50
Cellular Dynamata

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1. Introduction

At the end of the sixties, our program for dynamics lost momentum, and some of us turned to applications for inspiration. My own attempts at modeling in the biological sciences led me to *complex dynamics*, a blend of our own style of dynamical systems theory with system dynamics. A complex dynamical system is a directed graph, with a dynamical scheme (dynamical system with parameters) at each node, and a coupling function on each edge, expressing the control parameters of one scheme as a function of the states of another.[1]

A related type of structure is a *cellular dynamaton* or CD. This is a special kind of complex dynamical system, in which the graph is embedded in a physical space. To my knowledge, the first CD was the one-dimensional array of oscillators introduced by Anderson in 1924 as a model for cat gut.[2] Chris Zeeman introduced me to the CD idea with his model for memory, which he described to me in Amsterdam in 1972.[3] As I understood him then, there would be a two-dimensional array of Duffing cells, with a spatial pattern of attractors corresponding to a memory engram. This is close to Walter Freeman's recent model for memory in the olfactory cortex.[4]

2. Definitions

A CD is specified by three sets of data,

A. *the standard cell*, a single dynamical scheme, envisioned as a response (or bifurcation) diagram,

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[1] For more details, see (Abraham, 1984).

[2] For this and other early references, see (Kopell, 1983).

[3] See his own description of a single memory cell in (Zeeman, 1977).

[4] See (Freeman, 1991).

B. *the spatial substrate*, a regular lattice in a manifold, usually Euclidean space, and

C. *the connections*, a set of coupling functions, one for each node in the lattice, expressing the local control parameters as functions of the states at all other nodes. These data determine a single, massive, dynamical scheme, which has its own response diagram.[5] There are three worlds of CDs, depending on whether the standard cell is generated by a map, a diffeomorphism, or a vector field. Throughout this brief survey, we will assume the latter.[6]

3. Examples

The first and most studied CD is a lattice of oscillators, in a one-dimensional substrate. Here, the coupling is usually to nearest neighbors only, as in Anderson's model for peristalsis. Van der Pol oscillators are the usual choice for the standard cell. Freeman's olfactory bulb model belongs to this class.

Another early class of models comes from the discretization of PDEs, especially the wave equation, the heat equation, or reaction-diffusion equations.[7]

A simple neural net (given an arbitrary spatial substrate if neccesary) is also a CD, in which the usual standard cell is the simplest possible dynamical scheme (linear vector field in one dimension, plus a constant for control parameter), with global, linear coupling. These nets have simple cells, but complex coupling, and are capable of very complex behavior (that is, lots of static attractors with thin basins).[8]

Closely related to the simple neural net is the cuspoidal net, in which the standard cell is a cusp catastrophe, and the coupling is local and linear. As two coupled cusps can oscillate, as Kadyrov has shown,[9] a cuspoidal net can behave either as a simple neural net, or as a lattice of oscillators. They may be useful for spatial economic models[10] as well as for artificial intelligence.

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[5] For more details, see (Abraham, 1986).

[6] For practical reasons, the iterated map CD is the type most commonly explored in computer simulation. For a review of the results with one-dimensional substrates, see (Crutchfield, 1987). A color atlas of spatial patterns may be seen in (Abraham, 1991b).

[7] For a history of this class, see (Abraham, 1991c).

[8] On the existence of static attractors, see (Hirsch, 1989).

[9] See (Abraham 1991a). This is similar to Steve Smale's two-cell oscillator (Smale, 1976).

[10] See (Abraham, 1990b), (Beckmann, 1985), and (Beckmann, 1990).

4. The Future

So far, the development of this new subject, now in its early period, depends crucially on explorations. Currently, explorations of two-dimensional CDs with massively parallel supercomputers are on the frontier. Besides the computational cost, these explorations tax our cognitive strategies for the recognition of basic concepts of behavior of attractors and bifurcations. New methods of reducing space-time patterns to symbol sequences are needed.[11] A number of ingenious ideas based on classical analysis have evolved, especially in the context of oscillator CDs.[12] A surprising feature on the current frontier is the role played by applications. Many of the new ideas are coming from modelers in the biological and social sciences.[13] Morphogenesis remains the Everest of the scientific modeling art.

