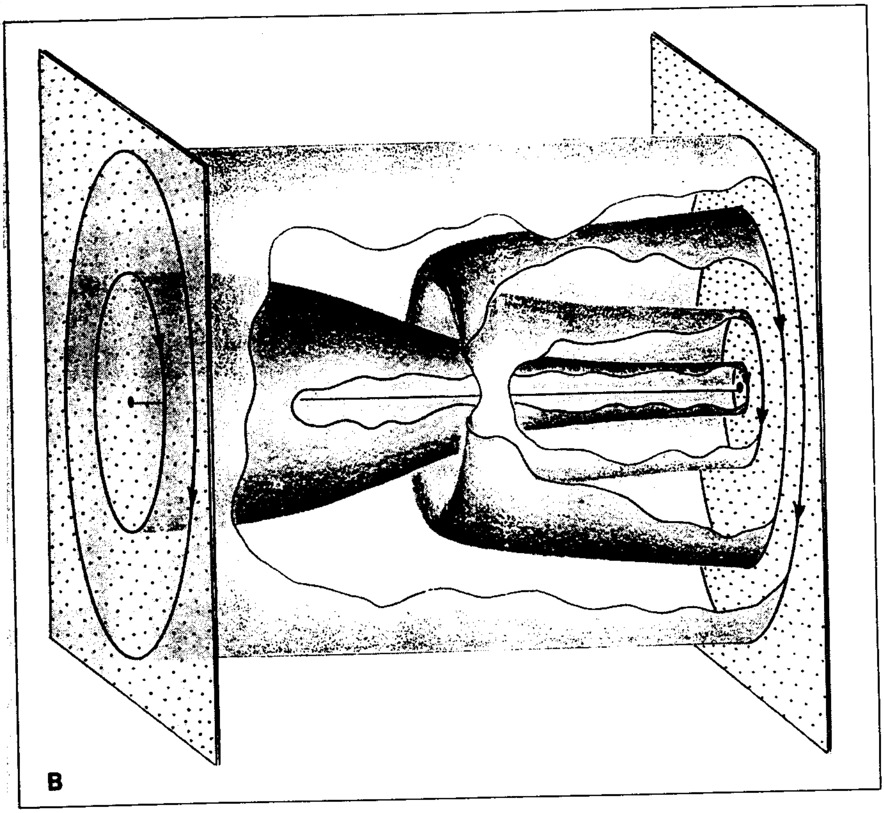


Figure 45. (*continued*)

Creation and Annihilation of a Static Attractor. This form of creation applies to the appearance of a new static attractor within a basin as a control parameter changes. It might occur within the basin of a point attractor. (Figure 45a). As the control parameter changes, the locus of these points and their trajectories may change a bit, and then suddenly, a second attractive point with its own basin and separatrix appears. The moment of occurrence is called the *bifurcation point* in the control space. The event may be called *static creation* or the *fold catastrophe*. When the control parameter is changed in the reverse direction, there is a corresponding disappearance of the attractive point that is called *annihilation*. In this event, the attractive point drifts toward the saddle point in its separatrix, collides, and vanishes into the blue along with its separatrix and basin!

Creation and Annihilation of a Periodic Attractor. Analogously, there could initially be a periodic attractor within a basin bounded by a periodic separatrix on the outside and a repelling point (virtual separatrix) on the inside. As a control parameter changes, there is the catastrophic appearance of a second periodic attractor outside the first one, and there is a periodic separatrix between them occurring at the bifurcation point (Figure 45b). This is but one possibility for the creation of an oscillation. Running the control parameter in reverse annihilates the second attractor, separatrix, and basin.

Creation and Annihilation of a Chaotic Attractor. This is the catastrophic appearance of a chaotic attractor. For example, there might be two Rössler bands in 4D, an attractor and a saddle. The inset of the chaotic saddle bounds the basin of the chaotic attractor, a *fractal separatrix*. Changing the control parameter can result in annihilation, as the attractor drifts toward the saddle and collides.

Subtle Bifurcations

With subtle bifurcations, an existing attractor changes rather than appearing out of or disappearing into the blue. The type of the attractor may change or stay the same (see Table 2 and Thompson & Stewart, 1986).

Excitation of an Oscillation. A point attractor in 2D destabilizes, as its CE is crossing the imaginary axis. At the bifurcation point, a tiny periodic attractor is born that grows in amplitude a the control parameter continues to change. (Figure 46). It is also called a Poincaré-Andronov-Hopf bifurcation, or, more commonly, a *Hopf bifurcation.*

There is a possible application in learned behavior under interval schedules where the control parameter might be the variability of the reinforcement–reinforcement interval (going from random to fixed). If the appearance of scalloping in the cumulative response recording (change in the interresponse times from fairly equal to systematically

Table 2. Type of Subtle Bifurcation (Attractor Type Following Bifurcation Event)

Type prior to change	Static	Periodic	Chaotic
Static		Hopf	Ueda
Periodic	Reverse Hopf	Octave jump Neimark	Excitation of chaos
Chaotic	Reverse Ueda	Relaxation of chaos	

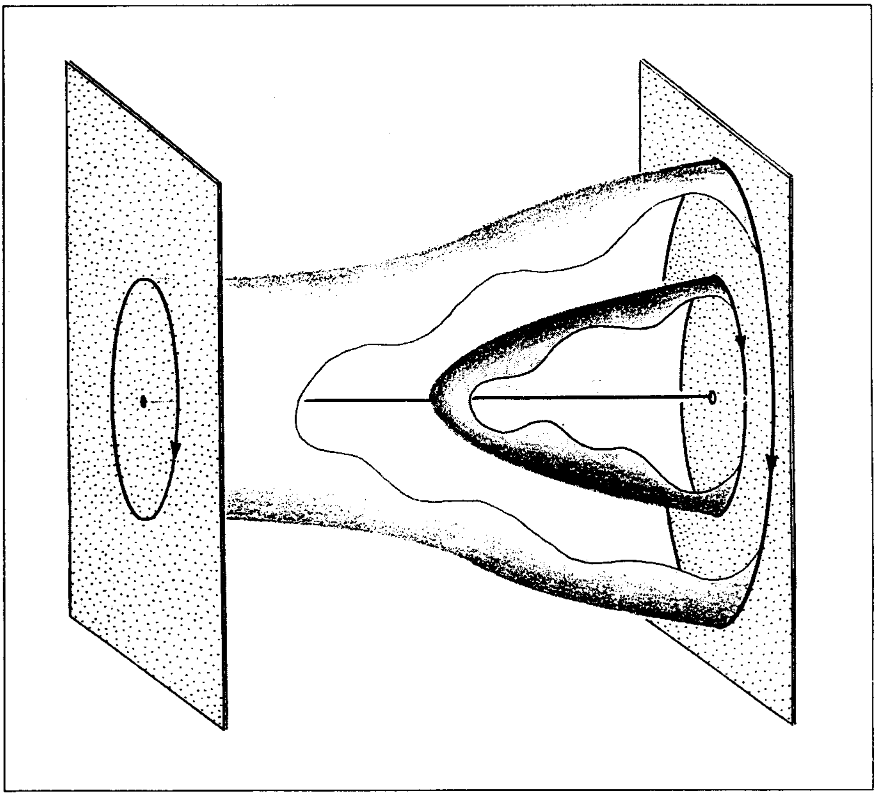


Figure 46. Poincaré–Andronov–Hopf bifurcation: the point attractor on the left bifurcates to a periodic attractor located on the parabolically expanding cone on the right; the cyclic repellor remains outside the attractor as an actual separatrix; the point repellor appears at the bifurcation and is a virtual separatrix (from Abraham & Shaw, 1987, © Plenum Press).

decreasing as the reinforcement time approaches) were rather sudden or at least well-behaved, it could represent a change from a static to periodic condition.

Excitation of Braids. Excitation of a torus occurs when a periodic attractor in a 3D basin destabilizes, through a transit of its CMs through the unit circle. A thin, attractive torus is emitted by the limit cycle, which becomes a central repellor within the new torus. The amplitude of the torus increases parabolically as the control parameter continues to change. On the attractive torus is a braid of periodic attractors. These fluctuate among many different, topologically inequivalent types of braids (Figure 23b) immediately after the creation of the torus. This multiple bifurcation event, sometimes called a *Neimark bifurcation*, includes a Cantor set of bifurcation points in the control space. Such bifurcation events are also called *thick bifurcations*. This bifurcation could provide a model for the genesis of Saturn's braided rings.

Octave Jumps. The octave jump is another example of a subtle periodic catastrophe (Figure 47). Changing embouchure tension on a brass mouthpiece, larynx tension in yodeling, or exhalation strength on a flute mouthpiece are common examples in music. Period doubling and subharmonics have appeared in frequency-driven visual evoked potentials (Makeig & Galambos, 1982) and EEG frequencies as well. Many common behavioral cycles involve frequency and period doubling.

Excitation of Chaos. This is a class of bifurcations in which a simple point or periodic attractor becomes a chaotic one, or one kind of chaotic attractor becomes another. Remember, this is how Rob Shaw, in analog simulation, first showed the Birkhoff Bagel while playing with the ring model for the Van der Pol forced oscillator creating the pleats that folded onto the torus (Figure 37). Before the bifurcation, there is a braid of periodic attractors on the torus, and afterwards they become lost amid the thick toroidal chaotic attractor. Another example is that of octave cascades (periodic excitations). Successive period doubling and subharmonics are eventually followed by the onset of chaos, a situations that seems to have been observed in the stimulation of cardiac cells (Guevara, Glass, & Shrier, 1981) and dopamine neurotransmission (King, Barchas, & Huberman, 1984). Many other scenarios for the subtle excitation of chaos have been established.

