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A BRIDGE BETWEEN WHITNEY'S FOLD AND CUSP POINTS AND THE CRITICAL CURVES LC_{-1} AND LC IN TWO-DIMENSIONAL ENDOMORPHISMS

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1. Introduction.

The important role of the critical curves (LC_{-1}, LC, and their images), as introduced by Gumowski and Mira since thirty years ago (see [Gumowski and Mira, 1980], and references therein), is now emerging. These have been used to understand the dynamical properties of two-dimensional noninvertible maps, and their bifurcations, in particular for global bifurcations. For a recent survey of the references on this topic see [Mira et al, 1996], while a recent survey of the results and properties on two-dimensional noninvertible maps, obtained making use of the critical curves, can be found in [Abraham et al, 1996]. These are only the very first steps towards the understanding of the dynamics of such maps, and it is our believe that we are moving towards the right direction.

Another pioneering work by Whitney (1955) started, a parallel field on the classification of map's singularities. After this seminal paper singularity theory developed rapidly, and a high level of classification has now been reached. Also catastrophe theory may be dated to start with that work, see [Arnold et al., 1986, Arnold, 1991]. The fundamental notions introduced by Whitney are those of fold points and cusp points for maps of the plane, proving that these are stable under perturbations, and proving that these are "generic" as every singularity of a smooth map may be suitably perturbed into fold and cusp.

The objective of the present work is to get a bridge between the two kinds of studies, by a comparison of the definitions, and an analysis of the properties, in order to reach a unifying approach. In our opinion this may be a source of interesting results. Our main goals are to obtain a better understanding of the geometrical behaviour of a map when an arc crosses, through some particular point, LC_{-1}, and a better understanding of the bifurcations occurring in the foliations of the plane, that is changes of structures related to changes in the dynamics. For example, the lip structure, the dovetail (or swallowtail) and other situations have already been described [see Mira et al. paper to appear, and new book], but also the other structures already classified in the singularity theory have a corresponding situation in the plane foliation and related qualitative different dynamics.

2. Critical curves LC_{-1} and LC.

Let us first give the definition of the critical curves of rank-0 (LC_{-1}) and of rank1 (LC), let us say "in Mira's sense", for short. This notion is exactly the two-dimensional extension of the notion of local extrema in the one-dimensional case, as these are the points which qualifies the dynamical behaviours and bifurcations of a map. This is now well known in the one-dimensional case [Mira, 1987, De Melo and van Strien, 1991, Sharkovsky et al, 1993, Robinson, 1995], but not so well know in higher dimensions.

Def.1 *A point $c_{-1} \in A$ is called a critical point of rank-0 of T if T is not locally invertible in it.*

The image of a critical point c_{-1} of rank-0 is called critical point of rank-1, c, and denoted by c, and so on for their images c_k of increasing rank. The sets of such points are denoted by LC_{-1}, LC, ... , LC_k .

By definition we have that in any neighbourhood U of a critical point c_{-1}, T: U $\rightarrow$ T(U) is not one-to-one.

Remark. The definition given above holds for any continuous map T: $A \subseteq \mathbb{R}^n \rightarrow A$, $n \geq 1$ (differentiable or not). It holds also for a map which is not continuous in some subset of A of zero Lebesgue measure, with finite jumps, but in this case it is necessary an heavier notation in the critical points of rank ≥ 1 when there is a jump in c_{-1}.

Remark. If we define a point $p \in A$ as regular if T is invertible in p, that is, a neighbourhood U of p exists such that T: U $\rightarrow$ T(U) is one-to-one, then a point is critical of rank-0 if it is not regular.

Let us consider T in the plane. It is clear that assuming T smooth we can analyse some necessary or sufficient condition to have a critical point c_{-1} or c. As when the sign of the jacobian determinant $J(x,y)=\det(DT(x,y))$ is not zero the map is locally invertible it follows that *a necessary condition (not sufficient) for $p=(x,y)$ to be a critical point of rank-0 is $J(x,y)=0$.*

While in the one dimensional case, for a point x such that $f'(x)=0$ the change in sign of f' in neighbourhood of x is a sufficient condition to have a critical point c_{-1}, in dimension two *the change of sign of the jacobian determinant $J(x,y)$ on the right and left side of a curve where $J(x,y)=0$ (which is a necessary condition for LC_{-1}) is not sufficient to state that the curve belongs to LC_{-1}.*

Clearly $T^{-1}(c) \supseteq c_{-1}$. When the equality does not hold then the points of $T^{-1}(c)\backslash c_{-1}$, when not critical of rank-0, are called extra preimages of c. An extra preimage curve $LC_{-1,e}$ is a locus of such points.

Remark. The definition "a point c_{-1} is critical of rank-0 for T, and $c=T(c_{-1})$ is critical of rank-1, if at least two coincident rank-1 preimages of c are merging in c_{-1}" which can be considered the original one [Gumowski and Mira, 1980], is to be interpreted as equivalent to the one given above.

