DYNAMICS OF NORTH-SOUTH TRADE AND THE ENVIRONMENT

by

Ralph H. Abraham
Mathematics Department
University of California
Santa Cruz, CA 95064

Graciela Chichilnisky
Economics Department
Columbia University
New York, NY 10024

and

Ronald J. Record
Mathematics Department
University of California
Santa Cruz, CA 95064

Abstract. We propose dynamical systems for the economic behavior of the North-South model, defined by iterated functions for the annual change of macroeconomic variables. Simulation reveals the bifurcation diagrams. We discuss the implications for business cycles, and the interaction of economic and environmental variables.

Dedicated to John von Neumann

CONTENTS

1. Introduction. The North-South model is an equilibrium model for the macroeconomic interaction between two regions, and was introduced in (Chichilnisky, 1981), which we refer to as GC81. The present paper develops several alternative dynamical models, based on the idea of slowly changing the capital stock variable in the static model, and assuming the approach to equilibrium follows rapidly. The models which result bear some similarity to one presented by John von Neumann in 1932 (Von Neumann, 1938) for which see also Ch. 3 of (Goodwin, 1991.) They are variants of the coupled logistic maps studied in several recent papers [Invernizzi, Gaertner, Gardini]. The idea is to alter GC81 to allow capital accumulation through time, assuming that the approach to equilibrim follows rapidly. This means that new equations are introduced, which are not found in GC86, for the evolution of capital stock through time, by accumulation and depreciation (see 4.1.1). The static North-South model GC81 is recalled in Section 3 below.

1.1. The concepts of the dynamic North-South model

Two fundamental equations are added to GC81, describing capital accumulation through time in each of the two regions. Our goal is to obtain, from these two new equations, a two dimensional discrete dynamical system, generated by an endomorphism of the plane, $T : R^2 \mapsto R^2$, and to describe its qualitative properties. The two equations are:

$$K_N(t+1)^+ = s_N(GNP_N) + (1-\delta_N)K_N(t) \qquad (1.1.1)$$

$$K_S(t+1)^+ = s_S(GNP_S) + (1-\delta_S)K_S(t) \qquad (1.1.2)$$

The equation (1.1.1) describes capital accumulation through time in the North, and (1.1.2) in the South. These equations are standard, and are interpreted as follows. Equation (1.1.2) explains *capital stock* at time $t+1$ in the North (N) as the sum of: capital stock in the previous period in the North, $K_N(t)$, minus the part of this which is depreciated (δ_N is the depreciation factor in the North) plus *savings* (savings rate in the North is s_N) times its *gross national product*, GNP_N.

In order to determine our two-dimensional discrete dynamical system we need to define from these equations an endomorphism of the plane, $T : R^2 \mapsto R^2$. For this we need to determine the variables in these two regions. The main variables in these equations are the GNP values in each region, because depreciation and savings rate are typically exogenous parameters. But how do we determine GNP in the two regions for any given values of the capital stocks in each, considering that they trade with each other through the international market?

The solution to this problem is the main contribution of our paper: the specifications of the GNP variables as the solutions of two simultaneous market equilibrium problems. Here is where we use GC81. The combination of equations (1.1.1) and (1.1.2) with the North-South trade model is done here for the first time, and we call this the *dynamic North-South model.*

Although the system is complex and has chaotic dynamics in a large range of parameters, some of its qualitative properties can be explained in simple and useful terms. These and further investigations of the properties of the dynamic North-South model have been carried out by us (reported below) and by M. di Matteo [3].

How do we obtain an endomorphism of the plane from the two equations for capital accumulation?

We start with initial values of the two capital stocks, one for each region, K_N and K_S. The static North-South model solves the world economy from the following initial parameters: *capital* and *labor supply* equations, *technologies* and *demand* in each region. Here, instead, we assume that *labor supply* and *technologies* are initially given in each region, and define *demand* endogenously, as a function of GNP, by a new equation,

$$I^D = GNP(1-\gamma)$$

(see 5.3.1 below) and leave the last remaining variable, *capital supply* to be determined by the capital stock at time t.

Therefore, for each level of capital in each region at time t, we solve fully the North-South model at time t and obtain GNP at time t. From this, in turn, we compute the capital stocks, at time $t+1$, using our new dynamic equations (1.1.1) amd (1.1.2).

The procedure can be summarized as follows. The static North-South model determines endogenously five price variables and sixteen quantity variables. It has two goods traded internationally (basic goods, B, and industrial goods, I) and two factors of production (capital, K, and labor, L.) The price variables are the *international terms of trade* for the two traded goods B and I, denoted by P_B and p_I, (these are reduced to one by the normalising assumption $p_I = 1$, and henceforth, $p = p_B$), and the *prices of labor* and *rental of capital* in each region, denoted w and r. Technologies are different in the two regions so that the rewards to labor and to capital are also different. The sixteen quantities which are endogenously determined are: *supply* and *demand* for the basics and industrial goods, *employment factors* in the two sectors, imports and exports, each in the two regions. From these endogenous variables we obtain an expression for the desired GNP in each region. By definition, GNP is the value of the gross national product, that is, the value of all the production (of B and I) computed at international market prices, p, the prices at which all markets clear. This means that part of the production of each country is consumed in the other county, and that relative prices p have adjusted to permit this trade, so that imports equal exports in each of the two traded goods. The result is an equilibrium level of GNP in each region,

$$GNP_N = pB_N^S + I_N^S \tag{1.1.3}$$

$$GNP_N = pB_S^S + I_S^S \tag{1.1.4}$$

Here p, B^S, and I^S are determined as the solution of a system of 22 simultaneous equations in 22 variables, as in the static North-south model. This is explained in Section 5.2 below. *Therefore, for each value of capital stock we have assumed an instantaneous adjustment to an equilibrium in the static North-South model.*

From all this we obtain the GNP in each region at time t. The two dynamic equations (1.1.1) and (1.1.2) then provide capital stocks in the two regions at the next period, $t+1$. Our plane endomorphism, T, is now well defined.

The equations describing GNP in each region are nonlinear. Therefore the endomorphism T is nonlinear as well. In the following we shall study its qualitative properties and experiment with simulations depicted graphically. But before analysing the model, it will be useful to explain the connections with the environment.

1.2. North-South trade and the environment

The environment appears in this model as one of the inputs, or *factors* of production. While in the original North-South model the two factors of production are *labor* and *capital*, recently (GC85) the model has been extended to three factors of production, one of which is a *natural resource.* Furthermore, in GC91, one of the factors of production is a common property resource, such as an aquifer, or fish from a common body of water, or wood from a common forest. In the original North-South model the behavior of a parameter α – representing the supply response of a factor to its price – is shown to be crucial in explaining the patterns of trade between the two regions, including the terms of trade and the gains from trade. Furthermore, in GC91, the absolute value of this parameter in the South, α_S, is proven to vary with the *property rights regime* for the resource (such as land.) This resource is used as an input for the production of the traded goods (such as cash crops: coffee, cotton, palm oil.) It is therefore of interest to simulate the behavior of the North-South model with *different property rights* for this environmental resource, that is, different values of α_N and α_S. As an example, GC91 predicts that a regime of property rights which gives better rights to the locals of the rainforest (for example, in Guatemala) could improve the terms of trade on cash crops and control the overexploitation of the rainforest.

We now apply our model to explain the fundamental connection between the environment and trade. We will look at the environment as a common property resource which is used as an input to production in both regions. Examples are: rainforests, bodies of water, or fisheries. These are inputs to the production of goods which are internationally traded, such as: wood products, industrial output, cash crops (cotton, coffee, soya beans, palm oil.) In our model, we shall now reinterpret L as an *environmental input* used, together with the other input, K, to produce basic and industrial goods, B and I. Thus, we rename L as E for the remainder of this section.

A crucial parameter in the North-South model is α, the response of the supply of E to its relative price, w/p. In GC81 and GC86, this parameter α was shown to determine the properties of the solutions (equilibria). Here, α will play a similar role: it represents the *property rights* on the environmental resource, E: α is smaller when the property rights are well-defined, and larger when they are ill-defined. For example: if the local population has well-defined property rights on the biodiversity of a rainforest, which is an input to the production of pharmaceuticals, then the wood input E will be treated more carefully. To supply larger quantities of E will require a larger increase of the price of E, p_E. Thus, α is smaller when the property rights on the rainforest are well-defined. The theory and the analytics proving this fact appear in GC92 as Lemma 1. When property rights on the rainforest are ill-defined, α is large: this means that a lot more wood will be "cut-off" and the forest destroyed for smaller increases in prices. The *price* represents the value of the input. Well-defined property rights lead to a proper valuation of scarce resources. Good examples are provided by Merck Pharmaceuticals, Inc. and Shaman Pharmaceuticals, Inc. These companies have entered into agreements to advance cash and to share the profits from prospecting biodiversity samples in Costa Rica and in South American countries. The biodiversity samples are an input to the production of valuable pharmaceuticals (examples: *curare* and the more recently discovered *periwinkle* which treats Hodgkins disease and leukemia in children) sharing the profits with the locals. This amounts to improving the property rights of the local population on the common property resource: the rainforest's biodiversity. This scheme is not too different from the venture capital agreements which advance working capital to use intellectual property (software ideas) and share the rights subsequently with the entrepreneurs. By increasing the realized value of the common property input, these agreements increase the interest in conservation by those who would otherwise overuse or overexploit the resource beyond its biological steady-state "extraction rate."

All of these considerations are summarized in the North-South model by varying the parameter α in the South. This variation simulates the input of property right agreements in developing countries for their valuable common property resources. For the theories explaining the general impact of varying α in the static North-South model in GC86, see GC92. In this paper we address the *dynamic* North-South model, and ask the same questions. The problem is more complex since our model is dynamic, and we rely on simulation to provide our answers.

1.3. Organization of the paper

We begin by recalling the static North-South model. Then we will develop the equations for the general form of the dynamic North-South model in a sequence of steps. To reveal the mathematical structure of the problem, we will present, in the first of these steps, a very simplified one-dimensional dynamical version of our two-dimensional dynamical system. This is only a mathematical artifice, as the economics are embodied only in the full two-dimensional version. We then explain some qualitative properties of the dynamical model and present simulations which confirm our results and suggest possible extensions. We end with a proposal for a dynamical system linking our dynamic North-South model with the atmospheric chemistry of the carbon cycle.