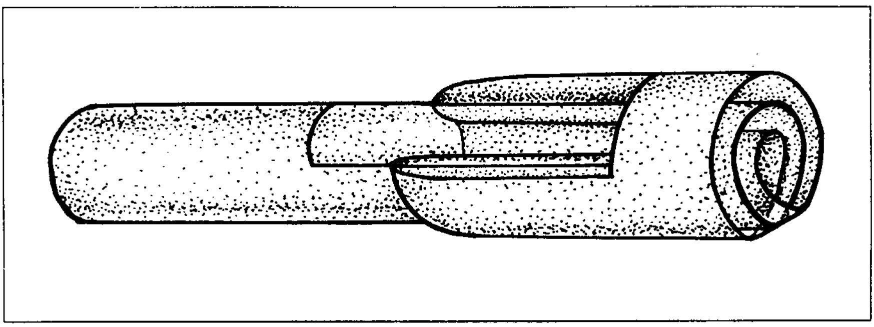


Figure 47. The octave jump: a single periodic attractor on the Mobiüs strip with rotation number 1 bifurcates to a two cyclic attractor with rotation number 2 and with a cyclic repelling separatrix between its lobes (from Abraham & Shaw, 1982–88, © Aerial).

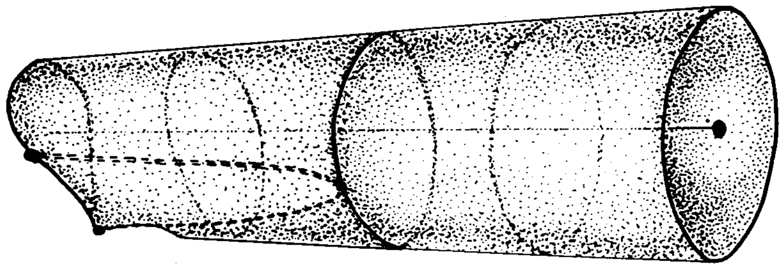


Figure 48. Explosive bifurcation: the outset of the saddle point forms a virtual separatrix in the basin of the point attractor below it; the saddle and attractor converge at the bifurcation point as the control moves to the right, with the point attractor taking over the loop of the separatrix as a cyclic attractor; a point repellor in the center is also a virtual separatrix (from Abraham & Shaw, 1982–88, © Aerial).

Explosive Bifurcations

These occur with a sudden change in magnitude, such as a small attractor suddenly becoming large. A prototypical example is a point attractor that explodes into a periodic attractor (Figure 48).

Hysteresis

Hysteresis occurs in a response diagram with multiple bifurcations of the catastrophe type. The original example, due to Duffing, has two folds. Moving the control parameter back and forth creates a "hysteresis loop" in the response trajectory (Figure 49). Examples of hysteresis are encountered in classical psychophysical measurement, especially with the method of limits. Hysteresis commonly occurs with catastrophic bifurcations, rarely with subtle or explosive bifurcations.

Bifurcations with Two Control Parameters

The *cusp catastrophe* is the best known example (Figure 22b) of bifurcations with two control parameters. The bifurcation diagram for the static cusp catastrophe (Figure 50) shows a one-dimensional portrait (vertical line) with two point attractors and a point repellor in between them, if both control parameters are in the area of the cusp and but one attractor is elsewhere. Hysteresis occurs. There is an analogous peri-

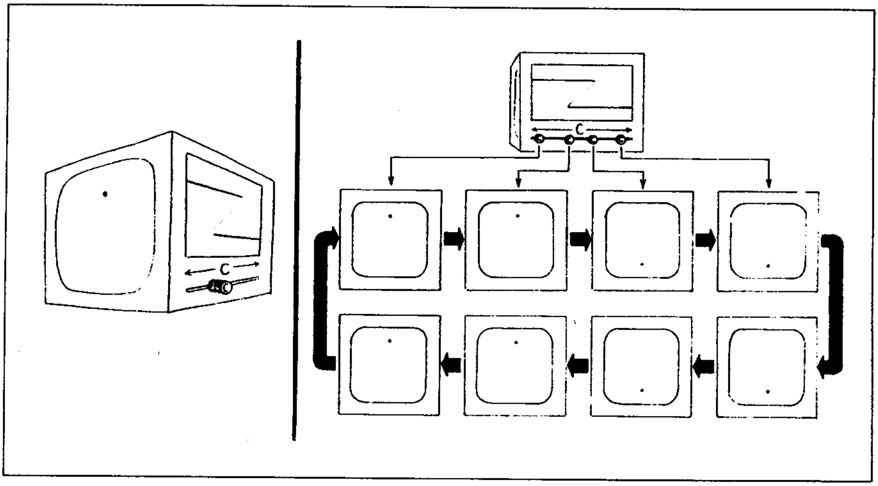


Figure 49. Hysteresis loop: as the knob starts on the left, the focal point attractor is high but jumps down as it gets to the bifurcation point; moving the control knob from right to left, the attractor jumps back up, but at a new bifurcation point to the left of the one for the right-going series (from Abraham & Shaw, 1987, © Plenum Press).

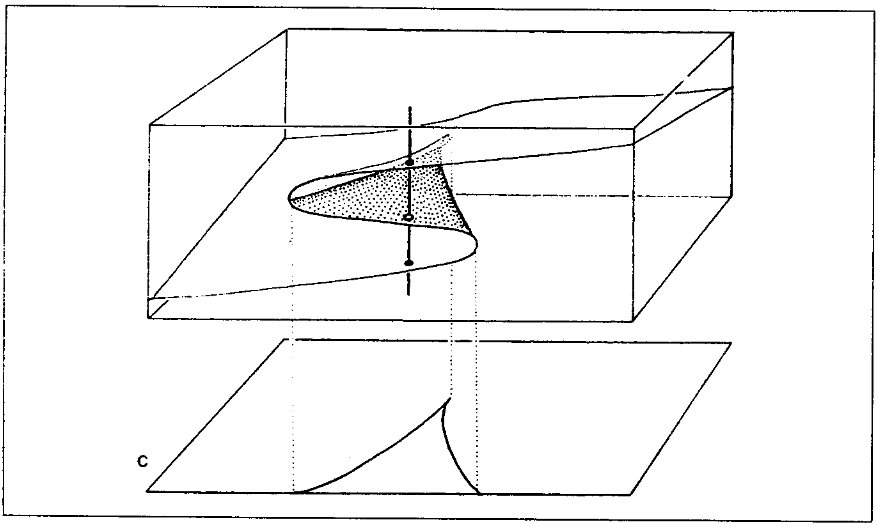


Figure 50. Static cusp catastrophe: the state space is 1D (vertical); the horizontal control plane is 2D; the resulting response diagram is thus 3D; within the cusp there are two point attractors separated by a point repellor; outside the cusp there is only one attractor; a frontal plane within the cusp displays the hysteresis loop of the double fold (from Abraham & Shaw, 1987, © Plenum Press).

odic cusp catastrophe in which a periodic attractor bifurcates into two periodic attractors and a saddle cycle whose inset is a separatrix for the basins of the two attractors (as in Figure 34). There is an 8D multicusp model with nice visual reduction of dimensionality for personality and affect (Callahan & Sashin, 1987). Many more generic bifurcation events are known, whereas many remain to be discovered. This should prove especially fertile territory for psychology and psychobiology.

Complex Dynamical Systems and Self-Organization

The extent to which dynamical systems theory may be applied to model complex natural systems remains to be seen, but there are some obvious possibilities. In general systems theory, it is common to analyze a complex system by breaking it down into a hierarchical network of simpler interacting systems involving various possibilities of coupling and feedback (Figure 51, 52). If each component system is modeled by a dynamical system with its own control parameters and response diagrams and these are coupled by mappings from states at one node to controls at another, the result would be a *complex dynamical system*. If a system influences its own control parameters, that is *self-regulation* or *self-control* or *self-organization*. Feedback loops of greater length or complexity could occur in such networks. A system with a Hopf bifurcation under the influence of a periodic control parameter, whether under external control or self-regulated, would exhibit a pattern of periods of static equilibrium alternating with exponentially increasing oscillations, a pattern commonly found in living organisms. Parabolically bursting neurons, regular spindling in EEG, activity and behavior cycles, are common examples. The involvement of subtle and catastrophic periodic bifurcations likewise seems to abound from biochemical, to physiological, to psychological levels, along with less well-understood examples of bifurcation sequences including static, period, and chaotic attractors.