Looking for the distinct rank-1 preimages of the points in the plane, we can divide the plane in regions, denoted by Z_k [Mira et al., 1996]. A region Z_k is such that all its points have the same number of distinct rank-1 preimages, obtained with the same inverses of T.
Then a critical curve LC belongs to the boundary of some region Z_k (but the converse is not necessarily true).
Such zones in which the plane can be subdivided correspond to the "foliation of the Riemann plane", and, together with the curves LC_{-1}, give a geometrical interpretation of the action of a map and of its inverses. If we consider the parameter space related to the constants defining T, we shall get a correspondence between zones in the parameter space and the qualitative structure of the regions Z_k of the foliated plane. Qualitative changes in the structure of these zones correspond to drastic changes (or bifurcation) in the functions defining T. We can try a "classifications of types of zones" and of their bifurcations, i.e. their appearance or destruction or qualitative changes. For example the transition of a region Z_2 into one of type Z_2-Z_4 due to one of the two sheets which folds again with a cusp. A few example are reported in [Mira et al., 1995], involving a "lip-structure" or a "dovetail (i.e. swallowtail)-structure" (see the figures included, taken from Mira et al., 1995). This terminology recall in mind some structures in catastrophe theory, but really the relation between the two theories is not only accidental. Such changes also implies, generally, bifurcations in the dynamics. As in fact the changes in the regions Z_k modify the inverses of T, and thus the structures depending on the preimages, like the

basins of attracting sets or the boundary of the set of unbounded trajectories ∂D_∞, ar modified.

We have mentioned above a cusp in the boundary of a zone. This is (almost) the definition in Mira's sense:

Def.2a We say that the critical point of rank-0, p_{-1}, is a cusp of LC_{-1} if its image p is a cusp point of the critical curve LC.

Def.2b We say that a critical point of rank-1, p, is of type fold (resp. cusp) of LC if the number of merging preimages $T_i^{-1}(p)=p_{-1}$ is even (resp. odd). The point p_{-1} of merging preimages is also called a fold (resp. a cusp) point of LC_{-1}.

That definitions 2a and 2b for a cusp are equivalent is a consequence of Whitney's work.

As the rank-1 preimages are obtained as solutions of algebraic equations, we have that the generic cases of a critical point p of rank-1 of type fold (resp. cusp) of LC is one having two (resp. three) merging preimages $T_i^{-1}(p)=p_{-1}$. And a point p having three merging preimages in p_{-1} can only occur if p_{-1} is a contact point between a critical curve LC_{-1} and a curve of extra preimages $LC_{-1,e}$, as qualitatively shown in Fig.1

3. Whitney's fold and cusp points.

We recall here the definitions given in [Whitney, 1955], with only a change in the notation. Consider the map $T: A \to A$, $A \subseteq \mathbf{R}^2$ open, as smooth as necessary, defined by

$$x'=f(x,y)\ ,\ y'=g(x,y).$$

By $DT(x,y)$ we denote its jacobian matrix at the point (x,y), the Jacobian, or determinant of $DT(x,y)$, is denoted by $J(x,y)$.

Def.W1 - A point $p=(x,y) \in A$ is good if

either it is non singular, $J(x,y) \neq 0$

or it is singular, $J(x,y)=0$, but with $\nabla J(x,y)=[\frac{\partial}{\partial x}J\ \frac{\partial}{\partial y}J] \neq 0$

Property.1 - It $p=(x,y)$ is good then the matrix $DT(x,y)$ is not the null matrix. Thus the space image of $DT(x,y)$ has dimension 2 if p is non singular and dimension 1 if p is singular.

Def.W2 - T is good if any $p \in A$ is good.

Property.2 - If T is good in A then the set $J(x,y)=0$ is made up of smooth curves.

Let $p=(x,y)$ be a singular point of A, T good, and denote by $\tau(x,y)$ the vector otogonal to $\nabla J(x,y)$, $\tau(x,y) = [\tau_1(x,y) \;\; \tau_2(x,y)] = [-\frac{\partial}{\partial y}J \;\; \frac{\partial}{\partial x}J]$ is the direction tangent to $J=0$ in p.

Def.W4 - Let T be good. A singular point $p=(x,y)$

is a fold point iff $DT\,\tau(p) \neq 0$

is a cusp point iff $DT\,\tau(p)=0$ and $\tau_1(p)\frac{\partial}{\partial x}DT\,\tau(p) + \tau_2(p)\frac{\partial}{\partial y}DT\,\tau(p) \neq 0$

Def.W5 - Let T be good. A point $p=(x,y)$ is excellent if either it is non singular or fold or cusp. T is excellent if any $p \in A$ is excellent.