2. Notational conventions. We will write K_N in place of the $K(N)$ used in GC86. We are going to encounter symbolic expressions in the variables:

$$K_N, K_S, s_N, s_S, \ldots$$

and so on. We will refer to K for example as a *root symbol,* and only when accompanied by a subscript N or S will the symbol denote a variable. Thus, we may write expressions or equations in these root variables, but they are symbolic only. When the appropriate subscripts are adjoined, they become expressions or equations of variables defined in our models. Let A be an expression of root symbols. Then A_N will denote the same expression in the corresponding variables of the North system, and likewise for A_S for the South, while A_T will be defined to mean $A_N + A_S$.

Note. Equation (GC2.21b) denotes equation 2.21b in the reference GC86.

3. Recalling the North-South model. We begin with the parameters, variables, and notations of the static North-South model [1]. The root symbols of the eight parameters in each region are: a_1, a_2, c_1, c_2, α, β, $\bar{K}$, and $\bar{L}$. Thus, we will encounter $a_1 = a_{1N}$, a_{1S}, etc. The crucial variables which determine the model are five price variables and sixteen quantity variables. The price variables are:

1. $p = p_B$ denotes the *price of basic goods, B.* Since the *price of industrial goods, I,* has been set to unity, $p_I = 1$, p is the *relative price of basics with repect to industrial goods.* It is also called the *terms of trade* since B and I are the only two goods in the international market. In a market equilibrium, p is the same in both regions, North and South, but all other price variables may differ in the two regions.

2. w denotes *wages.*

3. r denotes the *capital rental price.*

Since labor and capital are *not* traded internationally (that is, between the two regions), their values are determined by p according to local conditions (equations GC2.21b, GC2.4a) which are unequal in the two regions (because the two regions have different production technologies.) The five price variables, or *prices,* are p, r_N, r_S, w_N, w_S.

The *quantity variables* are the following.

4. K denotes *capital stock.* This is determined by r, see (GC2.4) and Fig. 3.1a below. This relationship is for the static model only. This K will be determined, in the dynamic models of this paper, by a discrete dynamical system modeling the annual variation of capital stock in each region.

5. L denotes *labor.* This is determined by w and p, see (GC2.3) and Fig. 3.1b below.

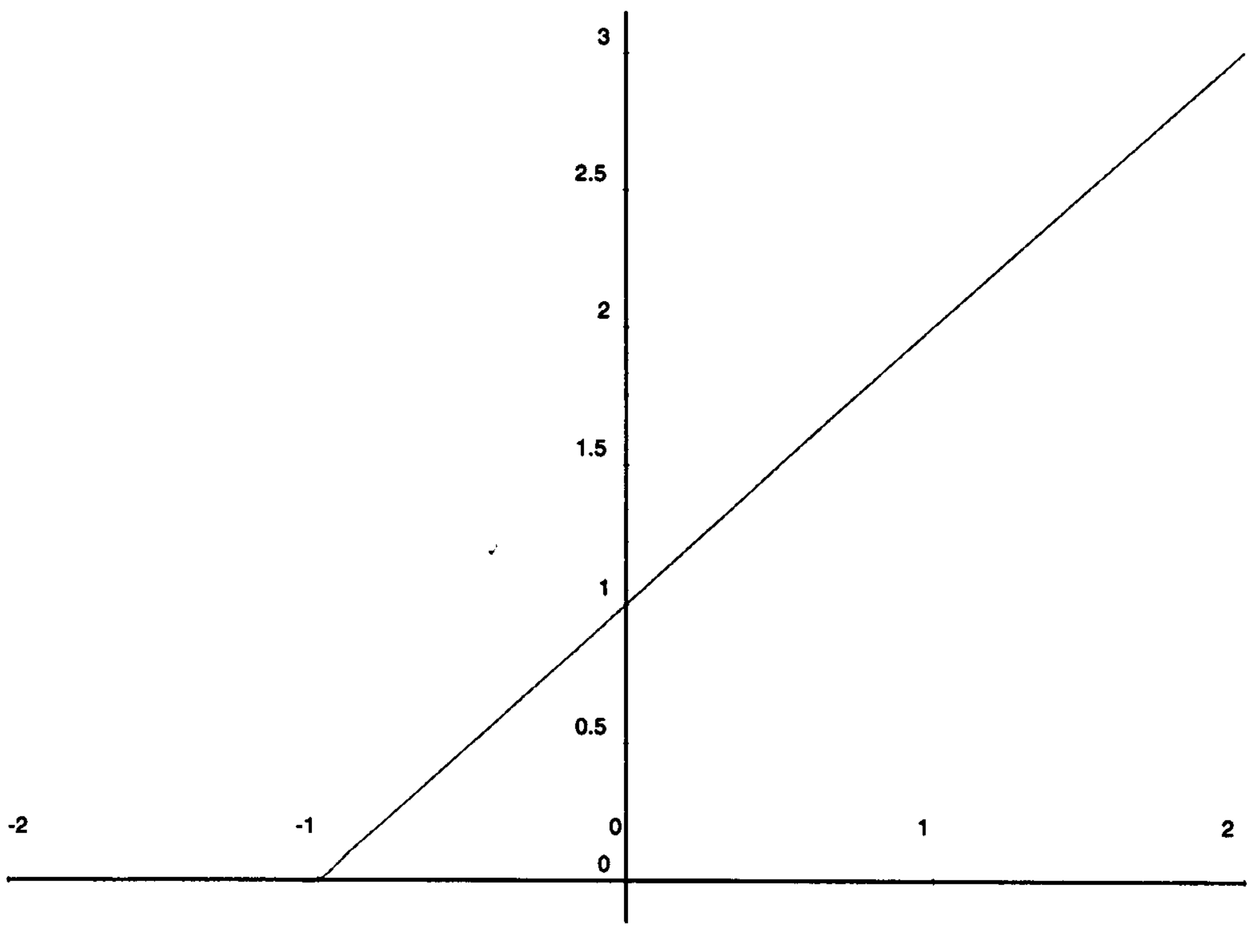

Fig. 3.1a. Graph of $K(r)$. The y-intercept is at $\bar{K}$, and the slope is β.

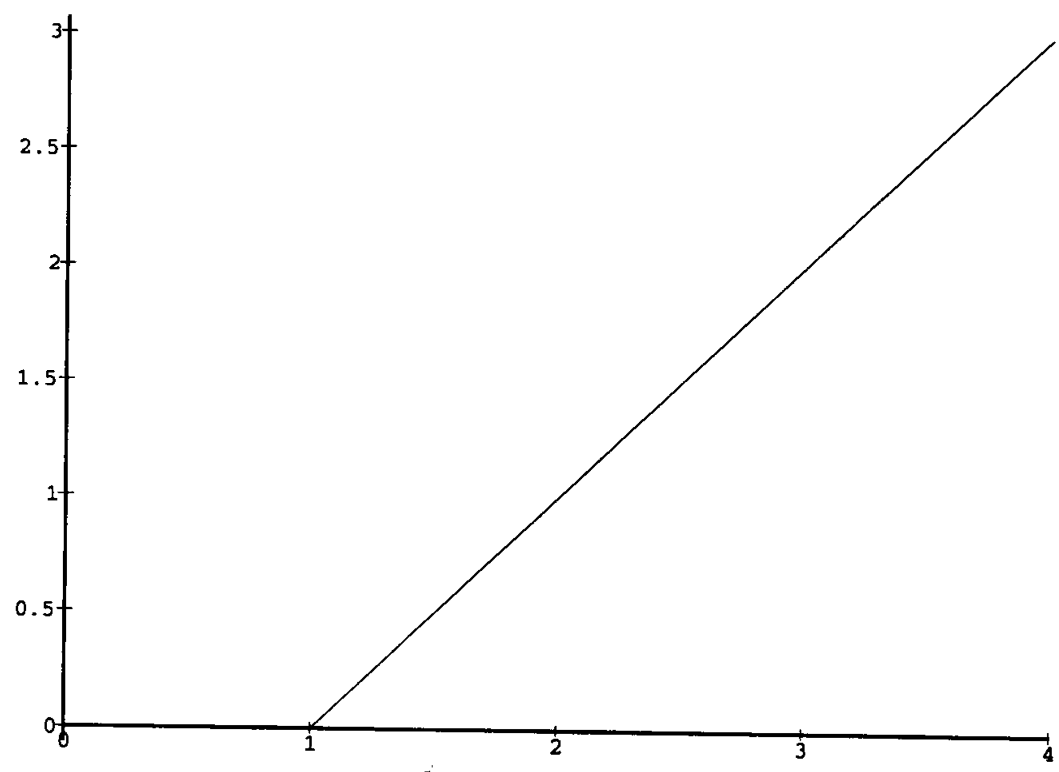

Fig. 3.1b. Graph of $L(w/p_B)$. The y-intercept is at $\bar{L}$, and the slope is α.

6. B^S and B^D denote quantities of *basic goods supplied* and *basic goods demanded.*

7. I^S and I^D denote quantities of industrial goods supplied and demanded.

8. $X_B^S = B^S - B^D$ and $X_I^S = I^S - I^D$ denote exports of goods, the excess of what is supplied over what is consumed in each region.

The sixteen quantity variables are: L, K, B^S, B^D, I^S, I^D, X_B^S, X_I^S, in each region. The diagram of Fig. 3.1 shows how p (and the parameters in each region) determine all of these other variables. Labor, L, and capital, K, are the inputs to production. Using labor and capital the two economies produce the two goods, or commodities, B^S and I^S. In each region, B^S is produced using labor and capital according to the formula,

$$B^S = min(L/a_1, K/c_1) \tag{3.1}$$

Therefore, efficient use of L and K requires that

$$B^S = L/a_1 = K/c_1$$

that is, labor and capital are used in fixed proportions for each level of output of B^S, or

$$L/K = a_1/c_1$$

where a_1 is called the *labor-output ratio* (since $B^S = L/a_1$) and c_1 is called the *capital-output ratio* (since $B^S = K/c_1$). Equation (3.1) is the *production technology* which determines how much B can be produced with the available K and L. Similarly, each region has a production technology for I,

$$I^S = min(L/a_2, K/c_2) \tag{3.2}$$

with the same interpretation for the parameters a_2 and c_2. Equations (3.1) and (3.2) give rise to (GC2.20). (GC indicates an equation number from Sec. 2 of GC86.)