Applications and Strategies in Psychobiology

The application of dynamical theory ranges from the loosest metaphorical level, through the application of previously developed models to new empirical domains, to the development of entirely new models. The interactive process between theory and experiment when dynamical systems are under investigation suggests supplementation of typ-

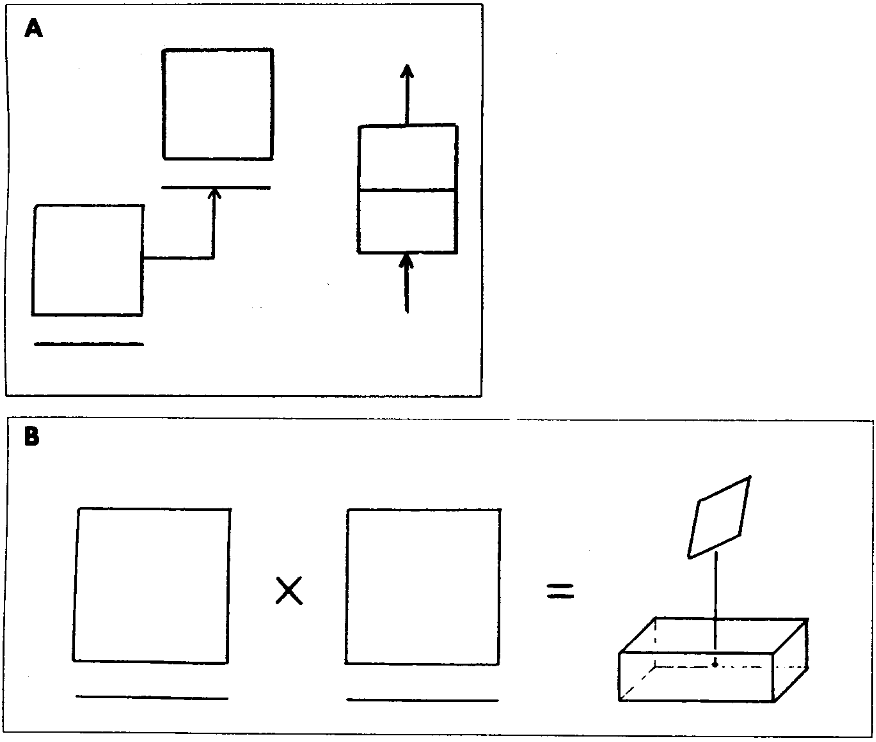


Figure 51. Complex dynamical systems: networks. (a) serial coupling diagrams (horizontal lines are control spaces; vertical lines are state spaces; simplified version at right); and (b) parallel coupling diagrams (a third control simultaneously influences both systems) (from Abraham & Shaw, 1987, © Plenum Press).

ical research strategies in psychology and social science or the innovation of new strategies to reveal dynamical systems and schemes, state spaces and bifurcations, as they evolve over time. Much of the remainder of these volumes are dedicated to this effort (see especially chapters by the Millers, Levine, and Tetrick); only the briefest introduction is included here to illustrate the dynamical approach and to offer some very simplified examples.

Consciousness and Transcendence

We hesitate to enter into the rich and personal domain of consciousness with a mathematical metaphor, and, in fact, the domain is much too rich to explore here. There are two aspects of this metaphor, both

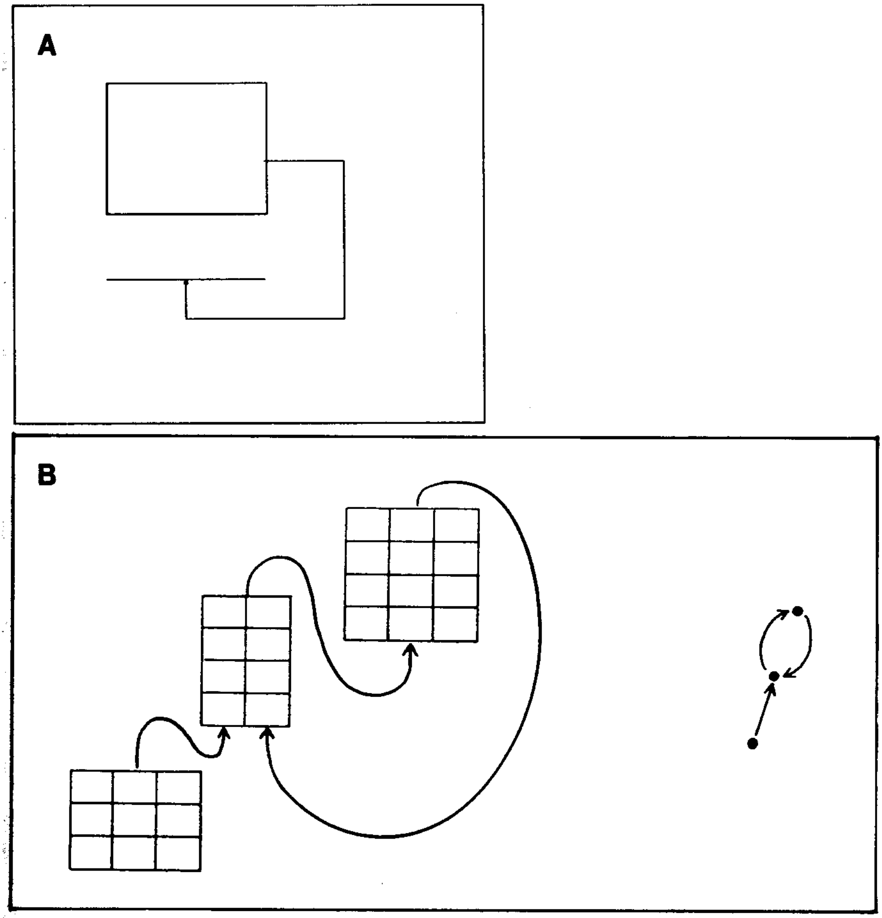


Figure 52. Self-regulation (alias self-control and self-organization): (a) dynamical system whose control parameter depends on the state of the system (from Abraham, Abraham, & Shaw, 1990, © Aerial); and (b) networks of complex dynamical systems with self-organizational feedback (from Abraham & Shaw, 1987, © Plenum Press).

important, but one much more profound. The first is to note analogies between the nature of consciousness and features of dynamical systems. Examples abound. William James, a contemporary of Poincaré, used examples of nonlinear physical systems to argue for saltatory steps (bifurcations) in the evolution of consciousness and for the emergence of conscious phenomenon from their neurophysiological substrates (1890). He considered consciousness as personal, continuous, constantly changing and never recurring exactly, able to identify objects separate from itself, and choosing from among its parts. There is the hint of basins, attractors, even chaotic attractors, and bifurcations.

> Consciousness . . . flows. A "river" or "stream" is the metaphor by which it is most naturally described. . . . Let us call it the stream of thought, of consciousness, or of subjective life (1890, p. 237).

In a remarkable passage describing this stream as something of a cross between a Lèvy flight (clusters of short flights punctuated by long ones) and a relaxation oscillator (the faster and slower portions being relevant here), consisting of

> resting places the "substantive parts" and the places of flight the "transitive parts," . . . Now it is very difficult, introspectively, to see the transitive parts for what they really are. If they are but flights to a conclusion, stopping them to look at them before the conclusion is reached is really annihilating them. Whilst if we wait till the conclusion is reached, it so exceeds them in vigor and stability that it quite eclipses and swallows them up in its glare. *Let anyone try to cut a thought across in the middle and get a look at its section, and he will see how difficult the introspective observation of the transitive tracts is.* (1890, pp. 243–244)

Despite his justifiable skepticism, we could call such a cut a James's section. In the 1930s, holistic psychology did formally discovery dynamical systems in the field theories of Lewin (1935) and Brown (1936). In field theory, goals of the organism were considered as point attractors, a point of view that ignored a great deal of mental subject matter by restricting spaces to that of goals and organisms moving with respect to those goals. It also got into knotty debates over teleogy. One of the great advantages of dynamical theory is to free biological modeling from the hegemony of Bernard's homeostatic point of view. There is a place for it, of course, but systems can grow, evolve, play, and love to gain information. Dynamical theory can handle both. Rössler (1986) has emphasized this advantage:

> Autonomous optimizers are dynamical systems that pursue goals in time and space, can be coupled, and are subject to interactional bifurcations. They are intelligent mathematically, can be implemented on simple, modular, parallel-processing hardware, are potentially immortal, and are bicameral like Zeus and Koko (Fraiberg-Lennenberg autism). A symmetry-breaking interactional bifurcation generating Mead symmetry (and consciousness in the sense of J. Jaynes, 1976) is a possibility worth exploring.

Among the many things he is saying is a reminder that the control spaces in modeling the mind could include optimizers. Pickenhain has recently (1984, 1988) recalled how Bernstein (1929) and Anokhin (1978) used feedback network goal optimizers in their modeling of motor control. Abraham has proposed dynamic schema for mind and memory that feature hierarchical coupling of resonant brain oscillators (macrons) independent of specific cellular neural networks (1985).

Abraham *et al.*'s (1973) speculations on the interactions of wave and unit events in the brain suggest a possible mechanism for such resonances (the evolution of EEG cospectra and coherence spectra are also given in a canonical state space following a major perturbation in brain function). Adey (1975) has pursued a similar concept of the effect of waves on units for many years, more recently extending the inquiry to molecular and ionic levels with nonlinear and Davydov soliton modeling (Lawrence & Adey, 1982).

The second, and more profound, aspect of the dynamical mind metaphor relates to self-awareness, self-control, and self-transcendence. These are stated so eloquently in the *Bhagavad-Gita* (Chapter II):

> The Veda's concern is with the three modes of nature. Become free of these three modes, O Arjuna, free from dualities, firmly fixed in purity, free from anxiety for possessions or safety, and he possessed of the Self. (Verse 45) But one who is self-disciplined, who moves with self-control among the objects of sense which are freed from attachment and aversion, attains grace (the joy and wholeness of pure consciousness). (Verse 64) One who has not established any self-control has no intellect or steady power of concentration, and thus no peace nor happiness. (Verse 66) As a ship is swept away on the water by a strong wind, so too is the intellect carried away when the uncontrolled mind is governed by the wandering senses. (Verse 67) What is dark night for most beings, therein is awake for the self-controlled; what is awake for most, is night to the wise. (Verse 69)

These verses are, among other things, about turning the mind on itself, and in so doing changing itself. Much of the *Gita*, and much of Eastern philosophy concerns the use of meditation and introspection to explore consciousness, to gain control of it, a process that it considers more important than life itself. The Callahan-Sashin model (1987) includes control parameters among its 8D structure; they suggest giving the therapist control of some of them for the correction of affective disorders. Eastern philosophy and now much of Western psychology places emphasis upon the individual's gaining insight and control of the processes of consciousness directly (Abraham, 1987) for normal healthy mental development and for the betterment of self and society (Abraham, 1975, 1988) and ecological harmony with the environment. And speaking of harmony, much of Bhuddist meditation makes use of music and rhythm. It seems that it is no accident that music, mind, meditation, mathematics, brain, and the universe have resonated and developed together (Abraham, 1986). And it is hopeful that we see all of them evolving to ever greater states.