Remark. T excellent $\Rightarrow$ T good, but the converse is not true. As we shall see, a good map T may have singular points which are neither fold nor cusp in the above definition.

Properties.

(a) If T is excellent the cusp points are isolated on the curve $J=0$.

(b) The curve $T(J=0)$ is smooth except for cusps at the images of cusp points.

(c) The canonical form of a map having a fold point in $p=(0,0)$ is $x'=x^2$, $y'=y$. The canonical form of a map having a cusp point in $p=(0,0)$ is $x'=xy-x^3$, $y'=y$. (i.e. if p is a fold point or a cusp point of T then a smooth change of coordinate exists such that in the new system T takes the canonical form)

(d) Arbitrarily near any map, there is an excellent map.

(e) The nature of fold and cusp points cannot be changed under "3-approximation" (i.e. maintaining the derivatives up to the third order). (In Whitney's paper [] are also described the changes allowed under less approximations).

Let us give a geometrical interpretation of the above definition. Let T be a good map. Then the locus $J(x,y)=0$ of singular points is smooth. Let $p=(x,y)$ be a singular point, then the jacobian matrix has an eigenvalue $\lambda_0(p)=0$ and another eigenvalue $\lambda_1(p)$. Let $V_0(p)$ be the eigenvector associated with $\lambda_0=0$ and $V_1(p)$ the eigenvector associated with the eigenvalue λ_1 if $\lambda_1 \neq 0$, or the generalized eigenvector if $\lambda_1=0$ too. V_0 and V_1 are linearly independent.

If the tangent vector $\tau(p)$ is not parallel to $V_0(p)$ then $DT(p)\tau(p)=V_1(p) \neq 0$. That is, the tangent in p to the set $J=0$ is mapped by T into the tangent in $T(p)$ to the set $T(J=0)$. p is a fold point.

Fig 2a

Fig 2b

Fig 3a

Fig 3b

Fig 3c

Moreover, any arc η crossing J=0 in p and not tangent to $V_0(p)$ is mapped by T into an arc tangent to V_1, that is $T(\eta)$ and $T(J=0)$ have the common tangent $V_1(p)$ in the fold point T(p) (see Fig.2a). *Extending Whitney's definition we call T(p) fold or cusp depending on p fold or cusp.* We only wish to know what happens to an arc η crossing J=0 in p and tangent to $V_0(p)$. As shown in [Mammana and Gardini, 1996], such an arc is mapped by T into an arc $T(\eta)$ having a cusp in the fold point T(p) of $T(\eta)$ (the tangent to the cusp depending on η) (see Fig.2b), however, due to propery (c) this behaviour can be seen immediately making use of the canonical form.

If the tangent vector $\tau(p)$ is parallel to $V_0(p)$ then $DT(p)\tau(p)=0$. But if Whitney's second order condition is satisfied, then $\tau_1(p)\frac{\partial}{\partial x}DT\tau(p) + \tau_2(p)\frac{\partial}{\partial y}DT\tau(p) \simeq V_1(p)$. T(p) is a cusp point of the curve T(J=0) and $V_1(p)$ is the vector tangent to the cusp (see fig.3a) (the terminology "tangent to the cusp" is in the geometric sense). If η is an arc crossing J=0 in p and transverse to V_0 then $T(\eta)$ is an arc tangent to V_1, i.e. $T(\eta)$ crosses the curve T(J=0) (see Fig.3b). While if η is an arc crossing J=0 in p and tangent to V_0 then the shape of $T(\eta)$ depends on the relative position of η with respect to the curve J=0 and the curve of extra-preimage through p, say $(J)_{ex}$ (see Fig.3c).

4. Merging of the fold and cusp points with LC_{-1} and LC.

It is clear that if p is a fold point or a cusp point in Whitney's sense, according to Def.W4, then it is also a fold or cusp point belonging to LC_{-1} in Mira's sense (in particular if T is excellent then J=0 is the curve LC_{-1}). The converse is not necessarily true. As in fact Whitney's definitions applies to a "generic" map, while Mira's definition, which are to be used when studying the dynamics of T as a function of its parameters, have to include also the cases which are "not generic" following Whitney.

For example, let us assume that T is good, and consider a point $p \in (J=0)$ such that $DT\tau(p)=0$ and $\tau_1(p)\frac{\partial}{\partial x}DT\tau(p) + \tau_2(p)\frac{\partial}{\partial y}DT\tau(p)=0$. Then this point is neither a fold nor a cusp in Def.W4 . However *it comes natural to call such a point a fold or a cusp if it turns out that "it behaves like a fold or a cusp"*. This is what we do following Mira's definition. As an example, consider the triangular family of maps

$$T\begin{cases} x' = \dfrac{\rho y x + f(x)}{1+\rho y} \\ y' = 1+\rho y \end{cases} \qquad (*)$$

defined in a suitable open set A [Bischi and Gardini, 1995].