Now α and β represent the responses of labor and capital supplies to changes in their prices: w and r. We postulate:

$$L = \alpha w/p_B + \bar{L} \tag{GC2.3}$$

with $\bar{L} < 0$, and

$$r = (K - \bar{K})/\beta \tag{GC2.4}$$

with $\bar{K} > 0$. Equation (GC2.3) means that as the real wage w/p_B increases, so does the supply of labor. And equation (GC2.4) means the same for capital. The negative value of $\bar{L}$ indicates the minimum wage needed for survival before people supply positive labor. See the graphs in Fig. 3.1. NOTE: These relationships are particular to the static model. Later in this paper, while retaining the static relationship (GC2.3), we shall replace (GC2.4) with a dynamic rule. Some further relationships are the following, all from GC86.

$$p_B = (a_1 - rD)/a_2 \tag{GC2.21}$$

$$B^S = (c_2L - a_2K)/D \tag{GC2.20}$$

$$I^S = (a_1K - c_1L)/D \tag{GC2.20}$$

and the intermediate functions are defined by

$$w = (p_B c_2 - c_1)/D \qquad (GC2.21)$$

all non-negative, and

$$D = a_1 c_2 - a_2 c_1.$$

All remaining symbols denote constants defined in GC86. Note that the superscript S in B^S and I^S denotes Supply (vs Demand), not South (vs North). Also, the subscript B in p_B indicates Basic (vs the subscript I for Industrial.) Henceforth, we will omit these subscripts when no confusion results (esp. in Sec. 4). Hence: L for L^S (we will not use L^D), p for p_B (we will not use p_I), B for B^S (we will not use B^D), and I for I^S (we will write I^D when we mean demand for industrial goods). Thus the equations above become:

$$p = (a_1 - rD)/a_2 \qquad (GC2.21a)$$

$$B = (c_2 L - a_2 K)/D \qquad (GC2.20a)$$

$$I = (a_1 K - c_1 L)/D \qquad (GC2.20b)$$

and the intermediate functions are defined by

$$L = \alpha w/p + \bar{L} \qquad (GC2.3a)$$

$$w = (pc_2 - c_1)/D \qquad (GC2.21b)$$

$$r = (K - \bar{K})/\beta \qquad (GC2.4a)$$

all non-negative, and

$$D = a_1 c_2 - a_2 c_1.$$

To close the model in GC86, two more variables were fixed:

$$I = \bar{I}^D$$

in each region.

This demand specification corresponds to a simple preference form which was defined and illustrated in GCH86. One can consider many other demand specifications without cahnging the structure of the model or its behavior, as shown in GCH81 and GCH86. Indeed, in the specification of our dynamical North-South model, the two-dimensional endomorphism is defined using a demand specification (5.3.1) which amounts to requiring that the demand for industrial goods I^D is a proportion $1 - \gamma$ of GNP. This last specification is useful in a North-South world, because typically industrial countries consume a higher proportion of their GNP in the form of industrial goods, while developing countries consume proportionately more basic goods. With our specification (5.3.1) it is also possible to simulate an economy where the proportion γ depends on the GNP level, with γ decreasing as a function of GNP. We now begin a step-by-step development of our two-dimensional dynamical system. The first step will be a simple one-dimensional model.

4. One-dimensional models. Here we introduce dynamics for macroeconomic variables of the North region, and enslave the variables of the South, according to the following.

PROPOSITION 4.1. In the North-South model, the South capital is obtained from the North by the affine isomorphism,

$$K_S = H_0 + H_1 K_N$$

where

$$H_1 = \frac{\beta_S a_{2S} D_N}{\beta_N a_{2N} D_S}$$

and

$$H_0 = \frac{\beta_S}{D_S}\left[-\frac{a_{1N}}{a_{2N}}a_{2S} + a_{1S}\right] - H_1\bar{K}_N + \bar{K}_S$$

Proof. From (GC2.4) we have

$$K_N = \beta_N r_N + \bar{K}_N \tag{4.0.1}$$

and

$$K_S = \beta_S r_S + \bar{K}_S \tag{4.0.2}$$

As we assume the terms of trade $p = p_B$ are the same in each region, $p_S = p_N$, or from (GC2.21a),

$$p = (a_{1S} - r_S D_S)/a_{2S} = (a_{1N} - r_N D_N)/a_{2N} \tag{4.0.3}$$

or, solving for r_S,

$$r_S = \frac{1}{D_S}\left\{(r_N D_N - a_{1N})\frac{a_{2S}}{a_{2N}} + a_{1S}\right\} \tag{4.0.4}$$

We now substitute (4.0.4) into (4.0.2) and obtain

$$K_S = \beta_S r_S + \bar{K}_S = \frac{\beta_S}{D_S}\left\{(r_N D_N - a_{1N})\frac{a_{2S}}{a_{2N}} + a_{1S}\right\} + \bar{K}_S$$

Using (GC2.4a) to replace r_N, we have

$$K_S = \frac{\beta_S}{D_S}\left\{\frac{a_{2S} D_N}{a_{2N}}\left[\frac{K_N - \bar{K}_N}{\beta_N}\right] - \frac{a_{1N}}{a_{2N}}a_{2S} + a_{1S}\right\} + \bar{K}_S$$

and simplifying, we get the proposition. ♣

Henceforth in Section 4, we will write K in place of K_N, and so forth.

4.1. The dynamics of the North-South model. We envision a dynamic in which changes in the capital stock in the North result, after a rapid transit to new static equilibrium, in new equilibrium values of the variables. We use discrete dynamics to model the annual reports of these variables. And now, equation (4.0.1) is understood as a demand equation, so that $\beta_N < 0$. *This differs from GC86.* The annual increment of K will be defined by a function, $f : \mathbf{R} \setminus \{\bar{x}\} \to \mathbf{R}$ (we will identify the excluded point $\bar{x}$ subsequently), so that for year $n+1$, we have $K(n+1) = f(K(n))$. Also, we write K^+ for $f(K)$. This function is assumed to be defined by

$$f(K) = (1-\delta)K + s(GNP),\ 0 < \delta, s < 1 \tag{4.1.1}$$

where the depreciation rate, δ, and the rate of savings, s, are constants with small, positive values, and

$$GNP = [pB + I] \tag{4.1.2}$$

As usual, GNP is the inner product of goods and prices, and again, $p_I = 1$ (GC2.16).

After substitution of the expressions in the preceding section, the endomorphism f may be written in the following form.

PROPOSITION 4.2. The function defined in (4.1.1) may be expressed as

$$f(K) = A_0 + A_1 K + A_2 K^2 + A_*/(K - K_0)$$

where the coefficients are given by

$$\begin{aligned} A_0 &= (s/\alpha)[1 + c_2\bar{K}/\beta](\bar{L} + \alpha c_2/D) - s a_2^2 c_1 \\ A_1 &= (1-\delta) + (s/\beta)\{-\bar{K} - (c_2/a_2)(\bar{L} + \alpha c_2/D)\} \\ A_2 &= s/\beta \\ A_* &= -s(c_1^2 a_2 \beta/D) \end{aligned}$$

and the singular point ($\bar{x}$ above) is

$$K_0 = \bar{K} + a_1\beta/D$$

4.2. Proof of Proposition 4.2. We will demonstrate the dynamical rule given above in six steps.

Step 1. First we observe:

$$p = u_1(K - K_0)$$

where $u_1 = -D/a_2\beta$, and $K_0 = \bar{K} + a_1\beta/D$.

Proof. From (GC2.21a) of Section 2 we have,

$$p = (a_1 - rD)/a_2$$

and substituting for r from (GC2.4a) above,

$$p = \frac{a_1}{a_2} - \frac{(K - \bar{K})D}{a_2\beta}$$

from which we obtain

$$p = u_0 + u_1 K$$

where u_1 is defined above, and

$$u_0 = \frac{a_1\beta + D\bar{K}}{a_2\beta}$$

Then Step 1 follows, with

$$K_0 = -u_0/u_1 = \frac{a_1\beta + D\bar{K}}{a_2\beta} + \frac{a_2\beta}{D} = \frac{a_1\beta}{D} + \bar{K}$$

Step 2. Continuing, we find:

$$pL = -\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1)$$

Note: Combining Steps 1 and 2, we have expressed L as a function of K. Combining with Proposition 4.1, we see that the evolution of all four of the primary variables, K_N, L_N, K_S, and L_S, are determined from our one-dimensional model.

Proof of Step 2. From (GC2.3a) of Section 2 we have,

$$pL = p(\alpha\frac{w}{p} + \bar{L}) = \alpha w + p\bar{L}$$

and substituting for w from (GC2.21b),

$$pL = \alpha\frac{pc_2 - c_1}{D} + p\bar{L} = \left(\frac{\alpha c_2}{D} + \bar{L}\right)p - \frac{\alpha c_1}{D}$$

Using Step 1,

$$\begin{aligned} pL &= \left(\frac{\alpha c_2}{D} + \bar{L}\right)p_1(K - K_0) - \frac{\alpha c_1}{D} \\ &= -\left(\frac{\alpha c_2}{D} + \bar{L}\right)\frac{D}{a_1\beta}K + \left(\frac{\alpha c_2}{D} + \bar{L}\right)\frac{D}{a_1\beta}K_0 - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{\beta a_1}K + \frac{\alpha c_2 + \bar{L}D}{\beta a_1}\left(\bar{K} + \frac{a_1\beta}{D}\right) - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \frac{\alpha c_2}{D} + \bar{L} - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1) \end{aligned}$$

completing the derivation.

Step 3. Next, see that:

$$pK = -\frac{D}{a_2\beta}K^2 + \left[\frac{D}{a_2\beta}\bar{K} + \frac{a_1}{a_2}\right] K$$

Proof. From Step 1 we have,

$$\begin{aligned} pK &= p_1(K - K_0)K \\ &= p_1K^2 - p_1K_0K \\ &= -\frac{D}{a_2\beta}K^2 + \frac{D}{a_2\beta}\left[\bar{K} + \frac{a_1\beta}{D}\right] K \\ &= -\frac{D}{a_2\beta}K^2 + \left[\frac{D}{a_2\beta}\bar{K} + \frac{a_1}{a_2}\right] K \end{aligned}$$

Step 4. Putting these together, we have:

$$pB = C_0 + C_1K + C_2K^2$$

where

$$\begin{aligned} C_0 &= \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1) \\ C_1 &= -\frac{\alpha c_2^2}{a_1\beta D} - \frac{c_2\bar{L}}{a_1\beta} - \frac{\bar{K}}{\beta} - \frac{a_1}{D} \\ C_2 &= \frac{1}{\beta} \end{aligned}$$

Proof. From Section 2 (GC2.20a) we have,

$$\begin{aligned} pB &= p\frac{c_2L - a_2K}{D} \\ &= \frac{c_2}{D}pL - \frac{a_2}{D}pK \end{aligned}$$

in which we may replace pL with Step 2, and pK by Step 3, obtaining

$$\begin{aligned} pB = &\frac{c_2}{D}\left\{-\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1)\right\} \\ &- \frac{a_2}{D}\left\{-\frac{D}{a_2\beta}K^2 + \left[\frac{D}{a_2\beta}\bar{K} + \frac{a_1}{a_2}\right] K\right\} \\ = &\frac{1}{\beta}K^2 - \left\{\frac{\alpha c_2^2}{a_1\beta D} + \frac{c_2\bar{L}}{a_1\beta} + \frac{\bar{K}}{\beta} + \frac{a_1}{D}\right\} K + \left\{\frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1)\right\} \end{aligned}$$

which is Step 4.

