ENDOMORPHISMS AND VISUALIZATION

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Abstract

The emerging role of nonlinear dynamical systems theory in the sciences, both in model building and data analysis, is leading to a uniform working strategy in all fields of science. Thus, compatible models in these fields may be combined into massively complex models for whole systems, such as Gaian physiology (land, ocean and atmosphere), human population growth and demographics, the world economy, or combinations of these. These massive models, though simpler than nature, may be too complex for our understanding. This problem is the basis for the new emphasis on scientific visualization in general, and *dynamical visualization* in particular. That is, given a continuous or discrete dynamical system of very high dimension, how can we visualize and understand its behavior? In this paper we will consider a special case of this problem, in which the massively complex dynamical system is a semi-cascade, that is, the iteration of non-invertible map, or *endomorphism*. The strategy of visualization of this system consists of projection of the trajectories onto a low-dimensional (especially, two-or three-dimensional) subspace. The new method of critical curves, discovered by Christian Mira in 1964 for the study of plane endomorphisms, provides tools to infer the behavior of the massive system from the simple observation of its projection onto a subspace.

Contents

1. Introduction

Rabbi Edgar Magnin presided over the creation of the Jewish community of Los Angeles, and the Hollywood film industry, from 1915 through World War II. A relative of the department store Magnins, he attended the Hebrew Union College in Cincinnati, Ohio, About his teachers, he said:

> I used to go to temple on Saturday morning, and one of the rabbis, you knew what he said before he spoke, and the other one you didn't know what he said

when he was all through. ... And I used to study these men to find out what not to do. I figured if there's anything I do that's the very opposite, then I can't go too wrong.[1]

As the computer revolution progresses, transforming our congnitive strategies for understanding nature, Rabbi Magnin's algorithm is moving to the foreground: traditional tools are stood on their heads.

Mathematics has been redefined, following the computer revolution, as the study of space-time patterns. And using complex dynamical models to mimick the space-time patterns of nature, we usually find these model patterns in a space-time manifold of very high dimension. Complex dynamical models are networks of dynamical schemes flows, cascades, or semi-cascades depending on parameters - typically with tens or hundreds of dimensions. Techniques of chaoscopy - attractor reconstruction, fractal dimension, symbolic dynamics, etc. - can automatically construct such models from laboratory data. These techniques may, in principle, extend our vision far beyond the traditional horizon of complexity, granting us new understanding of human and planetary life. Yet, while these new models are substantially less complex than nature herself, they may be still to complex for us to grok. Needed, then, is a strategy for the visualization of massively complex dynamical models, or space-time patterns (movies) in high-dimensional spaces. One possibility, explored in this paper, *the method of projection*, is based upon the new *theory of critical sets* of dynamical systems generated by an endomorphism. We briefly review the theory of critical sets, and then present the method of projection.

2. The theory of critical sets

There are three flavors of dynamical system: *flows*, generated by a vectorfied or autonomous system of ordinary differential equations of first order, *cascades*, generated by a diffeomorphism, or invertible map, and *semicascades*, generated by an endomorphism, or non-invertible map. Likewise there are three flavors of dynamical scheme, generated by a vectorified, diffeomorphism, or endomorphism, depending on parameters. The theory of critical sets applied only to semi-cascade schemes, and we will review the theory only in the simplest significant case, an endomorphism of the plane depending on a single real parameter. In fact, we will begin with a single smooth endomorphism of the plane, T, and define its critical sets.

Let C_{-1} denote the critical set of T, consisting of the *critical points* of the plane at which the Jacobian matrix of T is degenerate. According to the classical theory of singularities of Hassler Whitney of 1936, this will be a set of curves, generically, called *fold curves*. Not let C_0 denote the image of C_{-1} under T. Generically, this is a set of curves, possibly with cusps, consisting of *critical values* of T. More generally, for a natural number, k, let C_k denote the image of C_{-1} under k+1 applications of T.

Mira's theory of critical curves of a plane endomorphism makes use of these critical sets, C_k, to enclose and locate the chaotic attractors (and their basins) of the semi-cascade generated by T. In addition, in the context of a scheme generated by an endomorphism depending on parameters, the bifurcations of the scheme are predicted by changes in the pattern of intersections of the critical sets due to changes in the parameters. Not only are the calssical bifurcation (known from the parallel and earlier theory of cascades) indicated by these critical set patterns, but new types of bifurcation (special to the semicascade context) are revealed as well. Further, the critical sets are easier to locate, by computational means, than the attractors and their basins. In any case, the method of critical sets is an exciting frontier of current research, and many papers and experimental reports may be found in the recent literature [2]. Typically, in these papers, we may find a computer graphic histogram in which the attractors of an endomorphism of the plane are shown within a tangle of critical curves.

3. The method of projection

This method may be applied in the context of a cascade, or a semi-cascade, in a state space of high dimension. We simply project the data onto a plane section of the state space, and pretend (erroneously) that there is a semi-cascade in the plane which has produced this projected data. Off hand, this seems like a futile and hopless trick. However, a recent result of Noakes provides some hope. This result gives information on the topology of a two-dimensional attractor, given only the projection of its time-series data onto a one-dimensional subspace [3]. Our method may be tested with a cascade in three dimensions, projecting a trajectory onto a fixed plane. Interpreting the resulting histogram as that of a plane endomorphism according to the method of critical sets, some edges may be observed which correspond to folds in the three-dimensional context, others may be illusions, caused by the projection only. Further, some folds in the three-dimensional context may be invisible in projection. Nevertheless, examples support the method of projection as a visualization tool, in spite of these weaknesses.

4. Conclusion

Computer-graphic based tools of dynamical visualization are only at the beginning of an evolution which may be vital to the future of science and society. The progress of this evolution, shunned by most mathematicians, may be in the hands of artists or school children. The proliferation of personal supercomputers for virtual reality games may be the venue for these new directions of conceptual tools for the understanding of nature.

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