Gödel (1931) showed that paradoxes and tautologies could result in formal systems and logical languages when allowed the luxury of self-reference, which therefore must be denied them. Does dynamic modeling transcend these limitations? Will it fulfill its potential for describing self-organizational dissipative systems—those systems in which ". . . fluctuations play a central role . . . [forcing] . . . the system to leave a given macroscopic state and lead on to a new state" (Prigogine, 1976, p. 93) in which self-awareness and self-modification are a normal part of the control? We feel that dynamical theory is a part of the system, language, conscious process being modeled; that there is no boundary between the more formal aspects and the transcendent properties and that both consciousness and mathematics are synergistically evolving dynamically together as a bifurcating stream.

Dynamical Model of Psychological States

This is a simple model, unabashedly contrived as a didactic exercise illustrating some features of one mode of model construction. That mode is borrowing a simple model from an established application in one discipline and mapping variables and parameters from the new discipline fairly directly onto those of the existing model. Here the old model is that of the damped oscillator of the buckling column. Because many psychological systems behave as damped oscillators, it seems reasonable to explore the model of the buckling column as a generalized starting model for many of these systems.

Some potential target psychological systems for this exercise might include altered states of consciousness, emotional or mood swings (manic-depressive cycles when extreme), altered personality states (from normal variations to schizophrenic), and attitude changes of all sorts. This new discipline isn't so new really because it belongs to the heritage of the theory of cognitive spaces of Lewin (1943) and Brown (1936). It was inspired by the dynamical model of attitude change of Kaplowitz, Fink, and Bauer (1983), and Kaplowitz and Fink (Vol. 2, Chapter 14).

More specific potential systems drawn from many literary, philosophic, and psychological sources may be found in a remarkable and scholarly compendium, *Maps of the Mind*, by Hampden-Turner (1981). For example, in his Map 14, "The Divided Self: Jean-Paul Sartre to R. D. Laing," he summarizes aspects of Sartre:

> Existential being refers to a continuous dynamic flow of consciousness-through-action (praxis). . . . Interpersonal relationships were a perpetual struggle to assert the fluidity of our own existence against persistent attempts to objectify us.
>
> There is "no exit," . . . from the viscious circle (*Huis Clos*) of proffering in "bad faith" false versions of ourselves. (p. 60)

And aspects of Laing:

> The process of becoming schizophrenic begins with split or schizoid functioning. Persons ontologically insecure, that is those who have not been

allowed to experience themselves as continuously related to the world by moral action, may split themselves into two systems, a system of false selves presented as a mask to the world, and an inner self of authentic experience not revealed to others.

Schizoid organization is a question of degree. When I offer my true, embodied self to others for acceptance or rejection there is existential anxiety . . . [and so I] seek relief in the fabrication of false selves designed to gain acceptance.

Contemporary psychiatry [has] made a false objectification of psychic states. Patients seeking help find themselves further petrified by the viewpoints of psychiatry.

Patients can be helped back to a fusion of their subjective experiences with the social realities seen by others, only if these true selves are first accepted as legitimate bases to build upon.

When the continua of whole-part, subject-object, separation-relationship are cloven in a divided self, then all split-off ends are pathological, mutually excitatory, and wildly oscillating, the turned-on-hippie and the buttoned-down automaton alike, and both can only be healed by the integration of their extremities.

In the light of such models as catastrophe theory, Laing's work takes on renewed importance and the concept of a widening, catastrophic splitting in mind and behaviour, with jumps or oscillations between, becomes much more than a metaphor and is capable of mathematical expression and three-dimensional representation.

In his Map 16, "Left, Right and Centre: The View of Silvan Tompkins," he summarizes:

The basic ideological cleavage between, [hu]Man is a valuable end in [her/]himself (Left) and the valuable exists independent of [hu]Man, who should conform to it (Right), can be used to generate several derivative propositions.

The Left is more comfortable in the realm of feeling, especially interpersonal affections which are the roots of fraternity and equality. The Right fears that affection among people might hinder norm attainment, but generally approves of feelings when expressed towards objects symbolizing the norm.

It is therefore necessary to ask whether Left or Right ideologies are a sufficient basis for political life or learned disciplines,.. Thomkins suggests that the ideological centre may be the true repository of creative change-plus-continuity. He cites Kant who reconciled the subjective with the objective, rebellion with authority, and passion with discipline.

The Centre, so defined would not be a faint-hearted compromise between Left and Right, nor a hostile juxtaposition of extremeties which goad one another, *but a movement between the poles which encompasses their extremities while reconciling both within a single process.*

Perhaps the best models are parliamentary democracies where conflict exists within the context of cooperation, dissent within loyalty, private conscience within public accountability, open persuasion within secret balloting, freedom within the law, and Left-wing changes within Right-wing continuities (see Map 58). *It is a process of learning which moves from outer to inner in a cycle of eternal return. Ideologist should beware lest they*

> *become obsessed with one arc of a circle, having failed to appreciate the whole.* (Italics and brackets added)

Hampden-Turner cites many other examples of cognitive systems in which oscillation occurs, including Papez-MacLean's limbic system with its emotional polarities of "rage-fear, fight-flight, pleasure-pain, expectation-actuality, tension-relaxation, etc." (Map 22), Leary and Coffey's model of Sullivan's dynamisms (Map 34), and the moral structures of Osgood's semantic differential and its societal extensions by Hampden-Turner (Map 43). Also, Bateson's cybernetic–schismogenetic views and their application to alcoholism (Map 48), and their extension to schizophrenia, authoritarianism, and mental health by Jackson, Haley, Weakland, Sanford, and Hampden-Turner (Maps 49–51). Alcoholism has, in fact, been modeled as a dynamic system (Golüke, Landeen, & Meadows, 1983) in the general systems theory approach of Bertalanffy-Forrester (used also in Chapters 1, 5, 9, 10, 11, 12, and 13, Volumes 1 & 2). Lévi-Strauss's cognitive–social–mythical systems (Maps 57–58) and its extension to a program of social change are also oscillatory "cybernetic system[s] of figure ground reversal." They are reconciliatory in the Zen-like embracement of opposites, making them vulnerable to "run away and go into a catastrophic revolution." The ancient philosophy of *Tao* as evolved to the *Yin-Yang* of T'ai Chi (Map 3) he summarizes, "But the symbolism is less polarized than unified. Life is a rhythmic movement among opposites, a timeless ebb and flow in vibrating wave patterns." In addition to those models in which oscillation is explicitly recognized, many involve bipolar cognitive dimensions on which oscillation could be expected to occur, such as Williams's psychoreligious struggles (Map 5), Freud's instinctual-energy dimensions (Map 9), Jung's dynamic unities (Map 10), Fromm's growth–decay dimension (Map 11), and so on.

In most of these models, bifurcations resulting from increases in environmental and personality stress factors lead to cyclic attractors displaying underdamping of oscillations between extreme values. Such attractors represent unhealthy personal and social systems. Learning and development constitute damping factors (control parameters) that are responsible for bifurcations leading to static or damped periodic attractors representing a balance within normal limits between opposing values. Such attractors represent a healthy system. His is a most eloquent plea that wholesome balance is a dynamic and mature state of affairs, not a tepid compromise. His survey should give one a good survey of the richness of material available for exploration. Although we pick one now as an example for application, the real advances will be made when the application of dynamic modeling not only helps such models to mature but helps to reveal the rich commonality among them. Hampden-Turner recognizes this potential for synthesis in his summary of many applications of catastrophe theory to psychological and social systems (Map 56).

So now we pick one of these candidates, the right–left–center model of Tomkins, leaving the Laing or any of the others as an exercise for the reader. In this model we assume that there is some swaying back and forth along a disciplinary–ideological dimension, such as holistic humanism versus behavioristic reduction in psychological theory, or along a political–ideological dimension, such as individual liberty versus authoritarian conformity. This dimension may reside cognitively in an individual, estimable by some type of psychological probe, or sociologically in a population, estimable by some type of population measurement. The ideological position and its rate of change are the principal variables in the system.

One of the main parameters might represent perceived social pressure or other environmental demands to deviate from a central position independent of the direction, right or left, of the demanded deviation. It would be analogous to the weight on the buckling column; psychologically a weighting, importance, relevance, or amplification factor (R in Figure 53, also Appendix B4). Some psychologists might be passionate operational behavioral modifiers but disinterested in politics, and some citizens might be passionate about the behavior of their government but not of their scientists, and both might be interested in the educational style of their children's schools but not so interested in drug testing in professional sports. The greater the weighting factor, the greater the amplitude and rate of swaying and the deviancy of the final position, right or left, if "buckling" occurs; the smaller the weighting factor, then the less the amplitude and rate of swaying and the deviancy of the final position.