Step 5. Similarly, see that:

$$I = I_0 + I_1 K + I_*/(K - K_0)$$

where

$$I_0 = -\left[\frac{c_1 \bar{L}}{D} + \frac{\alpha c_1 c_2}{D^2}\right]$$
$$I_1 = \frac{a_1}{D}$$
$$I_* = -\frac{\alpha\beta a_2 c_1^2}{D^3}$$

Proof. From Section 2 (GC2.20b) we have,

$$\begin{aligned} I &= \frac{a_1 K - c_1 L}{D} \\ &= \frac{a_1}{D}K - \frac{c_1}{D}\left[\alpha\frac{w}{p} + \bar{L}\right] \\ &= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c_1}{D}\frac{w}{p} \\ &= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c_1}{D^2}\left[c_2 - \frac{c_1}{p}\right] \\ &= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c - 1 c_2}{D} + \frac{\alpha c_1^2}{D}\frac{1}{p} \\ &= \frac{a_1}{D}K - \left[\frac{c_1 \bar{L}}{D} + \frac{\alpha c_1 c_2}{D^2}\right] + \frac{\alpha c_1^2}{D^2}\frac{1}{p_1(K - K_0)} \end{aligned}$$

which is Step 5.

Step 6.

$$GNP = G_0 + G_1 K + G_2 K^2 + G_*/(K - K_0)$$

where

$$G_0 = C_0 + I_0 = \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \left(1 - \frac{c_1}{D}\right)\bar{L} + \frac{\alpha}{D}(c_2 - c_1) - \frac{\alpha c_1 c_2}{D^2}$$

$$G_1 = C_1 + I_1 = -\left[\frac{\alpha c_2^2}{a_1\beta D} + \frac{c_2 \bar{L}}{a_1\beta} + \frac{\bar{K}}{\beta}\right]$$

$$G_2 = C_2 = 1/\beta$$

$$G_* = I_* = -\alpha\beta a_2 c_1^2/D^3$$

Proof. From Section 4 (4.1.2) we have,

$$GNP = [pB + I]$$

in which we may replace pB by Step 4, and I by Step 5, obtaining

$$GNP = C_2K^2 + (C_1 + I_1)K + (C_0 + I_0) + I_* \frac{1}{K - K_0}$$

which completes our derivation. ♣

4.3. Preliminaries on quadratic maps. In the preceding sections we have obtained an endomorphism of real numbers, generating a semi-cascade (discrete dynamical system), for the dynamics of the North-South model. To relate this model to the well known logistic map, we will make use of the following [2].

PROPOSITION 4.3. A quadratic function, $f : \mathbf{R} \to \mathbf{R}$, defined by

$$f(x) = A_0 + A_1x + A_2x^2$$

with $A_2 \neq 0$, and the discriminant $\Delta^2 = (A_1 - 1)^2 - 4A_0A_2 > 0$, has a repelling fixed point at

$$B_0 = -\frac{(A_1 - 1)}{2A_2} + \frac{\Delta}{2A_2}$$

with its distinct preimage at $B_0 + B_1$, where

$$B_1 = -A_1/A_2 - 2B_0$$

The affine function,

$$x : \mathbf{R} \to \mathbf{R}; y \mapsto x(y) = B_0 + B_1y$$

is an affine isomorphism, and conjugates f into the canonical form for the quadratic family,

$$g(y) = x^{-1}(f(x(y))) = \mu y(1 - y)$$

with

$$\mu = 1 + \Delta$$

Furthermore, the usual domain of this logistic function, $y \in J = [0,\ 1]$, is mapped to an interval $x \in I = [B_0,\ B_0 + B_1]$, in the orientation preserving case $B_0 > 0$, else $x \in I = [B_0 + B_1,\ B_0]$, by this affine isomorphism.

Proof. To compute the next value of y under the conjugate map, we apply the inverse map to y^+,

$$\begin{aligned} y^+ &= -\frac{B_0}{B_1} + \frac{1}{B_1}x^+ \\ &= -\frac{B_0}{B_1} + \frac{1}{B_1}f(x) \\ &= -\frac{B_0}{B_1} + \frac{1}{B_1}[A_0 + A_1x + A_2x^2] \end{aligned}$$

and then with $x \mapsto y$,

$$y^+ = -\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1}(B_0 + B_1 y) + \frac{A_2}{B_1}(B_0 + B_1 y)^2$$
$$= \left[-\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1} B_0 + \frac{A_2}{B_1} B_0^2 \right] + [A_1 + 2A_2 B_0] y + (A_2 B_1) y^2$$

Now we equate this with the desired canonical form,

$$y^+ = g(y) = \mu y(1 - y) = 0 + \mu y + (-\mu) y^2$$

term by term.

For degree zero,

$$-\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1} B_0 + \frac{A_2}{B_1} B_0^2 = 0$$

and as $A_2 \neq 0$ and $B_1 \neq 0$,

$$A_2 B_0^2 + (A_1 - 1) B_0 + A_0 = 0$$

from which, by the binomial formula,

$$B_0 = -\frac{(A_1 - 1 \pm \Delta)}{2A_2}$$

NOTE. The quadratic equation for B_0 here is the condition for a fixed point of the map f, so the $\pm$ yields the two fixed points. As the slope of f at these two possible values for B_0 is

$$f'(B_0) = A_1 + 2A_2 B_0 = 1 \pm \Delta$$

we choose the positive sign for the repelling fixed point. If B_0^- denotes the other root, with the minus sign, then this is the paired fixed point, created by a fold bifurcation, and initially attractive, for Δ small and positive. Then its distinct preimage is $B_0^- + B_1^-$, where $B_1^- = -A_1/A_2 - 2B_0^-$. Also, note that the critical point is $x_e = -A_1/2A_2$.

For degree one,

$$\mu = A_1 + 2A_2 B_0$$

and for degree two,

$$\mu = -A_2 B_1$$

Subtracting these two expressions and solving for B_1,

$$B_1 = -\frac{A_1}{A_2} - 2B_0$$

completing the specification of the affine isomorphism. From the first expression for μ above we obtain its form in the proposition. ♣

COROLLARY. Given the function $f : \mathbf{R} \setminus \{\bar{x}\} \to \mathbf{R}$, defined by

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_*/(x - \bar{x})$$

with $A_2 \neq 0$, and $(A_1 - 1)^2 > 4A_0A_2$, then $y \mapsto x = B_0 + B_1 y$ with

$$B_0 = -\frac{A_1 - 1 + \Delta}{2A_2}$$

and

$$B_1 = -A_1/A_2 - 2B_0$$

is an affine isomorphism, and conjugates f to the canonical form $g : \mathbf{R} \setminus \{\bar{y}\} \to \mathbf{R}$, with

$$g(y) = x^{-1}(f(x(y))) = \mu y(1 - y) + \nu/(y - \bar{y})$$

with $\nu = A_*/B_1^2$, $\bar{y} = \bar{x}/B_1 - B_0/B_1 = x^{-1}(\bar{x})$, and $\mu + \Delta$ as above. And as above, the usual domain of the logistic function, $y \in J = [0,\ 1]$, assuming $\bar{y} \notin J$, is again mapped to the interval, $x \in I = [B_0,\ B_0 + B_1]$, by the affine isomorphism.

Proof. The quadratic terms are conjugated as shown, according to Proposition 4.3 above. For the last term, see that

$$(1/B_1)\frac{A_*}{x - \bar{x}} = \frac{\nu}{y - \bar{y}}$$

with which, the formula for g is obtained. ♣

REMARK. If the singular point $\bar{y}$ lies outside the interval J, then this interval as approximately the invariant interval defined by the initially repelling fixed point and its distinct preimage. In case the point $\bar{y}$ lies to the right of the interval J, the domain of g should be reduced to the subinterval J^* defined by the expanding fixed point and its nearby preimage, as shown in Fig. 4.1(a). In case $\bar{y}$ lies to the left of J, then the interval may be increased to J^*, as shown in Fig. 4.1(b). The case with $\bar{y}$ in the interval is shown in Fig. 4.1(c).

The invariant interval of g, J^*, is not identical to the reference interval, $J = [0,\ 1]$ unless $\nu = 0$. Likewise, we have an interval for f, I^*, not identical to the corresponding reference interval, $I = [B_0,\ B_0 + B_1]$.

4.4. Simulations. We begin by fixing values for the many parameters appearing in this dynamical system. First, let $\delta = 0.1$ and $s = 0.08$. For the others, our guide will be

Table (c) on page 44 of (GC86), except for the sign of β which we reverse. Thus, in the North,

$$
\begin{array}{ll}
a_1 = 2 & \bar{K} = 12 \\
a_2 = 0.15 & \bar{L} = 0.5 \\
c_1 = 1.8 & \alpha = 6 \\
c_2 = 1.7 & \beta = -9.7
\end{array}
$$

These are chosen so that $p,\ r,\ w,\ L,\ K,\ B,\ I > 0$ in each region. Note that the bifurcation parameter μ in the transformed dynamical system depends upon all of these values. The derived constants are then approximately:

$$D = 3.13$$

$$
\begin{array}{ll}
A_0 = -0.058524 & B_0 = 0.167727 \\
A_1 = 1.350306 & B_0^- = 42.306847 \\
A_2 = -0.008247 & B_1^- = 79.110881 \\
A_* = 0.120491 & B_1 = 163.389119
\end{array}
$$

with the singularity at $\bar{x} = 5.801917$ and the attracting fixed point at 42.316339. All these features are shown on the graph of the function f corresponding to these parameters in Figs. 4.2a, b.