From the spectral properties of T we find the jacobian $J(x,y)=\rho\frac{\rho y+f'(x)}{1+\rho y}$. It is immediate to see that any map T in the class (*) is "good" in Whitney's definition, as in any point (x,y) of its domain A, it is either $J(x,y) \neq 0$ or $J(x,y)=0$ but $\nabla(J(x,y)) \neq 0$. In fact, $\nabla(J(x,y)) = \rho\ [\ \frac{f''(x)}{1+\rho y}\quad \frac{\rho(1-f'(x))}{(1+\rho y)^2}\ ]$ has the second component always different from zero when J(x,y)=0. Moreover for the class of maps T in (*)

we have that LC_{-1} is the set defined by J(x,y)=0 and we can write

$$(x,y) \in LC_{-1} \Leftrightarrow y = -\frac{1}{\rho} f'(x) \quad \text{and} \quad f'(x) \neq 1$$

The vector tangent to LC_{-1} in one of its points (x,y) has the direction

$$\tau(x,y) = [\ \tau_1 \quad \tau_2\ ] = \frac{\rho}{1+\rho y}\ [\ -\rho \quad f''(x)\ ]$$

The eigenvectors of DT(x,y), when $(x,y) \in LC_{-1}$, corresponding to the eigenvalues $s_1=0$ and $s_2=\rho$, called V_0 and V_1 respectively, are given by

$$V_0(x,y) = [\ 1 \quad 0\ ] \quad , \quad V_1(x,y) = [\ \frac{x - f(x)}{(1-f'(x))^2} \quad 1\ ]$$

Thus when $f''(x) \neq 0$ the tangent to LC_{-1} in its point (x,y) is not parallel to the eigenvector $V_0(x,y)$. This is a sufficient condition for (x,y) be a fold-point of LC_{-1}. Its image by T is a fold-point of LC, with tangent vector in T(x,y) given by $V_1(x,y)$.

While at a point $(x_s,y_s) \in LC_{-1}$ such that $f''(x_s)=0$, the tangent to LC_{-1} is parallel to the eigenvector $V_0(x_s,y_s)$. Such a point (x_s,y_s) may be (but not necessarily) a cusp-point of LC_{-1} and its image by T a cusp-point of LC. By applying Whitney's second order condition in this point we get:

$$\tau_1 \frac{\partial}{\partial x}(DT\tau) + \tau_2 \frac{\partial}{\partial y}(DT\tau) = \frac{-\rho^4}{(1-f'(x_s))^2}\, f'''(x_s)\ V_1(x_s,y_s)$$

so that if $f'''(x_s) \neq 0$ it is a cusp, with $V_\rho(x_s,y_s)$ tangent to the cusp $T(x_s,y_s)$.

We note that, depending on f(x), we may have a map T which is good but not excellent. In fact, if f(x) is such that at a point x we have $f'(x_s) \neq 0$, $f''(x_s)=f'''(x_s)=0$, then $(x_s,y_s) \in LC_{-1}$ is good but neithed fold nor cusp.

However, in the geometrical sense such a point may be either a fold or a cusp point. For example, considering $f(x)=x^4-c$, T is a map of type Z_0-Z_2 and the point $(0,0) \in LC_{-1}$ is of fold type, while considering $f(x)=x^5-x$ then $(0,-1) \in LC_{-1}$ is of cusp type.

Observing that in Whitney's notation, if $\phi(t)$ is a parametrization of J=0 such that a point p=(x,y) is get at t=0, then

$$\frac{d}{dt}T(\phi(t))|_{t=0} = \nabla_\tau T(p) = DT\,\tau(p)$$

$$\frac{d^2}{dt^2}T(\phi(t))|_{t=0} = \nabla_\tau \nabla_\tau T(p) = \tau_1(p)\,\frac{\partial}{\partial x}DT\,\tau(p) + \tau_2(p)\,\frac{\partial}{\partial y}DT\,\tau(p)$$

we conjecture that a possible extention of Whitney's definition to good points is as follows:

let $p \in (J=0)$ be a good point of T and k the first integer such that

$$\frac{d^k}{dt^k}T(\phi(t))|_{t=0} = \nabla_\tau \ldots \nabla_\tau T(p) \neq 0$$

then p is a fold point or a cusp point depending on k odd or k even respectively.

It is easy to see that for the class of maps given in (*) we have, for the first non vanishing derivative,

$$\frac{d^k}{dt^k}T(\phi(t))|_{t=0} \simeq f^{k+1}(x_s)\, V_1(x_s,y_s) \neq 0$$

and in this example the rule given above gives the correct result.