One's position may be pushed and pulled ideologically right or left by two types of forces, one emotional, the other intellectual (LE, LI, RE, and RI in Figure 53). Emotional forces may be given greater weight by making the accelerative coefficients of a term of the third power of the ideological position a function of them, while making the accelerative coefficients of the linear term of the ideological position a function of the intellectual forces. These accelerative forces are also a function of two other forces. One is a stiffness or centralizing tendency opposing the importance factor and tending to send the individual to a central or near-central position (S in Figure 53). The other is a decentralizing or uniqueness tendency opposing the centralizing tendency (U in Figure 53). These forces represent a safe, tension-reducing, return-to-center tendency on the one hand, and a tendency to move away from the center to a less safe position, a deviation-from-center tendency. Addi-

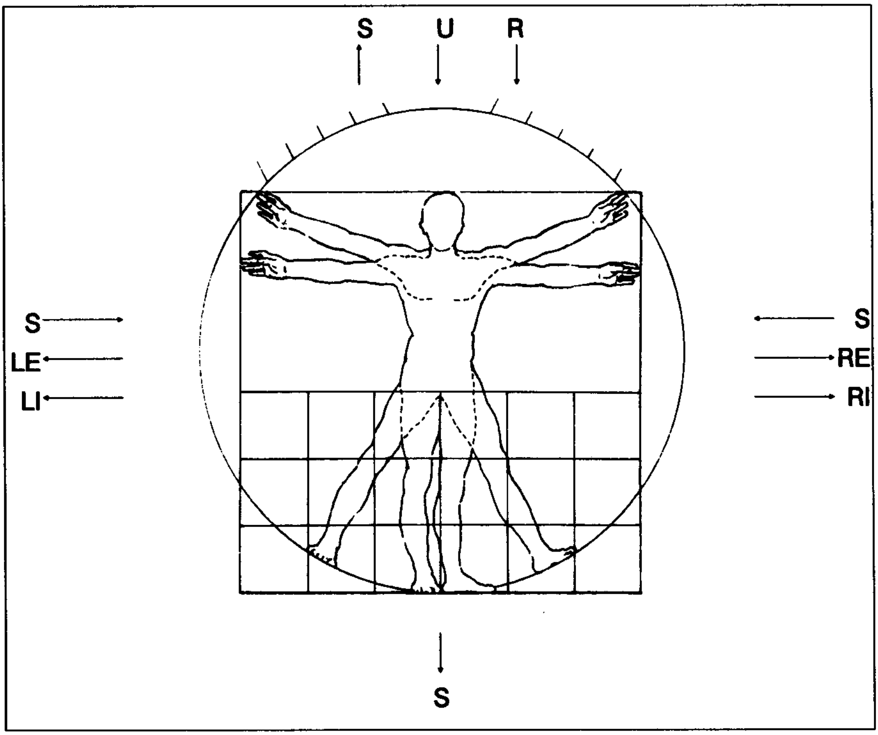


Figure 53. Tompkins left–right–center ideological paradigm. (Forces acting on the system: LE, left emotional; LI, left intellectual; RE, right emotional; RI, right intellectual; S, stiffening or centering; U, unique or individualizing; R, relevant.) (From Abraham, Abraham, & Shaw, © 1990, central part of figure is by Dave Fernandez, from *Maps of the Mind* by Hampden-Turner, © and courtesy Hampden-Turner & Beazley).

tionally there is a damping coefficient that combines with the rate of change of position to affect acceleration.

A basic model, that is, its ODEs given in Appendix B4, will behave pretty much like the buckling column (Appendix B1 and Figures 23–24). There are two sources of biasing a final position away from center, which the general model of the buckling column shown did not have. One is the possibility of differential magnitude of the restorative emotional and intellectual forces attached to each noncentral position. An equivalent feature could be given the buckling column if it were made of a sandwich of two bonded leaf springs of different strengths. This would be equivalent also to saying that there were built in biases toward the left or right. Even with a weight less than the critical buckling value for the importance of that ideological dimension, the resting state

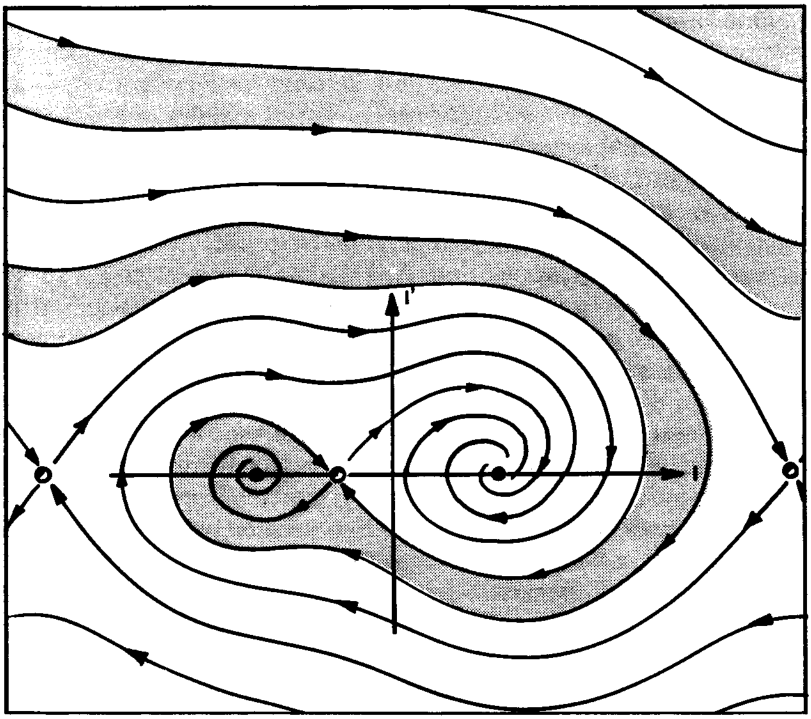


Figure 54. Phase portrait of ideological system showing right bias (I, right–left ideological dimension; I', rate of change of I. basin for right focal point attractor greater than that for the left that defines bias) (from Abraham, Abraham, & Shaw, 1990, based on Abraham & Shaw, 1982–88, © Aerial).

would be off center. The portrait would look like an asymmetrical version of that for the buckling column (Figure 54; compare to Figure 15d).

Contingent Operant Behaviors

Operant behaviors are those that are performed under specific environmental conditions that are rigged to change after the occurrence of the response (an arrangement called "reinforcement"). In operant learning, response probabilities change over time as a joint function of these contingencies (associative factors) and several state variables (drive or regulatory factors) of the animal (Abraham *et al.*, 1972). In some cases, the environmental change that is contingent upon the occurrence of the response provides the opportunity for another (contingent) response, which in turn is followed by another environmental change. Translation: One response may reinforce another. The rat presses the bar (instrumental response), food is delivered, a consummatory response occurs (the contingent response), and sensory consequences to that follow.

Premack (1969) sought essential parameters of reinforcement existing between such pairs of instrumental and contingent responses. He suggested that the different responses could be considered as existing hierarchically along a continuum of response probability. He made another very interesting proposition. Any pair of responses could be considered as existing in a Lewinian "life space" that we could construe more simply as a state space having two dimensions representing the probabilities for each response. He proposed that the reinformement value of any response was proportional to its probability and examined how the velocity vectors of a point in the space were different when the contingencies were introduced, when the probabilities were independently manipulated, and when the contingencies were reversed. Raising the base rate of say, pinball playing, could give it the ability to reinforce eating candy, whereas candy eating might lose its ability to reinforce pinball playing.

One of the really exciting aspects of this work was that it was counterintuitive. It demanded adjustments to major theories of learning and even countered conventional empirical wisdom, to wit, that biologically driven rewards such as food were considered more potent than sensory rewards resulting from manipulatory, exploratory, and other behaviors. Although many experiments have shown some constraints upon these generalizations since the inauguration of this experimental paradigm, it has provided a base for further theoretical work, some using dynamical approaches.

One such model, based on drive-regulatory (homeostatic, equilibrium) rather than associational features of learned performance, suggests the use of the difference between current and baseline states for each response as key components in the nonlinear, coupling terms in the differential equations of the dynamical system (Hanson & Timberlake, 1983):

$$I' = b(C_o - C)I - I/(a_i I_o)$$
$$C' = -(1/b)(I_o - I)C - C/(a_c C_o)$$

where

I is some measure of the strength of the instrumental response,
C is some measure of the strength of the contingent response,
o, the subscripts that label the baseline value (set points or homeostatic balance points) of each response,

a is a resistive coefficient for each response, and
b is an amplification factor for the linking or coupling component.

So, the rate of change in the instrumental response is proportional to the product of the current level of that response and the difference between the base and current levels of the contingent response, being positive when the current level is below the base level, and negative when the current level is above the base level. It is diminished by the ratio of the current level to the base level of the instrumental response itself. This ratio is larger the greater the current level, and smaller the greater the base level. The relative importance of each of the two components of the equation is controlled not only by the current and base levels of the two responses but on the constants a and b characterizing the response system. Similarly, the rate of change in the contingent (relatively constrained) response is proportional to the same type of cross-coupled term and linear term, but now the cross-coupled term is negative if the current instrumental level is less than the baseline level, positive when it is greater. Further, the amplification factor, b, which made the coupling term and the rate of change greater (when $b > 1$) for the instrumental response, is inversely diminishing the influence of the coupling component for the contingent response (unless $b < 1$).