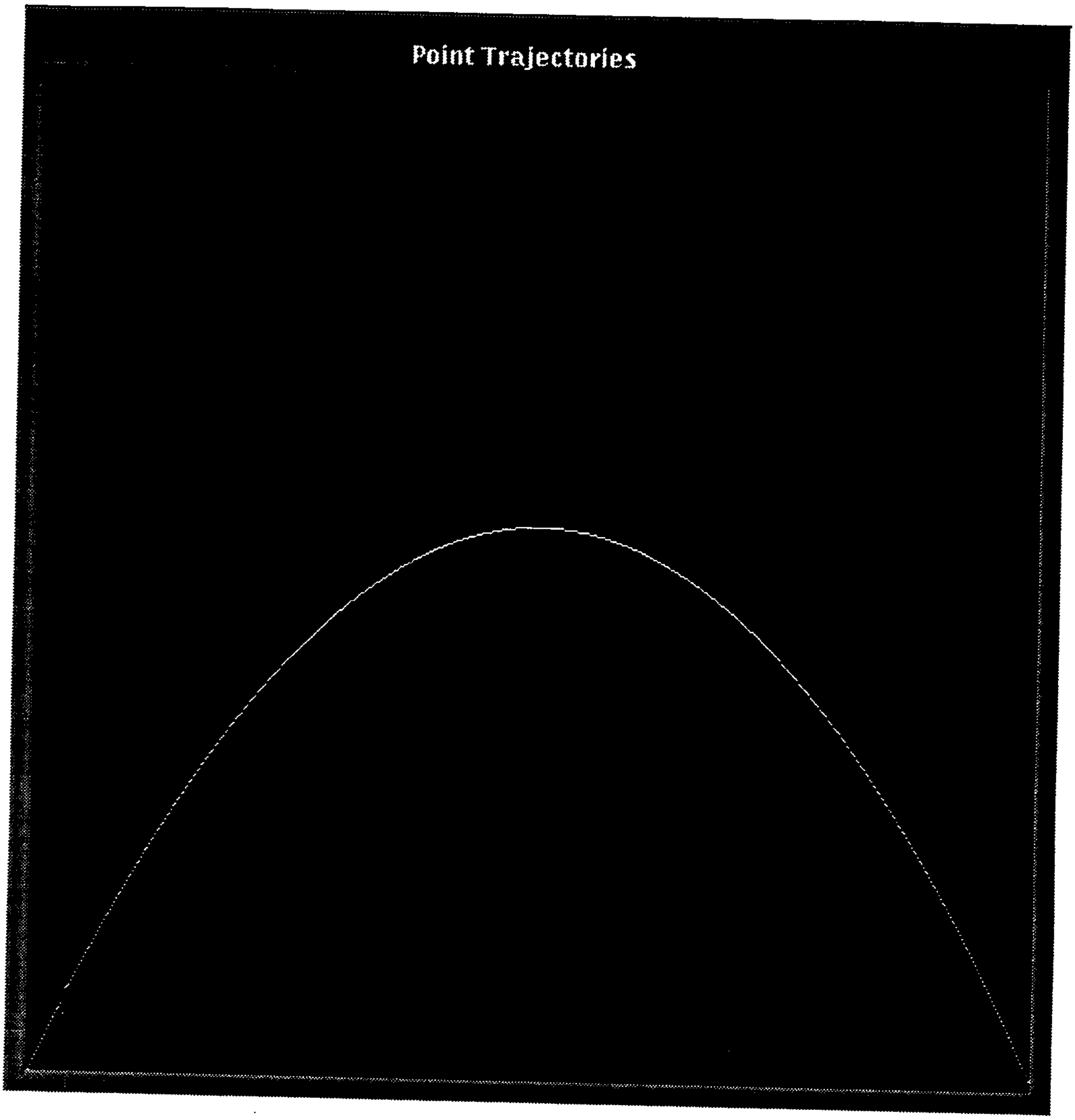

Figure 4.2a. Graph of the one-dimensional model with $0 < x < 163$. Note the gap in the graph at the left. This is the singularity, shown enlarged in Fig. 4.2b.

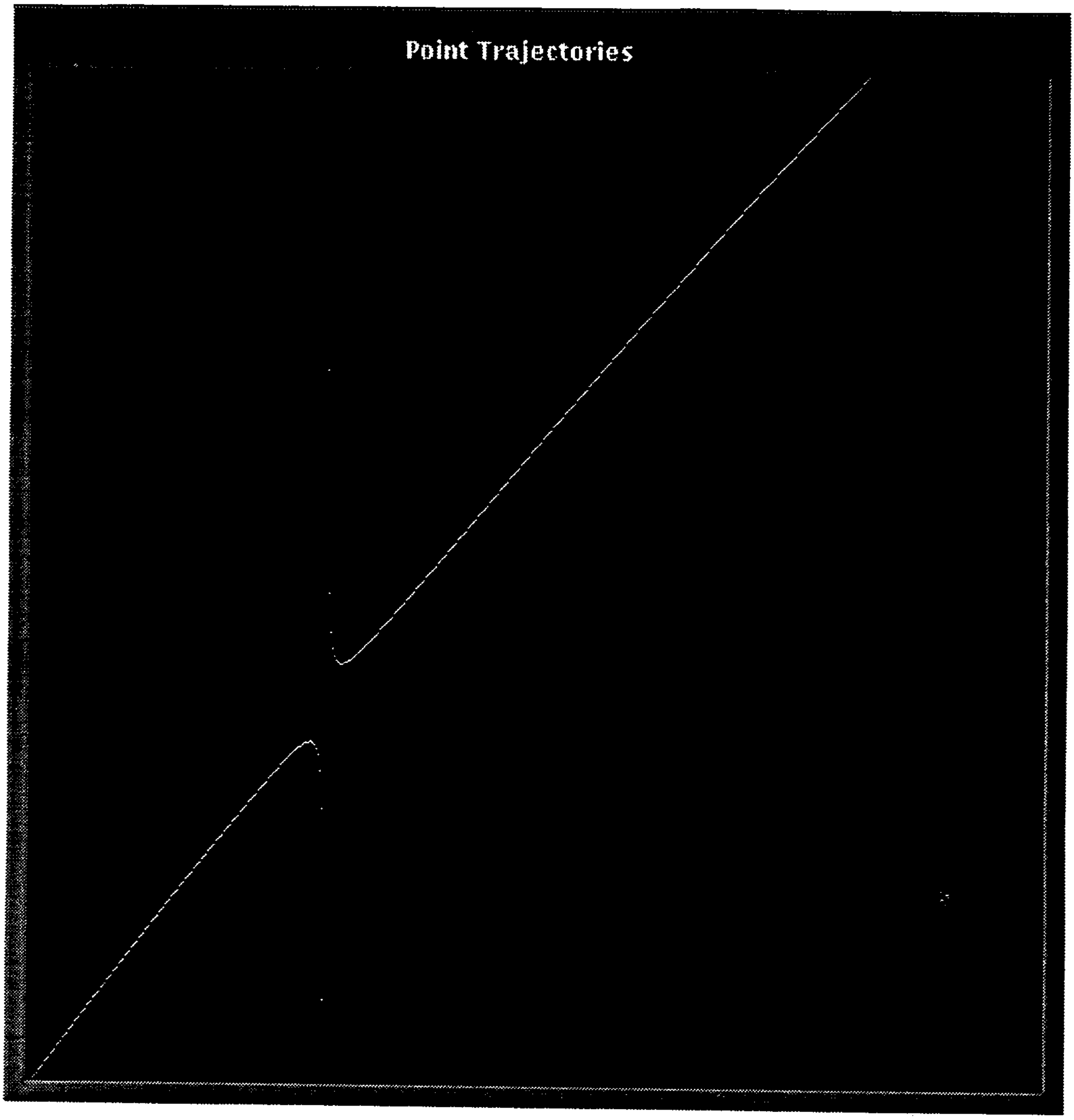

Figure 4.2b. Graph of the one-dimensional model with $0 < x < 20$ illustrating the singularity in the map.

The bifurcation diagram for function f of (4.1.1) – with all the parameters fixed with these values except for α, which is regarded as the control parameter in the simulation – is the familiar orbit diagram for the quadratic family, as shown in Fig. 4.3.

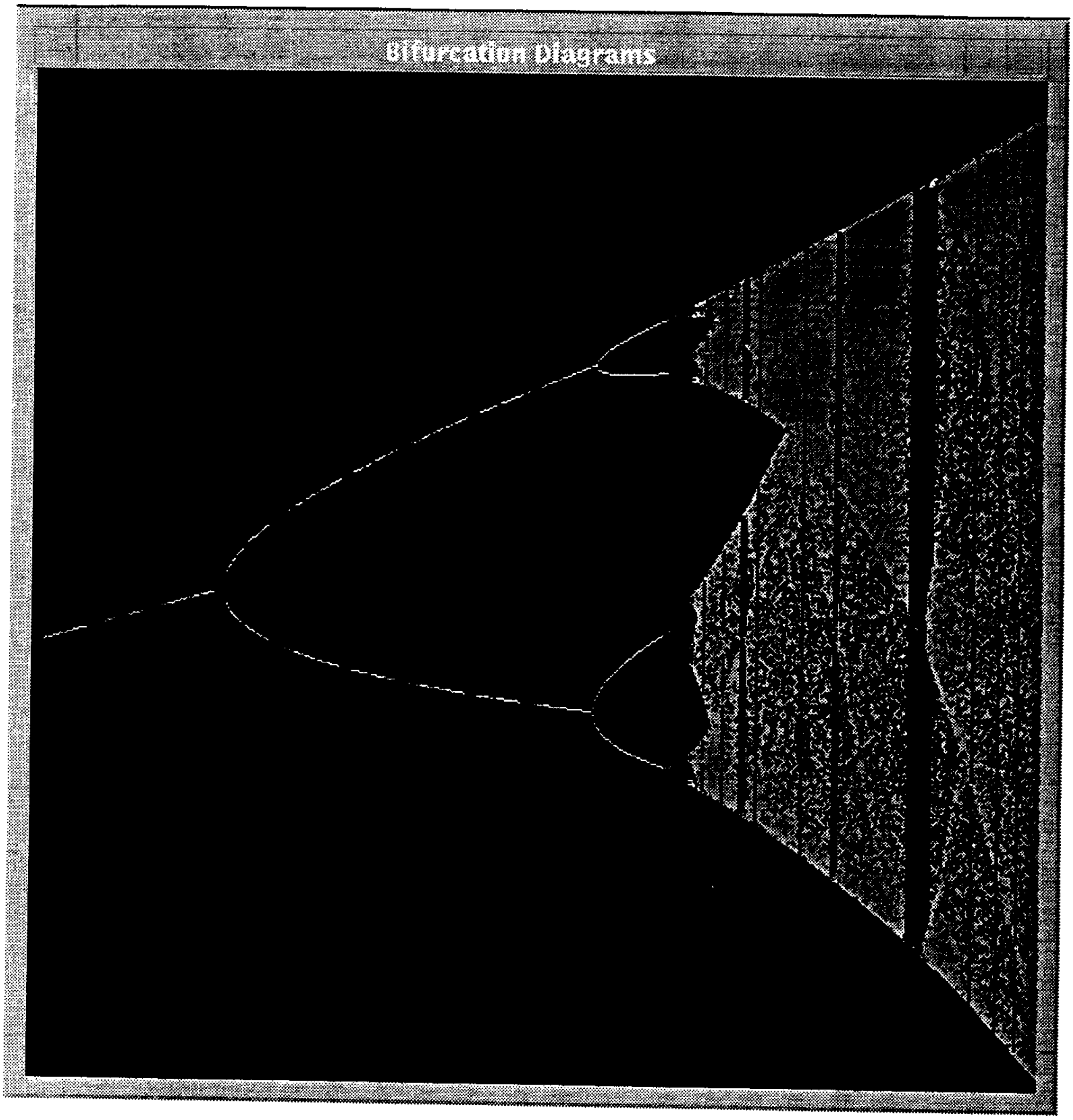

Figure 4.3. Response diagram for the one-dimensional model with $6 < \alpha < 8$. This is the familiar figure for the quadratic family. The vertical axis is the domain of the one-dimensional dynamical system. Each value of the control parameter α determines a vertical interval, and a particular map generating the dynamic. The white point (or set of points) is the unique attractor of the dynamical system for the given value of the control parameter: a point attractor (as in equilibrium theory), periodic attractor (as in business cycles), or a chaotic attractor (as in economic data).