5. Non good points.

However, the main difference between Mira's definition of critical points and Whitney's definition occurs at points which are not good. At a point

$p \in (J=0)$ such that $\nabla J(p)=0$

i.e. not good, we have no information. And we cannot perturb T if we are really interested in the dynamics of that map. It can be seen, through examples, that such a point may be of any type:

(i) p may be not a critical point in Mira's notation (i.e. a point not belonging to LC_{-1}),

(ii) $p \in LC_{-1}$ may be of fold type,

(iii) $p \in LC_{-1}$ may be of cusp type,

(iv) $p \in LC_{-1}$ may be neither of fold nor of cusp type.

Furthermore, avery point of LC_{-1} located in the intersection of different branches of LC_{-1} is a non good point whose geometrical properties must be investigated in each particular case.

It is clear that the points p belonging to the class (iv) are the more interesting ones. They may correspond to situations of bifurcation, but may be persistent in the map to study. They are particular and the properties of "folding", when an arc η crosses through LC_{-1} in such a point, may not occur in the same way as for fold or cusp points.

Example 1: cases (i) and (ii). Let T be defined by

$x'=x^3$, $y'=x+y^2$

then $J(x,y)=6x^2y$ and $J=0$ on the axis $x=0$ and $y=0$, but only $y=0$ is the critical curve LC_{-1}. Any point $(0,y)$ with $y \neq 0$ is not good but it is not a critical point.

The particular point (0,0) is a critical point not good. However in this example (T is of type Z_0-Z_2), it

Fig 4 is a fixed point and a critical point of fold type (fig. 4).

Example 2: case (iv). Let T be defined by

$x'=x^2$, $y'=x+y^2$

then $J(x,y)=4xy$ and $J=0$ on the axis $x=0$ and $y=0$, both constituting the critical curve LC_{-1}. T is a

Fig 5 map of type Z_0-Z_2-Z_4 (see Fig. 5). The point (0,0) is a fixed point and a critical point of type neither fold nor cusp. A segment η crossing (0,0) transverse to τ, vector tangent to $LC_{-1,a}$ and transverse to V_0, is mapped by T into and arc tangent to V_1 but crossing the critical curve LC_a.

It is clear that the non good points are those related to qualitative changes in the plane foliation. They occur, for example, when a cusp point comes to contact a fold curve as some parameter of T is varied, and thus denotes a change in the structure of zones Z_i, to which there correspond, generally, changes in the dynamics, or in the structure and shape of some basin.

6. Applications.

The classifications of the critical points of T and the understanding of the action of the map T when some arc η crosses a critical curve very importand in the understanding of the dynamics of noninvertible maps. A few example are here recalled.

Fig 6

(a) If LC intersects LC_{-1} in a point a_0 and the tangent to LC in a_0 is not parallel to the eigendirection V_0 (of $DT(a_0)$), then LC_1 is tangent to LC in a_1 (and the common tangent in a_1 is parallel to the eigendirection V_1, of $DT(a_0)$). While a cusp in a_1 appear when the tangent to LC in a_0 is parallel to the eigendirection V_0, and after a loop, or knot, on LC_1 appear (see Fig. 6). This explains why the boundary of a chaotic area, given by the images of critical arcs, may be smooth or not, and when the transition from one situation to the other occurs.

(b) The situation depicted in Fig.6 can occur when the arc η crossing through LC_{-1} belongs to the unstable set of some cycle. This explains the formation of loops in invariant sets. This occurrence is also related to the inset of chaos [Lorenz, 1989] [Mira et al., 1996] [Frouzakis et al., 1996]

(c) The occurrence of such particular events in closed invariant curves Γ born by Neimark-Hopf bifurcation is a phenomenon related to the appearance of chaotic behaviour [Lorenz, 1989] [Mira et. al., 1996].

- If Γ crosses transversally LC_{-1} and the tangent to Γ in p is not parallel to V_0, then Γ is tangent to LC in $p_1=T(p)$, and tangent to the successive images of LC, that is, to arcs of critical curves LC_i for $i \geq 0$. This is a mechanism, typical of non invertible maps, which creates the "oscillations" often observed in the shape of Γ.

- If the tangent to Γ in p becomes parallel to V_0 then Γ ought to develop a cusp in $p_1 \in \Gamma \cap LC$. But this occurrence, considering the dynamics of T as a function of a real parameter λ, implies that after, say for $\lambda>\bar{\lambda}$, the closed invariant curve Γ would have a "loop" or "knot", which is not possible (for a closed invariant curve of finite lenght, homeomorphic to a circle). That is, the value $\bar{\lambda}$ denotes a bifurcation causing the destruction of the closed invariant curve Γ, or the closed invariant curve was already broken. Sequences of this kind may be at the basis of the transition to complex behavior. A few example are shown in [Mira et al., 1996].

(d) The analysis and classification of bifurcations in foliation, as those appearing in [Mira et al., 1995], (see figures).