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References

Abraham, 1984
Ralph H. Abraham, Complex dynamical systems, *Mathematical Modelling in Science and Technology*, eds. X.J.R. Avula, R.E. Kalman, A.I. Leapis, and E.Y. Rodin, Pergamon Elmsford, NY, pp. 82–86.

Abraham, 1986
Ralph H. Abraham, Cellular dynamical systems, *Mathematics and Computers in Biomedical Applications, Proc. IMACS World Congress, Oslo, 1985*, eds. J. Eisenfeld and C. DeLisi, North-Holland, Amsterdam, pp. 7–8.

Abraham, 1990a
Ralph H. Abraham, Cuspoidal nets, *Toward a Just Society for Future Generations*, eds. B.A. and B.H. Banothy, Int'l Society for the Systems Sciences, pp. 66–683.

Abraham, 1990b
Ralph H. Abraham, Visualization techniques for cellular dynamata, *Introduction of Nonlinear Physics*, ed. Lui Lam, Springer-Verlag, Berlin.

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[11] See (Abraham, 1990a) for one such proposal, based on the response diagram of the standard cell.

[12] See, for example, (Kopell, 1983) or (Kaneko, 1990).

[13] See, for example, the double-logistic map lattices arising in economic models, in (Gaertner, 1991).

Abraham, 1991a
Ralph H. Abraham, Gottfried Mayer-Kress, Alexander Keith, and Matthew Koebbe, Double cusp models, public opinion, and international security, *Int. J. Bifurcations Chaos* **1**(2), 417–430.
Abraham, 1991b
Ralph H. Abraham, John B. Corliss, and John E. Dorband, Order and chaos in the toral logistic lattice, *Int. J. Bifurcations Chaos* **1**(1), 227–234.
Abraham, 1991c
Ralph H. Abraham, Cellular dynamata and morphogenesis: a tutorial, preprint.
Beckmann, 1985
Martin Beckmann and Tonu Puu, *Spatial Economics: Density, Potential, and Flow*, North-Holland Amsterdam.
Beckmann, 1990
Martin Beckmann and Tonu Puu, *Spatial Structures*, Springer-Verlag, Berlin.
Crutchfield, 1987
James P. Crutchfield and Kunihiko Kaneko, Phenomenology of spatio-temporal chaos, *Directions in Chaos*, ed. Hao Bai-Lin, World Scientific, Singapore.
Freeman, 1991
Walter Freeman, The physiology of perception, *Sci. Am.* **264**(2), 78–85.
Gaertner, 1991
Wulf Gaertner and Jochen Jungeilges, A model of interdependent consumer behavior: nonlinearities in R2, preprint.
Hirsch, 1989
Morris W. Hirsch, Convergent activation dynamics for continuous time networks, *Neural Networks* **2**, 331–349.
Kaneko, 1990
Kunihiko Kaneko, Simulating Science with Coupled Map Lattices, *Formation, Dynamics, and Statistics of Patterns*, ed. K. Kawasaki, A. Onuki, and M. Suzuki, World Scientific, Singapore.
Kopell, 1983
Nancy Kopell and G.B. Ermentrout, Coupled oscillators and mammalian small intestines, *Oscillations in Mathematical Biology*, ed. J.P.E. Hodgson, Springer-Verlag, Berlin, pp. 24–36.
Smale, 1976
Steve Smale, A mathematical model of two cells via Turing's equation, *The Hopf Bifurcation and its Applications*, eds. J.E. Marsden and M. McCracken, Springer-Verlag, New York.
Zeeman, 1977
E. Christopher Zeeman, Duffing's equation in brain modelling, *Catastrophe Theory*, ed. E. Christopher Zeeman, Addison-Wesley, Reading, MA, pp. 293–300.