5. *Two-dimensional models.* In the first dynamical system studied above, we had an

evolution in the North variables, while the South variables were to be determined from their Northern siblings by an algebraic relation. We now want to consider a more symmetric dynamic, in which the corresponding variables in both regions are in mutual coevolution.

5.1. A preliminary model. Here we rewrite the one-dimensional model as a two-dimensional model without changing the dynamics for K_N. That is, instead of obtaining K_S from K_N after each timestep by conjugation with the affine isomorphism of Proposition 4.1, which assumed a rapid settling to static equilibrium, we will derive a semi-cascade for K_S parallel to that of K_N.

From Proposition 4.1 we have

$$K_S = H_0 + H_1 K_N \tag{5.1}$$

while from Proposition 4.2,

$$K_N(n+1) = f(K_N(n))$$

or writing f_N in place of f,

$$K_N^+ = f_N(K_N) \tag{5.2}$$

Note that the inverse of Proposition 4.1 is

$$K_N = \frac{K_S - H_0}{H_1} \tag{5.3}$$

We now apply the map of (5.1) to the left-hand side of (5.2), and its inverse (5.3) to the right-hand side, as in the proof of Proposition 4.3, with the following result.

PROPOSITION 5.1. The dynamic (5.2) for K_N implies a conjugate dynamic for K_S, which may be expressed,

$$K_S(n+1) = f_S(K_S(n)) \textit{ or } K_S^+ = f_S(K_S)$$

where the generating endomorphism is,

$$f_S(y) = A_{0S} + A_{1S}y + A_{2S}y^2 + A_{*S}\frac{1}{y-\bar{y}}$$

and the coefficients are given by,

$$\begin{aligned}
A_{0S} &= H_0 + H_1 A_0 - A_1 H_0 + A_2 \frac{H_0^2}{H_1} \\
A_{1S} &= A_1 - 2A_2 \frac{H_0}{H_1} \\
A_{2S} &= \frac{A_2}{H_1} \\
A_{*S} &= H_1^2 A_* \\
\bar{y} &= H_0 + K_N^0 H_1
\end{aligned}$$

NOTE. Given K_S and all the parameters, we obtain all the variables. But, we will use different values for the parameters in the South: Again, as in Section 4.4, we let $\delta = 0.1$ and $s = 0.08$. For the others, we again refer to Table (c) on page 44 of (GC86), except for the sign of β which we reverse. Thus, in the South,

$$\begin{array}{ll} a_1 = 4.5 & \bar{K} = 2.7 \\ a_2 = 0.02 & \bar{L} = -2 \\ c_1 = 0.01 & \alpha = 75 \\ c_2 = 3 & \beta = -0.025 \end{array}$$

Again, these are chosen so that $p,\ r,\ w,\ L,\ K,\ B,\ I > 0$ in each region. Note that the bifurcation parameter μ in the transformed dynamical system depends upon all of these values. The derived constants are then approximately:

$$D = 13.5$$

$$\begin{array}{ll} A_0 = -750.642844 & B_0 = 2.691218 \\ A_1 = 558.498719 & B_0^- = 2.694575 \\ A_2 = -103.512843 & B_1^- = 0.006303 \\ A_* = 0.0000000008 & B_1 = 0.013018 \end{array}$$

with the singularity at $\bar{x} = 2.691667$ and the attracting fixed point at 2.694576. All these features are shown on the graph of the function f_S corresponding to these parameters in Fig. 5.0.

Proof. From PROPOSITION 4.1 we have

$$K_S = H_0 + H_1 K_N$$

with inverse

$$K_N = \frac{K_S - H_0}{H_1}$$

while from Proposition 4.2,

$$K_N(n+1) = f_N(K_N(n))$$

As in the proof of PROPOSITION 4.3, we now apply the affine isomorphism and its inverse to this equation, getting

$$
\begin{aligned}
K_S^+ &= H_0 + H_1 K_N^+ \\
&= H_0 + H_1 f(K_N) \\
&= H_0 + H_1 f(\frac{K_S - H_0}{H_1}) \\
&= H_0 + H_1 A_0 + H_1 A_1 \frac{K_S - H_0}{H_1} + H_1 A_2 \left[\frac{K_S - H_0}{H_1}\right]^2 \\
&+ H_1 A_* \frac{1}{(K_S - H_0)/H_1 - K_0} \\
&= H_0 + H_1 A_0 + A_1 (K_S - H_0) + \frac{A_2}{H_1}(K_S^2 - 2H_0 K_S + H_0^2) \\
&+ H_1^2 A_* \frac{1}{K_S - H_0 - K_0 H_1}
\end{aligned}
$$

from which the proposition follows. ♣

We may apply the Corollary of Proposition 4.3 independently to each of the dynamical systems (4.2) and (5.1), obtaining the (uncoupled) two-dimensional logistic endomorphism,

$$
\begin{aligned}
k_N^+ &= \mu_N k_N (1 - k_N) + \nu_N / (k_N - k_{N0}) \\
k_S^+ &= \mu_S k_S (1 - k_S) + \nu_S / (k_S - k_{S0})
\end{aligned}
$$

both on the unit interval, with

$$
\begin{aligned}
\mu_N &= 1 + \sqrt{(A_{1N} - 1)^2 - 4A_{0N} A_{2N}} = 1 + \Delta_N \\
\mu_S &= 1 + \sqrt{(A_{1S} - 1)^2 - 4A_{0S} A_{2S}} = 1 + \Delta_S \\
\nu_N &= A_{*N} / B_{1N}^2 \\
\nu_S &= A_{*S} / B_{1S}^2
\end{aligned}
$$

That is, we have in this model a minor modification of two (uncoupled) logistic maps, each of the form

$$f(K) = (1 - \delta)K + s(GNP)$$

or equivalently,

$$f(K) = (1 - \delta)K + s(pB + I).$$

We now seek to couple them through p.

5.2. The main model. We will work with an endomorphism of the plane,

$$T : R^2 \to R^2; (K_N,\ K_S) \mapsto (K_N^+,\ K_S^+)$$

defined as in the one-dimensional model by

$$K_N^+ = s_N(pB_N + I_N) + (1 - \delta_N)K_N \tag{5.2.1}$$
$$K_S^+ = s_S(pB_S + I_S) + (1 - \delta_S)K_S \tag{5.2.2}$$

where the terms of trade, p, are the same in both regions, because markets are competitive. These equations predict growth of capital stock in one fiscal period. As before, $pB + I$ is the GNP (gross national product), s is the savings rate, and δ is depreciation. In our simulations, we will use $s \approx 12/100$, and $\delta \approx 10/100$ for both regions.

The time evolution of all of the variables in each system is to be found by the iteration of the mapping T, beginning with any initial state, $(K_N^0,\ K_S^0)$. To complete the definition of the endomorphism T and thus the dynamics of the model, we explain the determination of the intermediate variables, $p,\ B,\ I$, in each region. These are determined by equation (GC2.22) of GC86 modified as follows:

$$\beta_N = 0,\ \bar{K}_N = K_N;\ \beta_S = 0,\ \bar{K}_S = K_S.$$

We recall, from GC86, the equation,

$$A_T p^2 + (C_T + I_T^D)p - V_T = 0 \tag{GC2.22}$$

where here, $A = \beta a_1 a_2/D^2$, and C and V are defined below. Equation (GC2.22) then becomes, with $\beta = 0$ in each region,

$$(C_T + I_T^D)p - V_T = 0 \tag{5.2.3}$$

using the convention of Section 2. Here, the symbolic expressions C, V and I^D, are defined by

$$C = (1/D)[c_1\bar{L} - a_1K + \alpha c_1c_2/D] \tag{5.2.4}$$
$$V = \alpha c_1^2/D^2 \tag{5.2.5}$$
$$I^D = GNP(1 - \gamma) \tag{5.2.6}$$

where $\gamma \in (0,\ 1)$. In fact, we will choose $\gamma \approx 60/100$. In any case, we would like $s + (1-\gamma) << 1$. Note that C is a function of K in each region, V is a constant, and GNP in the expression for I^D is to be determined from the formula $GNP = pB + I$. Equation (5.2.6) is the assumption that demand for industrial goods is proportional to GNP, as described above, in each region. This treats the two goods, B and I, symmetrically. Note that the values of B and I are directly computed as function of K (in each region) by the equations (3.1) and (3.2), but the value of p in this expression is not directly available. We obtain this value, assuming the rapid approach to equilibrium in the static model as described in Section 1, as described below.

Once p is determined, we obtain the GNP, which is given by equation (4.1.2), and equations (GC2.20a,b), (GC2.21a), and (GC2.3) from GC86, as:

$$\begin{aligned} GNP &= p(c_2L - a_2K)/D + (a_1K - c_1L)/D \\ &= p[\alpha c_2^2/D^2 + c_2\bar{L}/D - a_2K/D] \\ &\quad + [-2\alpha c_1c_2/D^2 + a_1K/D - c_1\bar{L}/D] + \alpha c_1^2/D^2p \end{aligned} \qquad (5.2.7)$$

for each region. Note that equation (5.2.3) determines p if GNP is known, but our expression (5.2.7) above requires p. When this circularity is resolved, we obtain a quadratic equation for p with all coefficients known.