So how does this system behave? Note that these are mass action equations, very similar to those we saw for ecological populations. But here we have response competition rather than population competition. The state spaces consist of two-dimensional plots, the instrumental response as a function of the contingent response, the dimensions being some measure of the strength of the responses (probability, rate, duration). Because we have mentioned that the simplest population model yielded an unstable center (Figure 17a) but that the addition of a small perturbation might shove the system to a more stable static or periodic attractor (Figure 17b,c), we therefore might expect such attractors here. Interestingly, though, this model is more complex, having a saddle at the origin, and retaining the unstable center in the upper right-hand quadrant, its locus depending on the experimentally determined coefficients (Figure 55). An example from their own experiments where $a_i = .013$, $a_c = .10$, $b = 96$, $I_o = 8.4$, and $C_o = 84.8$, the center of the center is at (19.7, 84.8). Besides the retention of this instability, this model illustrates a couple of other obstinate features. One, its stiffness, common in behavioral systems, seemingly caused mainly by the reciprocal use of the coefficient b, sorely taxed the resolving power of our simulation program, Dynasim (Abraham, 1984), until the equations were rescaled. Think of stiffness as numerical awkwardness created by the wide numerical range of the coefficients. The other feature is that the trajectories are not confined to positive values despite the fact that

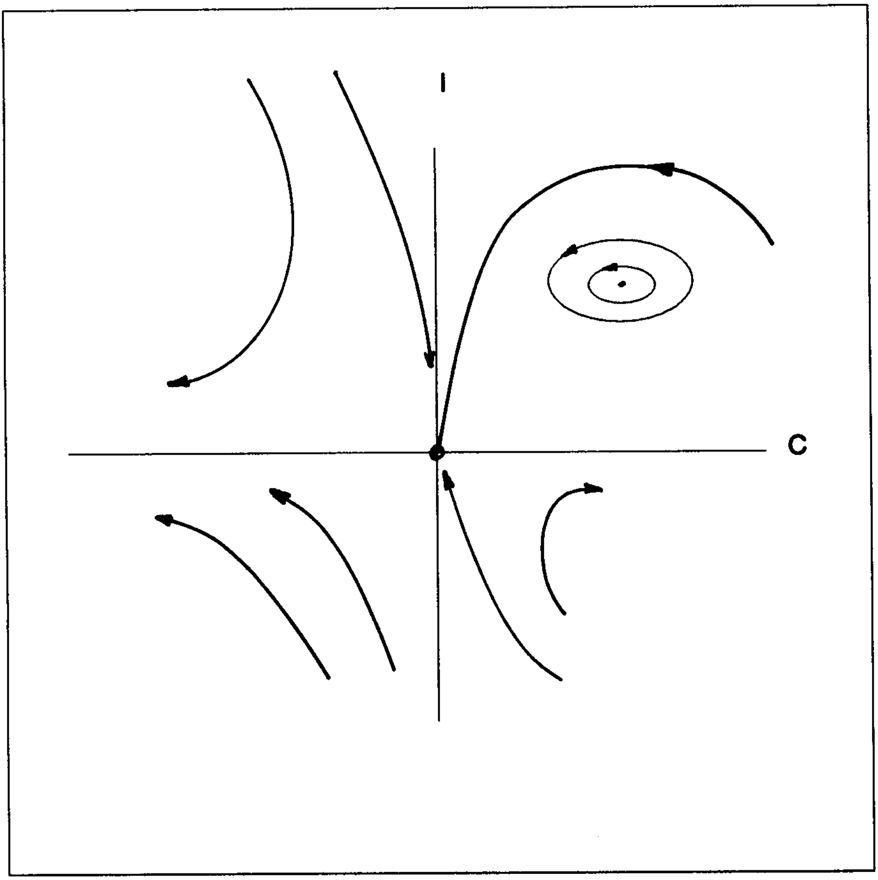


Figure 55. Contingent operant behaviors: phase portrait (I, time spent on the instrumental response; C, time spent on the contingent response) (From Abraham, Abraham, & Shaw, 1990, © Aerial).

the behavioral measurements obviously must be. Either the coefficients or the measurements need corrective transforming to translate the space and measurements to conform to each other, or the phase space could be clipped, but that would complicate interpretation.

Nonlinearites have had a long, liberalizing, but nonetheless perplexing refractory influence on the reductive, linear, equilibrium approach of the historical mainstream of behavioral analysis. Terms like *induction* (Pavlov, 1927; Segal, 1972), *contrast* (Skinner, 1938), *elation* and *criss-cross Crespi effect* (Crespi, 1942), *contrast* and *interaction* (Catania, 1961, 1963) suggest that when learned response systems are coupled they often display nonlinear dynamics. We could further suggest that learning generally involves nonlinear dynamics. One-trial learning (Abraham, 1967; Estes, 1960; Guthrie, 1930; Voeks, 1954; even Skinner, 1938) could be considered a dynamic reorganization, a catastrophic bifurcation, rather than a random stochastic event. And Hull (in a letter quoted in Hilgard & Marquis, 1940) stated, "As I see it, the moment one expresses in any very general manner the various potentialities of behavior as dependent upon the simultaneous status of one or more variables, one has the substance of what is currently called field theory." And while in the behaviorist tradition and cast in the language of hypotheoretico–deductive theory the use of which he pioneered in Psychology, he described learning and the performance of learned behavior in a series of coupled differential equations (Hull, 1943), that could easily be recast into modern dynamical format. We do not mean to imply that the drive reduction and equilibrium approaches of Hull and Hansen and Timberlake are necessarily correct, and in fact, prefer the contiguity approaches of Guthrie, Voeks, and Estes (Hull himself confessed to Voeks of leaning that way also) but are suggesting that the dynamical systems approach could yield an elegant approach to learning theory.

Coupled Circadian Oscillators

It has not been uncommon to model coupled circadian oscillators (see note 7, Czeisler *et al.*, 1986). One such model has considered two principal mutually coupled Van der Pol oscillators, one a more fundamental endogenous circadian pacemaker, x, reflected in core temperature measurements, and a second more labile one, y, reflected in sleep–wake activity cycles (Gander, Kronauer, Czeisler, & Moore-Ede, 1984; Kronauer, Czeisler, Pilato, Moore-Ede, & Weitzman, 1982). These may entrain to environmental exogenous drivers such as the sun, called zeitgebers, z. The couplings between x, y, and z may vary so that they may show varying degrees of independence. The differential equations for one of their models are:

$$[x'' + u_x(-1 + x^2)\,x' + \omega^2 x] + F_{y->x}\,y' = 0$$

$$[y'' + u_y(-1 + y^2)\,y' + \omega_y\,y] + F_{x->y}x' = 0 \text{ or } F_{z->y}z$$

where ω is frequency, u is internal oscillator stiffness, and the F's are coupling forces. This model entrains y to z; y thereby mediating z's influence on x. A principal experimental paradigm removes the effect of z and observes a gradual decline in ω (increase in length of the circadian cycle). Garfinkel and Abraham (1985) applied the visual dynamical approach to see if features of this model might be made clearer than the use of the traditional raster plots. The state space is a 2D torus. The phase portraits of the model (collapsed into a planar format visually) for successive 25-day segments of the 100-day-long experiment exhibit dramatic changes (Figure 56). The response diagram shows the bifurcation points as the braids on the torus change under the influence of the decline in ω_y (Figure 57).

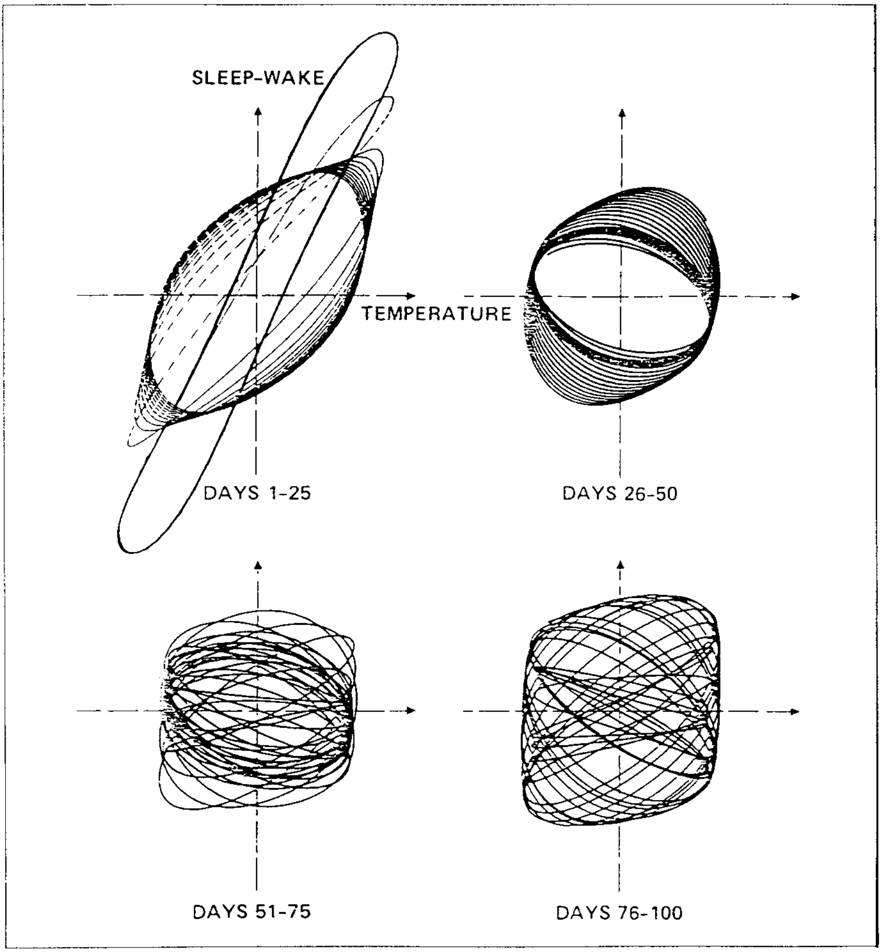


Figure 56. Coupled circadian oscillators freerunning with zeitgebers removed (from Garfinkel & Abraham, 1985, and from Abraham, Abraham, & Shaw, 1990, © Garfinkel and Aerial).