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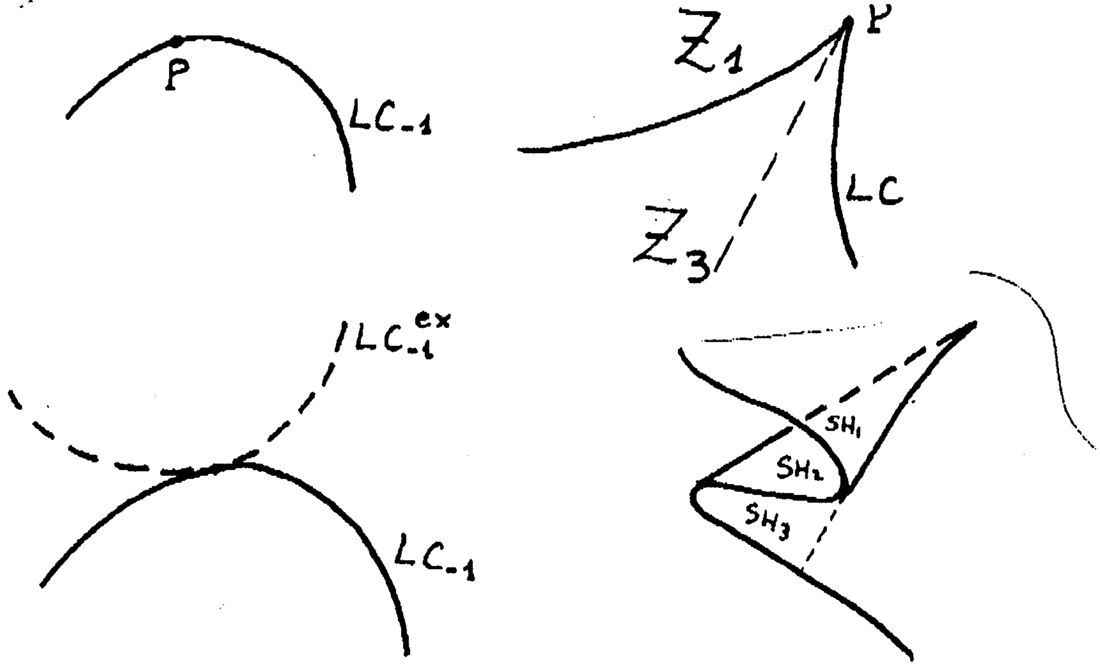
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Fig. 1