We begin by rewriting (5.2.3), using (5.2.3) and (5.2.6), in the form,

$$p[C_T + (1-\gamma)GNP] - W_T = 0 \qquad (5.2.8)$$

and using (5.2.7), this yields,

$$E_Tp^2 + (C_T + F_T)p + (G_T - W_T) = 0 \qquad (5.2.9)$$

where

$$E = (1-\gamma)[\alpha c_2^2/D^2 - (a_2K + c_2\bar{L})/D], \qquad (5.2.10)$$

$$F = (1-\gamma)[-2\alpha c_1c_2/D^2 + a_1K/D - \bar{L}c_1/D], \qquad (5.2.11)$$

and

$$G = (1-\gamma)\alpha c_1^2/D^2. \qquad (5.2.12)$$

Thus, computing L from K in each region, all the coefficients of the quadratic equation (5.2.9) are known. We solve this equation, and in case of two real roots, we choose the larger one for the current value of p. Then we have GNP in each region, and the specification of the map T is complete.

An interesting simplification to our main model results from substituting rK for GNP in the dynamical rules for the 2D endomorphism, equations 5.2.1 and 5.2.2. This third model has been studied by Di Matteo (this volume) and we may return to it in a future publication.

5.3. Simulation results. For model 1, the bifurcation diagram is shown in Fig. 5.1. Throughout this section, the values of all the constants are as given in Section 4.4 (for the North) and Section 5.1 (for the South) except as noted in the Figure captions.

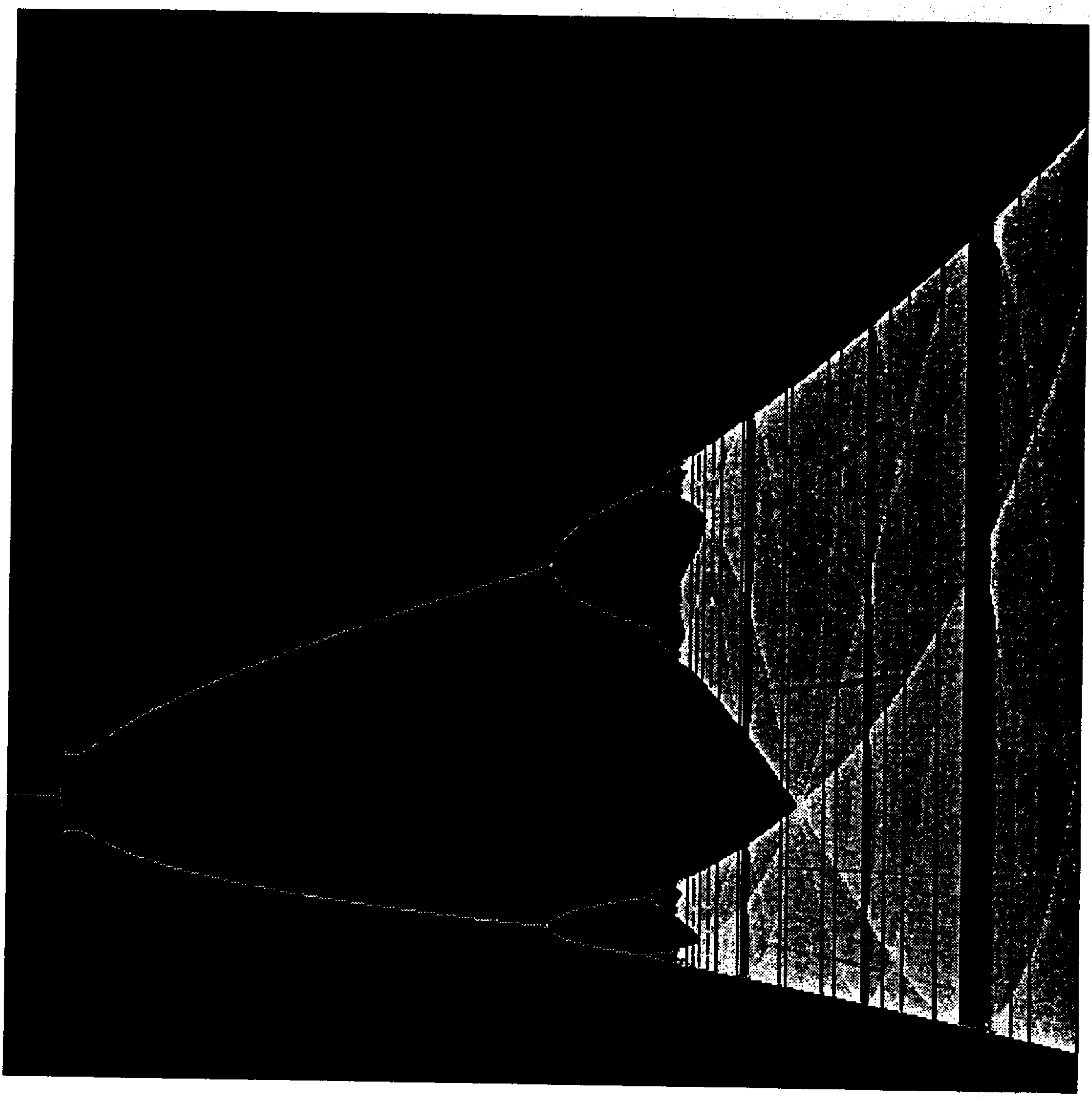

Figure 5.1. Response diagram for the first of the two-dimensional models. Here we vary the North's α from 31 to 49 while holding the South α fixed at 20. The x-axis is the North alpha (bifurcation) parameter, while the y-axis represents the North capital supply, K_N, after several iterations. The interpretation of this diagram is identical to that of Fig. 4.3, except that here the vertical axis is a one-dimensional projection of a two-dimensional

state space.

But the second two-dimensional model is our main goal in this paper. And for this model, the bifurcation diagram is shown in Fig. 5.2.

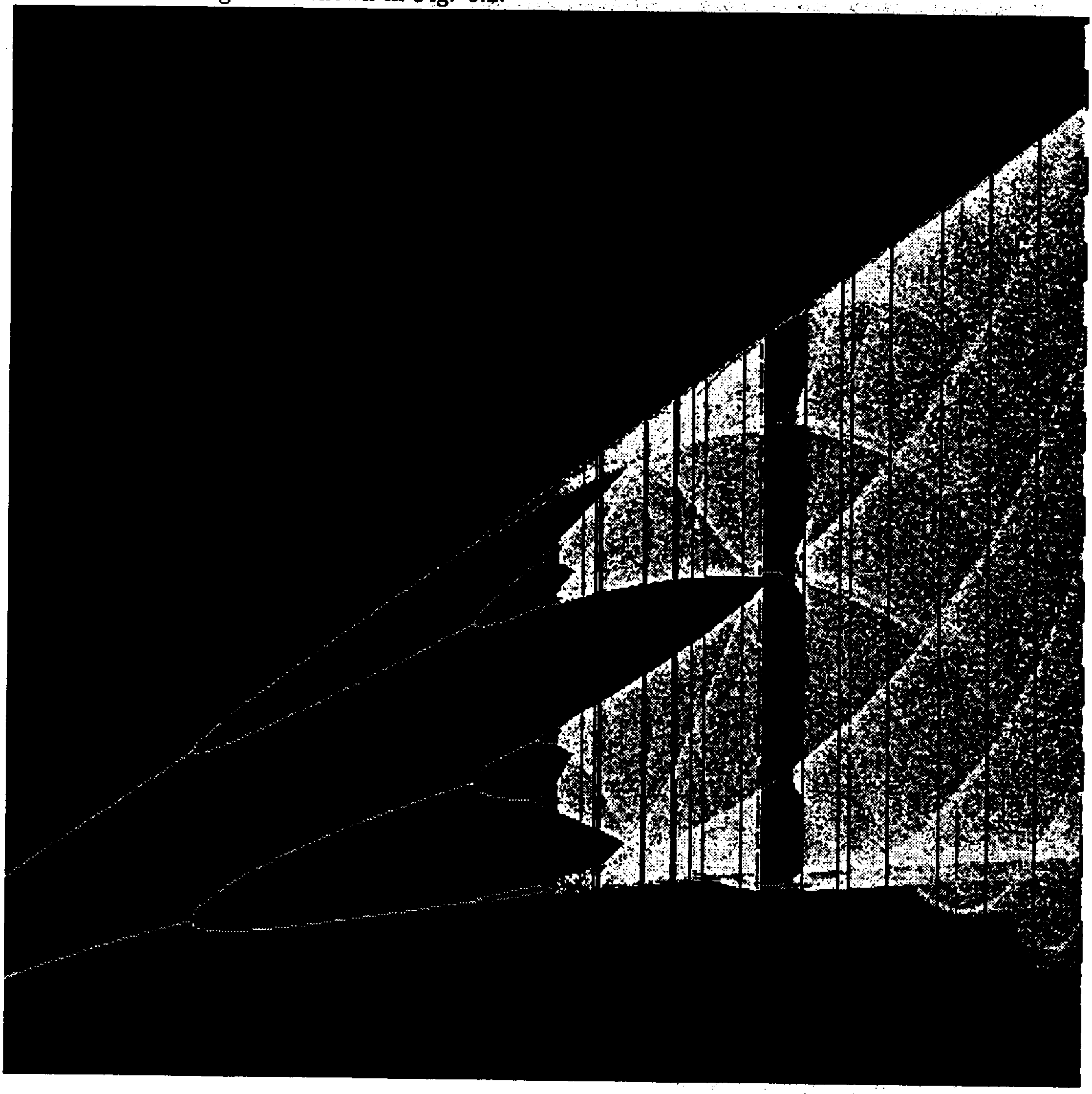

Figure 5.2. A histogram of the attractor in the two-dimensional state space, for a particular

value of the control parameter, $\alpha_S = 80$. The horizontal axis represents values of K_N, the vertical, K_S. The color bar shows the gray scale code, from black (no points of the trajectory in a unit area) to white (maximum number of trajectory points in a unit area).

For some values of the various parameters, we find a single basin, with a chaotic attractor. The attractor portrait for one such case is shown in Fig. 5.3.