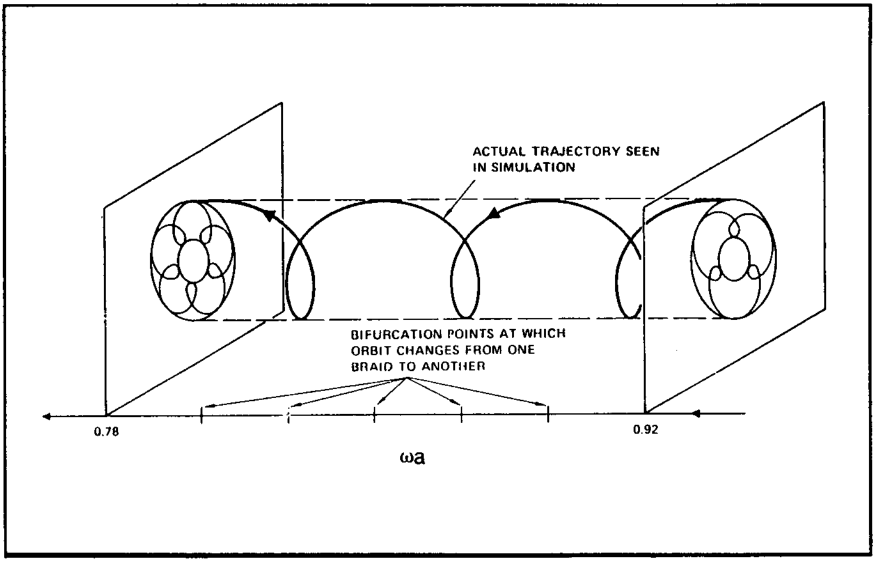


Figure 57. Response diagram of coupled circadian oscillators with frequency of the activity oscillator, ω_a, as the control parameter; the attractors bifurcate between different braids with complex winding ratios or chaotic orbits, but the trajectory is making a smooth, albeit complex, flight during the progression (from Garfinkel & Abraham, 1985, and from Abraham, Abraham, & Shaw, 1990, © Garfinkel and Aerial).

The model of coupled Van der Pol circadian oscillators already existed, of course. The benefit of adding the generalization of using dynamical system theory has been merely to relate it to the general concept of dramatic reorganization within the context of a dynamical scheme as revealed visually using the phase portrait and response diagram. These models have direct clinical as well as theoretical and experimental application, as in the suggested use of light as a zeitgeber in sleep disorders (Czeisler *et al.*, 1986; Czeisler & Allan, 1987; Sinclair, 1987).

The neuroendocrine system provides many other examples of coupled circadian oscillators to which dynamical systems theory has been successfully applied (Rössler, Götz, & Rössler, 1979). In particular, the hypothalamic, pituitary, gonadal systems involving LH (luetenizing harmone) (Abraham, Kocak, & Smith, 1985; Smith, 1980) and adrenal systems ACTH/Cortisol (Abraham & Garfinkel, 1986) have been modeled. The LH model uses complex dynamical schemes, a serial network (hypothalamus–pituitary–gonad) with parallel short (pituitary–hypothalamus) and long (gonad–hypothalamus) feedback between dynamical schemes connected by piecewise linear functions (Sparrow, 1981).

This complex dynamical scheme exhibits catastrophic, subtle, and explosive bifurcations including static, periodic, and chaotic cusp catastrophes. It also provides a model for intermittency and noise amplification in a serially coupled system. Using age as a control parameter, it has been used to model puberty as a Hopf bifurcation. Its clinical relevance includes control of ovulation. The cortisol model is a bit more sophisticated biologically in including secondary messenger and ligand/receptor systems and experimentally in suggesting efficient hardware/software realizations. Although still somewhat biologically simplistic, these models probably represent some of the more interesting and useful applications of dynamical systems theory in psychobiology and hopefully serve as metamodels for the application of such theory to research on complex psychobiological systems.

Neuropsychobiological Models

At a more metaphorical level, there have been a few examples of dynamical concepts applied to neurochemical, pharmaclological, and electrophysiological bases of behavior (Basar, Basar-Eroglu, Rosen, & Schutt, 1984; Ehlers & Havstad, 1982; Freeman & Skarda, 1985; Makeig & Galambos, 1982; Mandell, Russo, & Knapp, 1982). More explicit models, now classics, are those of Zeeman (1977) on neuronal excitability and behavior. Some other contemporary examples include investigation of the fractal dimensionality of EEG (Mayer-Kress & Layne, 1987; Watt & Hameroff, 1987) and evoked potentials (Röschke & Basar, 1987).

One of the most original and innovative applications investigates nigrostriatal dopaminergic (DA) control of neuronal firing rates as a substrate to behavioral stability under schizophrenia (King, Barchas, & Huberman, 1984). There are four differential equations involving variables of neuronal firing rates and concentrations of nigral and striatal stored and released DA. Parameters include those related to short and long feedback loops, external depolarization of nigral input, and various synthesis, release, degredation, and reuptake constants. The nonlinearities derive mainly from a second-order relationship between DA synthesis and the firing rate of the DA neurons. The system exhibits several periodic bifurcations and a chaotic one and a cusp catastrophe, and sensitivity to initial conditions. We are currently running some simulations on this model using their original equations instead of their reduction to a single logistics equation. This system is being investigated with *in vivo* voltammetry (Justice, Nicolaysen, & Michael, 1987). We hope to see this model extended formally to include secondary messenger dynamics at one end, and to include additional behaviors at the other end (Aranow, 1987).

Strategies

We hope we have illustrated that mathematics and psychology are in the midst of a chaotic catastrophe of their own, converging in a turbulent period from which we will see emerging a new era of scientific maturity. Both fields are increasingly attempting to model complex cooperative systems evolving through multiple modes of dynamical equilibrium. Philosophically, we hope they provide powerful metaphors upon which to predicate paths to creative, fulfilling, and productive society and the protection of individual rights of its members (F. D. Abraham, 1975; R. Abraham, 1981). Scientifically, we hope to see a revolution not only in this convergence of the cooperation of mathematics and science but a revolution in the nature of experimental design in the behavioral sciences. Just as the elegant Markovian models of the learning process led to increased use of within-subject dependent variables, so should the current theorizing that also stresses temporal sequences in behavior. The use of factorial and related designs may have to subjugate themselves to experiments that stress starting at a great number of different initial conditions and collecting many such varied replications to reveal phase portraits and attractors, especially multiple attractors, and under several parametric conditions in order to pin down bifurcation points of dynamical schemes. Greater respectability will accrue to the smaller *n* designs typical of psychobiology, notoriously shakey compared to the large *n* designs typical of purely behavioral research; their tightness will come in producing more complete trajectories and in developing inferential tests of the parameters characterizing limit sets, characteristic exponents, characteristic multipliers, and fractal dimensionality. Another revolution that might be anticipated is the increased incorporation of on-line control of experimental independent variables by the model in response to the reading of dependent variables (Abraham, Betyar, & Johnston, 1968; Abraham et *al.*, 1973; Abraham, 1976; Garfinkel & Abraham, 1985).

Finally, although naturally no approach will hegemonically rule scientific progress, those of us enthusiastic about the dynamical theoretical approach forsee its attractiveness providing a more common language for a growing community of scientists, an observation made by Staddon (1984) for the field of psychology who felt that some theoretical controversies lay in communication failures deriving from models that were too idiosyncratic. As champions of individuality, but enthusiasts of the dynamical approach, we feel assured that increased use of the approach will be characterized by individuality and scientific multiplicity.

Appendices

A. Glossary

Actual separatrix. Separatrix that forms a boundary between basins.
Annihilation. A catastrophic bifurcation in which an attractor disappears.
Attractor. Irreducible limit set to which all nearby trajectories tend. Attractors may be static, periodic, or chaotic.
Average velocity vector. Bound vector divided by the time it takes for the trajectory to go from the first to last point.
Basin. A region of the state space containing all the trajectories that tend to a given attractor.
Bifurcation. When a phase portrait changes dramatically, qualitatively, into some topologically nonequivalent form as some control parameter moves past a bifurcation point.
Bifurcation point. A value of a control parameter at which a bifurcation occurs.
Bound vector. A vector from one point to another.
Braid. A finite number of closed trajectories, alternately attractive and repelling, winding around a two-dimensional torus.
Cantor set. An infinite set of points on a line taken by iteratively decimating equally spaced intervals such as the middle third of each remaining segment.
Catastrophic bifurcation. A bifurcation where a limit set suddenly appears (creation) or disappears (annihilation).
Center. A nest of closed trajectories around a central rest point.
Chaotic attractor. Attractor comprised of a chaotic limit set.
Chaotic limit set. Limit set that is neither a point nor a cycle.
Characteristic exponent. A complex number that measures the rate and character of approach and departure of trajectories with respect to a limit point.
Characteristic multiplier. A complex number that measures the rate and character of approach and departure of trajectories with respect to a limit cycle.
Closed orbit. Closed trajectory or limit cycle.
Closed trajectory. A trajectory that closes upon itself. A closed orbit or limit cycle.