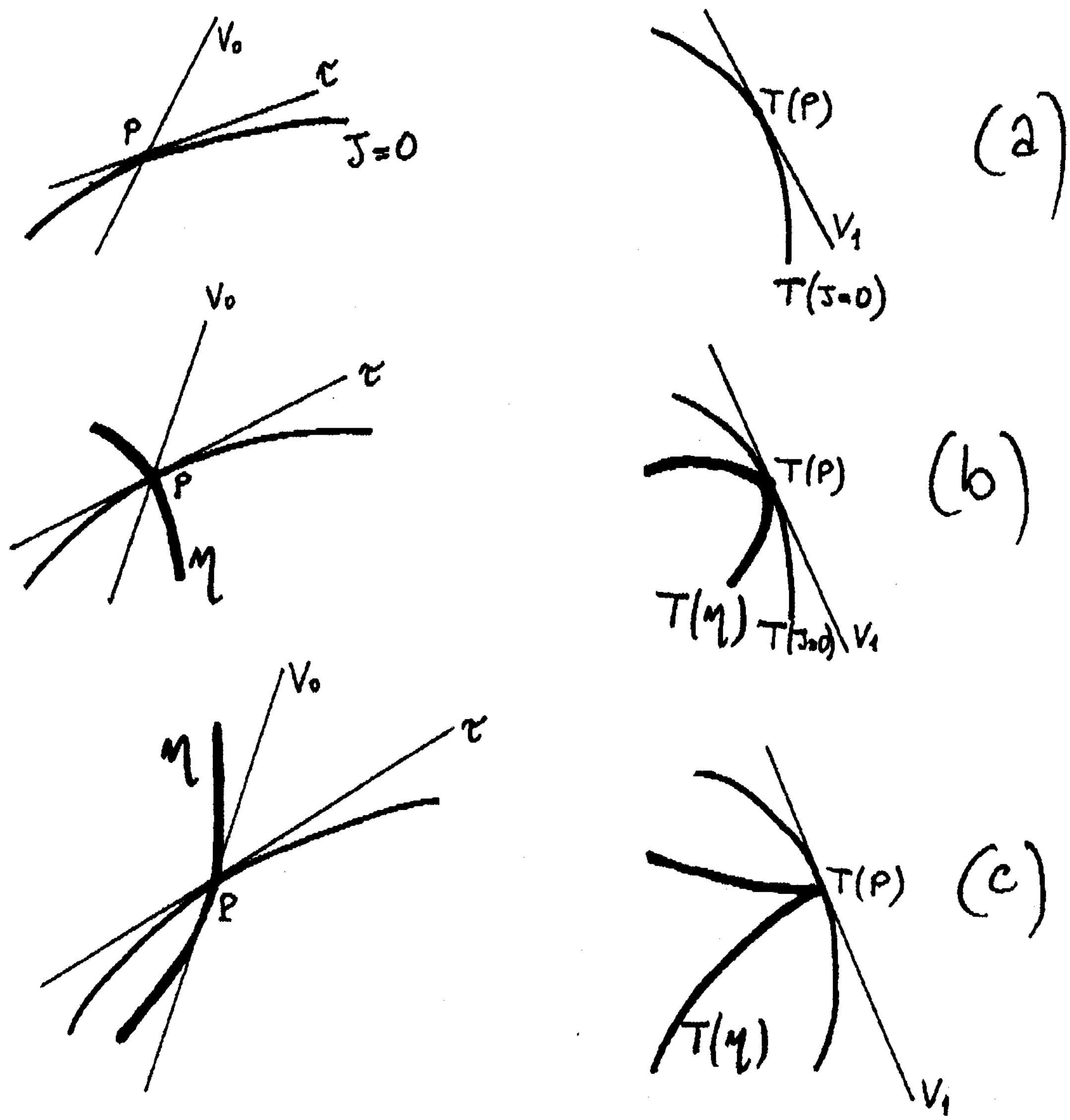
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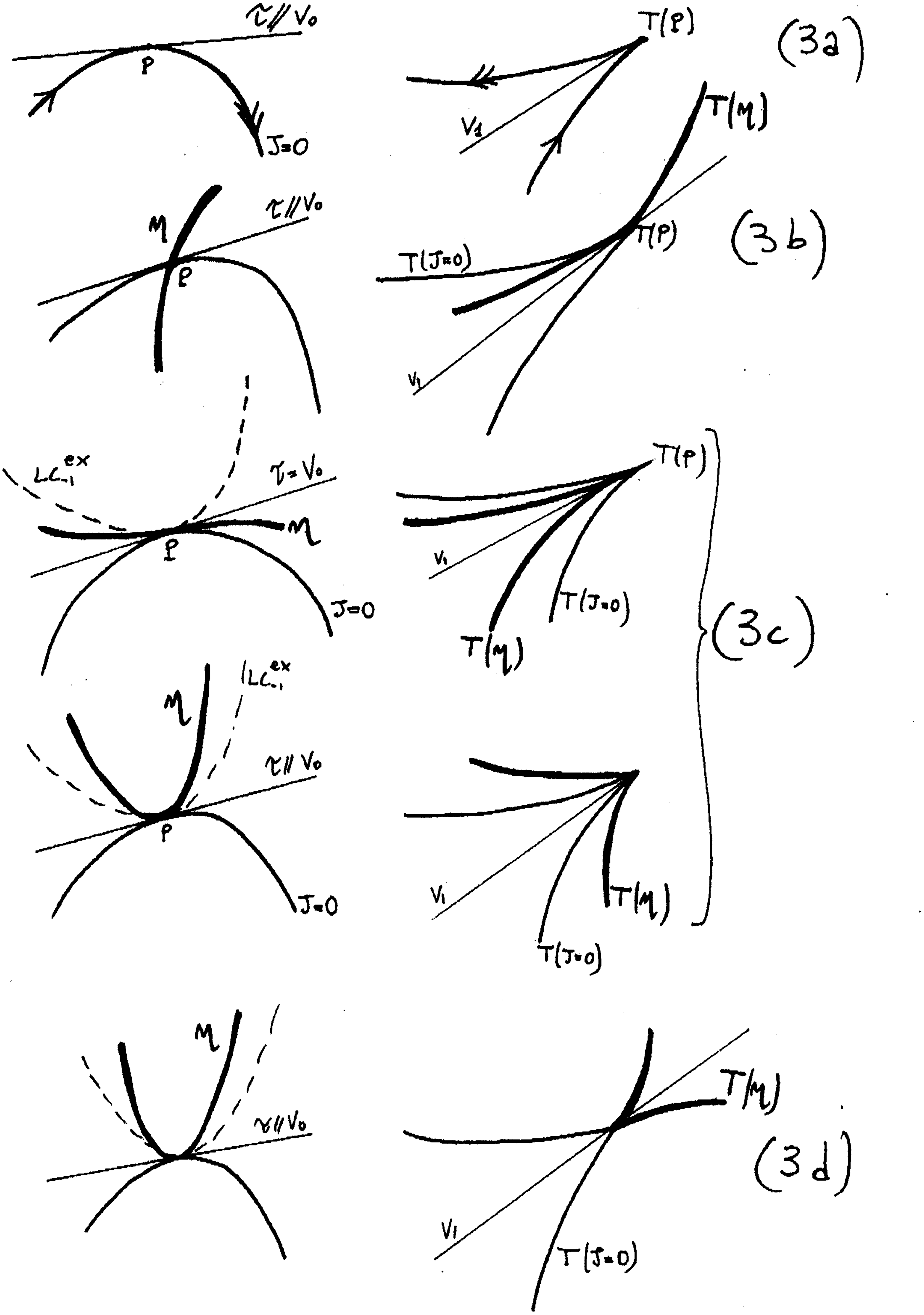
Fig. 2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LC-1 ex
τ // V0
P
J=0
V1
T(M)
T(J=0)
M
τ // V0
T(M)
(3d)
V1
T(J=0)