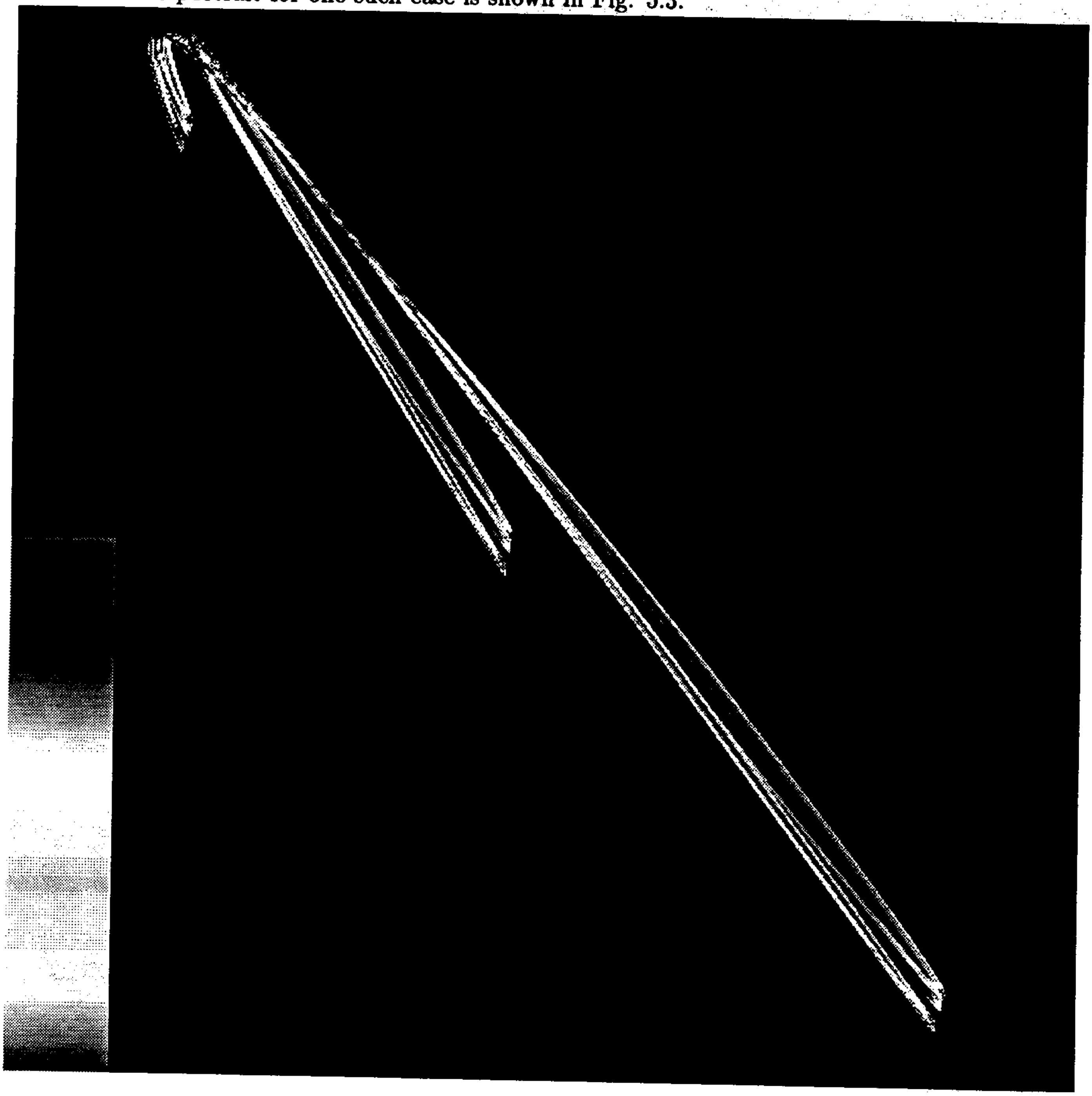

Figure 5.2. A histogram of the attractor in the two-dimensional state space, for a particular value of the control parameter, $\alpha_S = 80$. The horizontal axis represents values of K_N, the vertical, K_S. The color bar shows the gray scale code, from black (no points of the trajectory in a unit area) to white (maximum number of trajectory points in a unit area).

For other values of the parameters, we find two basins, each containing a point attractor, but the basins are separated by a fractal boundary. The basin portrait for one such case is shown in Fig. 5.4.

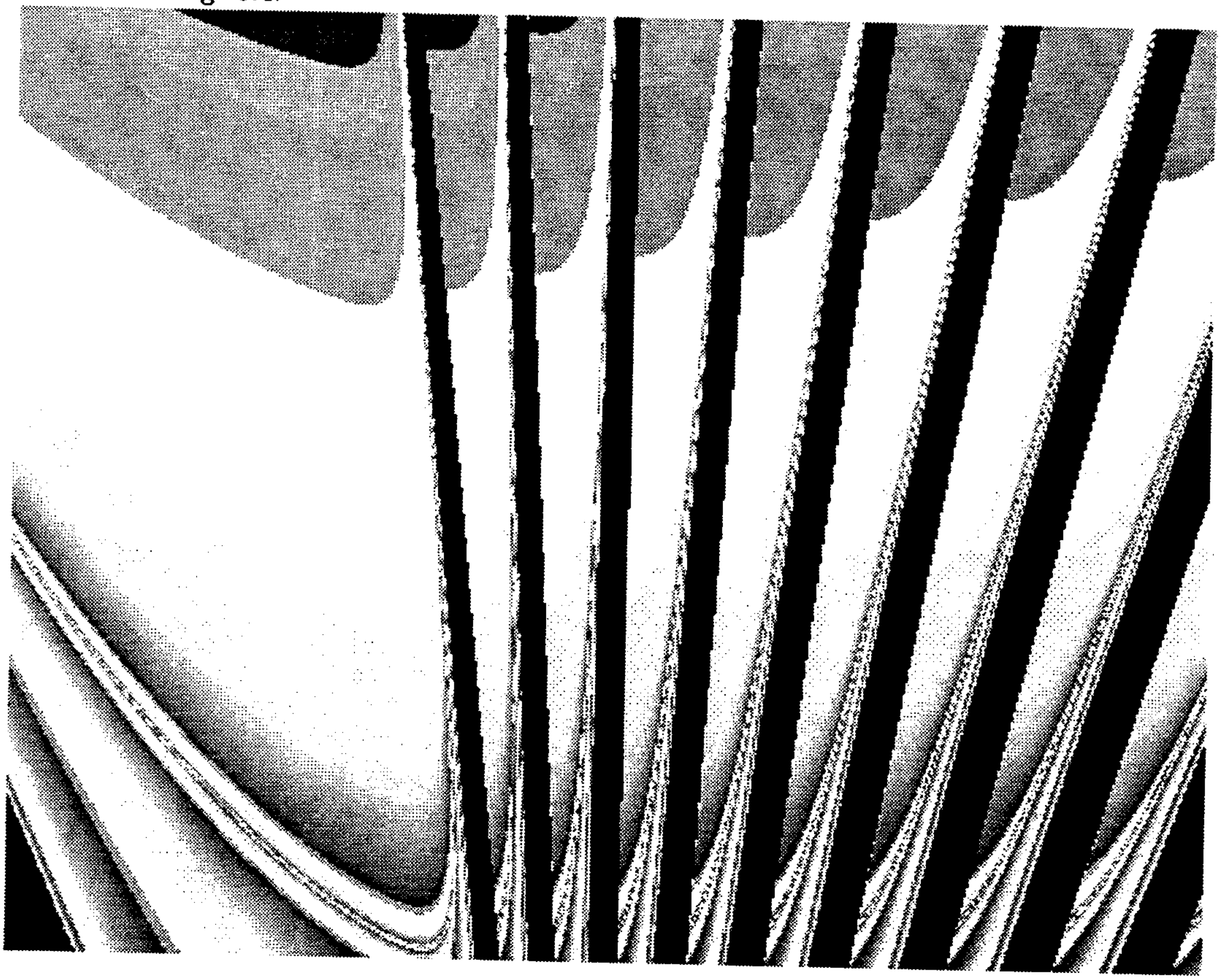

Figure 5.4. The two basins of attraction using the second of the 2-dimensional models with the South's α set to 17.5 and the North's to 1.5. In addition, the South's a_2 and c_1 are set to 0.05 and 0.04 rather than 0.02 and 0.01 as in Figure 5.3.

6. Conclusion. A future development will explore the global climate connection with international trade. In this context, the common property resource is the planet's atmosphere, which is used as an input of production, for example, in the combustion of fossil fuels (oil). A by-product of this combustion is CO_2. We are now interested in *two* separate but closely interacting dynamics: international trade and the biosphere (atmospheric chemistry, solar radiation, biological gas exchange, ocean dynamics, water reservoirs, climate, etc.) Especially, we will explore the greenhouse gas exchange between (1) the atmosphere, (2) human populations (which inhale oxygen and exhale carbon dioxide, both by breathing and by industrial activities) and (3) biomass and bodies of water, which act as CO_2 reservoirs. A simple biosphere model for beginning the study of this connection is the *daisy-world model* of Watson and Lovelock. This model achieves climate regulation with two cooperating species of daisies: black daisies (preferring cool but making warmth) and white daisies (prefering warm but making cool). We will replace one species of daisies by human industry.

We will then extend the analysis of this paper to consider two coupled dynamical systems: the dynamical North-South system and the modified daisy-world system just described. The dynamical North-South model will be extended to three dimensions: K, L, and E. See GC88 for this extension in a static framework. The daisy-world model will have two dimensions: population density, P, and green volume, G.

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NOTES

1. Defined in (Chichinilsky, 1986).

2. See (Devaney, 1991) for background on the quadratic family. A special case of this proposition may be found in Hao Bai-lin. See also Mira, 1987.

3. See (di Matteo, 1993.)

References

[1] Chichilnisky, G. (1981), Terms of trade and domestic distribution: export-led growth with abundant labor supply, *J. Development Economics*, **8**, 163-192.

[2] Chichilnisky, G. (1985), International trade in resources: a general equilibrium analysis, in: *Environmental and Natural Resource Mathematics, Short Course, Proc. Symp. Appl. Maths., Proc. Amer. Math. Soc.* **32**, 75-125.

[3] Chichilnisky, G. (1986), A general equilibrium theory of North-South trade, in: *Equilibrium Analysis, v.2*, (eds. W. Heller, D. Starrett, and R. Starr), Cambridge University Press, 3-56.

[4] Chichilnisky, G. (1991), *Global environment and North-South trade*, Stanford Institute for Theoretical Economics, Technical Report No. 31, Stanford University. To appear in *American Economic Review.*

[5] Chichilnisky, G. (1992), *Global environment and North-South trade*, Stanford Institute for Theoretical Economics, Working Paper No. 78, Stanford University.

[6] Chichilnisky, G. (1993), North-South trade and the dynamics of renewable resources, in: *Structural Change and Economic Dynamics*, Cambridge University Press.

[7] Chichilnisky, G. and G. M. Heal (1987), *The Evolving International Economy*, Cambridge University Press.

[8] Chichilnisky, G. and M. Di Matteo (1992), *Migration of labor and capital in a general equilibrium model of North-South trade*, Working Paper, Columbia University.

[9] Devaney, R. (1991), *An Introduction to Chaotic Dynamical Systems, Second Edition*, Addison-Wesley.

[10] Di Matteo, M. (1992), Dynamical properties of Chichilnisky's model of North-South trade, Working Paper, Università di Siena.