Complex dynamical system. A hierarchical network of simpler interacting dynamical systems involving various possibilities of coupling and feedback.

Constant trajectory. A trajectory that stays within a single point (which must be a critical, fixed, or rest point).

Control parameter. A parameter in a dynamical system that may be varied, changing the dynamical system. Its change past a critical value, the bifurcation point, may be responsible for a bifurcation.

Converging flows. Trajectories that converge. For example, as toward a fixed point attractor.

Creation. A catastrophic bifurcation in which an attractor appears.

Critical point. A state at which the velocity vector is zero. A rest point.

Cusp catastrophe. A three-dimensional response diagram with a one-dimensional phase portrait and two control parameters. There is a region within the bifurcation set (the cusp curve), with two point attractors and a point repellor in between. There is a single point attractor elsewhere. There is a catastrophic bifurcation (called a fold) as the control parameters cross the cusp.

Cycle. Periodic trajectory. Oscillation. Limit cycle.

Deformation. The addition of a weak vectorfield to a dynamical system with usually only minor consequences in its phase portrait.

Differentiation. The process of deriving the instantaneous velocity vector. The limit of the average rate of change of a variable. The limit of difference ratios between states and time as time shrinks. The reverse of integration. Differentiation of the phase portrait produces the vectorfield.

Differential equation. An equation that expresses the rate of change of a variable.

Divergent flows. Trajectories that diverge. For example, as from a point repellor.

Donor. The saddle from which a saddle connection departs.

Double fold catastrophe. A response diagram with one control parameter in which there is a region where there are two point attractors with hysteris (direction of change in the control parameter) determining which of the two bifurcation occurs. Outside the region of the double fold, there is but a single attractive point. A model for hysteresis.

Dynamical scheme. A dynamical system with control parameters.

Dynamical system. Technically, the vectorfield. More loosely applied to a system and the information adequate to produce its vectorfield. Also, loosely, the integral–differential pair, the phase portrait and the vectorfield.

Entrainment. Two oscillators are coupled, and one or more parameters, such as phase or frequency, become identical in both oscillators.

Epsilon equivalence. A deformation of a dynamical system in which trajectories remain within a specified distance of each other.

Explosive bifurcation. When there is a sudden change in magnitude of an attractor.

Fixed point. A critical or rest point.

Fixed point attractor. Attractor comprised of a fixed point.

Fold catastrophe. Creation of a static attractor and a companion separatrix.

Fractal dimension. A measure of fractal microstructure.

Fractal microstructure. A characteristic of the Cantor-process-like pattern of the trajectories of chaotic attractors as revealed in a Lorenz section of the attractor.

Fractal separatrix. A separatrix having fractal microstructure. For example, the inset of a chaotic saddle or a homoclinic tangle.

Generic. Properties shared by almost all dynamical systems.

Heteroclinic trajectory. A saddle connection for which the donor and receptor are different saddles.

Homeomorphism. Continuous "rubber sheet" deformation of the state space.

Homoclinic trajectory. A saddle connection for which the donor and receptor are the same saddle.

Hopf bifurcation. A subtle bifurcation with a static attractor changing to a periodic one. The excitation of an oscillation.

Hyperbolic. A limit point is hyperbolic if none of its CEs is on the imaginary axis. A limit cycle is hyperbolic if none of its CMs is on the unit circle.

Information gain (increase). The increase of information about past states with diverging flows.

Information loss (decrease). The loss of information about past states with converging flows.

Index. The dimension of the outset of a limit set.

Inset. The set of trajectories arriving at a limit set.

Instantaneous velocity vector. The instantaneous rate and direction of change in the state of the system at a point in time. One exists for each point on a trajectory and thus for every point in the state space. The tangent vector to a trajectory.

Integration. The limit of summing the rates of change in state for some interval of time. Reverses the process of differentiation in returning the values of states from rates of change information. As differentiation produces the slope or tangent (representing the rate of change) of some variable as a function of time, so integration produces the variable as a function of time. Complete integration of the vector field produces the phase portrait.

Limit cycle. Limit set consisting of a periodic trajectory.

Limit point. Limit set consisting of a critical point.

Limit set. Special asymptotic trajectories approached or departed by a trajectory in the phase space. They may be limit points, limit cycles, or chaotic limit sets.

Lorenz section. The pattern of trajectories of a chaotic attractor as revealed by a hyperplane within a Poincaré section. Makes clear the pattern of layering of trajectories crossing the Poincaré section.

Mass action, law of. Any (ordinary differential) equation in which the rate of change of a variable is proportional to the product of that variable and some other variable, such as the concentrations of two reagents in a chemical reaction, sizes of two populations of cooperating or competitive species, or strengths of interacting behaviors. May generalize to more than two variables, of course.

ODE. Ordinary differential equation.

Ordinary differential equation. Differential equation where the rate of change of a variable is expressed as a function of that variable. ODE.

Oscillation. Periodic trajectory. Limit cycle.

Outset. The set of trajectories departing a limit set.

Periodic attractor. Attractor comprised of a limit cycle.

Periodic trajectory. A trajectory that repeats itself endlessly with the same cycle time; the instantaneous velocity vectors at each point remain the same through successive cycles.

Phase portrait. The state space, filled with trajectories (only a few representative ones are usually shown).

Poincaré section. A hyperplane perpendicular to a trajectory, used to reveal the pattern of any repeated crossings by that trajectory.

Receptor. The saddle to which a saddle connection goes.

Repellors. Irreducible limit sets from which all nearby trajectories depart.

Response diagram. Diagrams showing the of bifurcations of attractors in a phase portrait as a function of the control parameters.

Rest point. Constant trajectory. Critical point. Fixed point.

Rotation number. The ratio of the frequency of a driven oscillator to that of the driving oscillator.

Saddle. Limit set that some trajectories approach and others depart.

Saddle connection. A saddle-to-saddle trajectory.

Sensitive dependence on initial conditions. Refers to the fact that in a given chaotic dynamical system, trajectories differing by only a small amount at one moment (the initial conditions) may diverge and differ by a large amount at a later time.

Separatrix. Points in a state space that are not in any basin. They may be actual or virtual.

Spectrum. The set of characteristic exponents of a limit point, or characteristic multipliers of a limit cycle. Also, the power distribution function for the frequency transform of a time series, which is better called the power spectrum.

State. A point in a state space representing the state of a system at a given moment.

State space. A geometric model for all the possible states of a system. Euclidian spaces, cylinders, spheres, and tori are frequently used.

Static attractor. Attractor comprised of a limit point.

Static Creation. The fold catastrophe for a point attractor.

Strobe plane. Poincaré section. Named because strobing the trajectory by a light or an oscilloscope trigger pulse reveals the repeated crossings of the plane by the trajectory.

Structural stability. The property of a dynamical system whereby all delta perturbations of it have epsilon-equivalent phase portraits.

Subtle bifurcation. Bifurcation with a sudden change in the type of an attractor, but in which the magnitude changes gradually.

System. A set of related or interacting variables that change over time.

Tangent vector. The instantaneous velocity vector at any point of a trajectory.

Tangles. Complex heteroclinic transversal intersections of saddle insets and outsets.

Time labeling. Labeling a trajectory with time marks or "ticks."

Time series. A variable that is a function of time. Graphically, time is the axis for the independent variable (usually horizontal).

Topological equivalence. A homeomorphism between two phase portraits, preserving trajectory curves and directions, but not times.

Trajectory. A curve connecting temporally successive states in a state space.

Transversality. A clean (nontangential) intersection between two hypersurfaces.

Vectorfield. The collection of all the instantaneous velocity vectors in the state space. A dynamical system.

Virtual separatrix. Separatrix that lies within one basin.

B. Differential Equations for the Model Systems

1. *The Buckling Column*

$$x' = v$$
$$v' = (-1/m)\,[a_3x^3 + a_1x + cv]$$

where

x is the horizontal displacement,

v is the instantaneous velocity of the column,
m is the mass,
c is a damping coefficient ($c = 0$ for the undamped frictionless case), and
a_1 and a_3 are also constants.

2. *The Sustained Oscillator*

$$i' = v$$
$$v' = (-1/CL)\,\{i + Bv^3 - Av\}$$

where
i is current, v is voltage, C is capacitance, L is inductance, and $Bv^3 - Av$ is the characteristic function of the tube (A and B are constants).

3. *The Forced Hard Spring*

$$x' = v$$
$$v' = (-1/m)\,[a_3x^3 + a_1x + cv] + F\cos(\theta)$$
$$\theta' = \omega.$$

where
x is the driven displacement,
v is the driven velocity,
θ is the driving phase,
ω is the driving frequency,
F is the coupling strength,
$a_3x^3 + a_1x$, $a_1 > 0$ is the restoring force of the spring, and
$a_3 > 0$, $= 0$, < 0 for hard, linear, or soft spring, respectively (see Stoker, 1950, for details).

4. *Dynamical Model of Psychological States for Tompkins's Left–Right–Center Ideological Paradigm*

$$x' = v$$
$$v' = (-1/R[a_3x^3 + a_1x + Cv)$$

where
$a_1 = (RI - LI + 2U/S)$,
$a_3 = (RE - LE + U/S)$,
x is a position on the right–center–left ideological dimension,
v is the rate of change of that position,
R is the importance or relevance of the dimension,

C is a linear damping coefficient representing resistance to change,
LI, *LE*, *RI*, and *RE* are the intellectual and emotional attractive forces of the Left and Right,
U is the uniqueness force, and
S is the centralizing force.

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