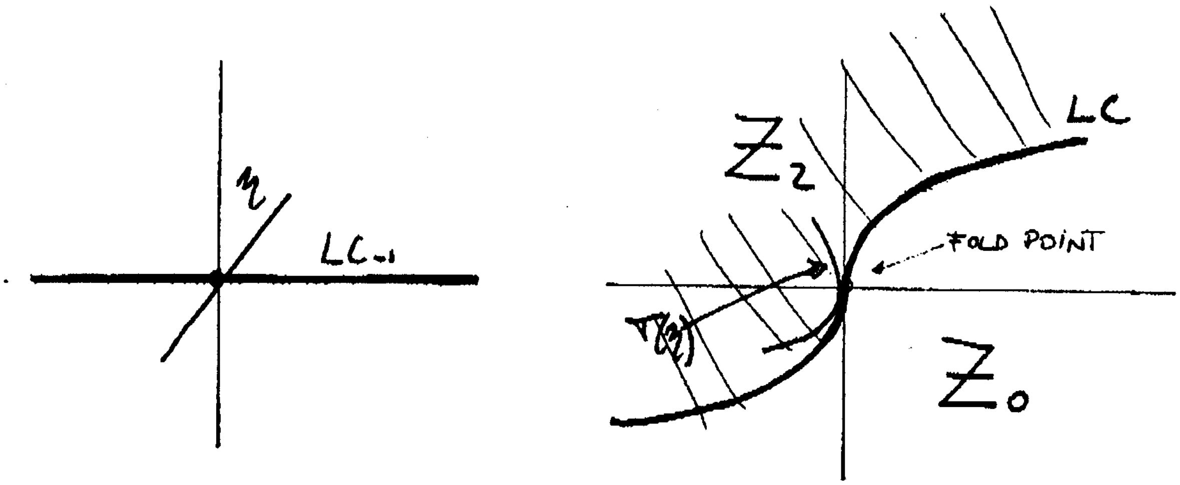


Fig. 4

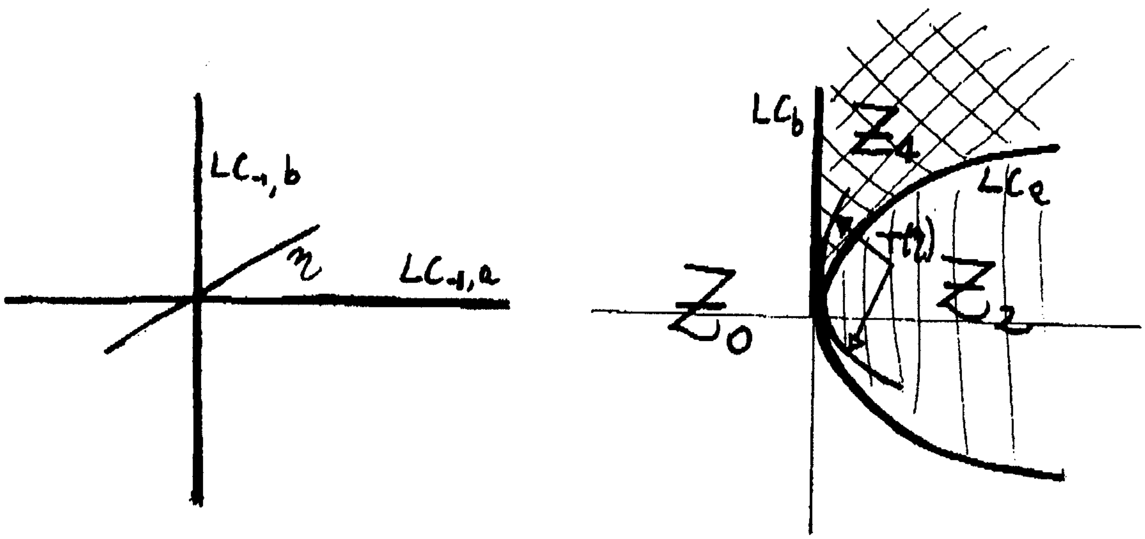


Fig. 5

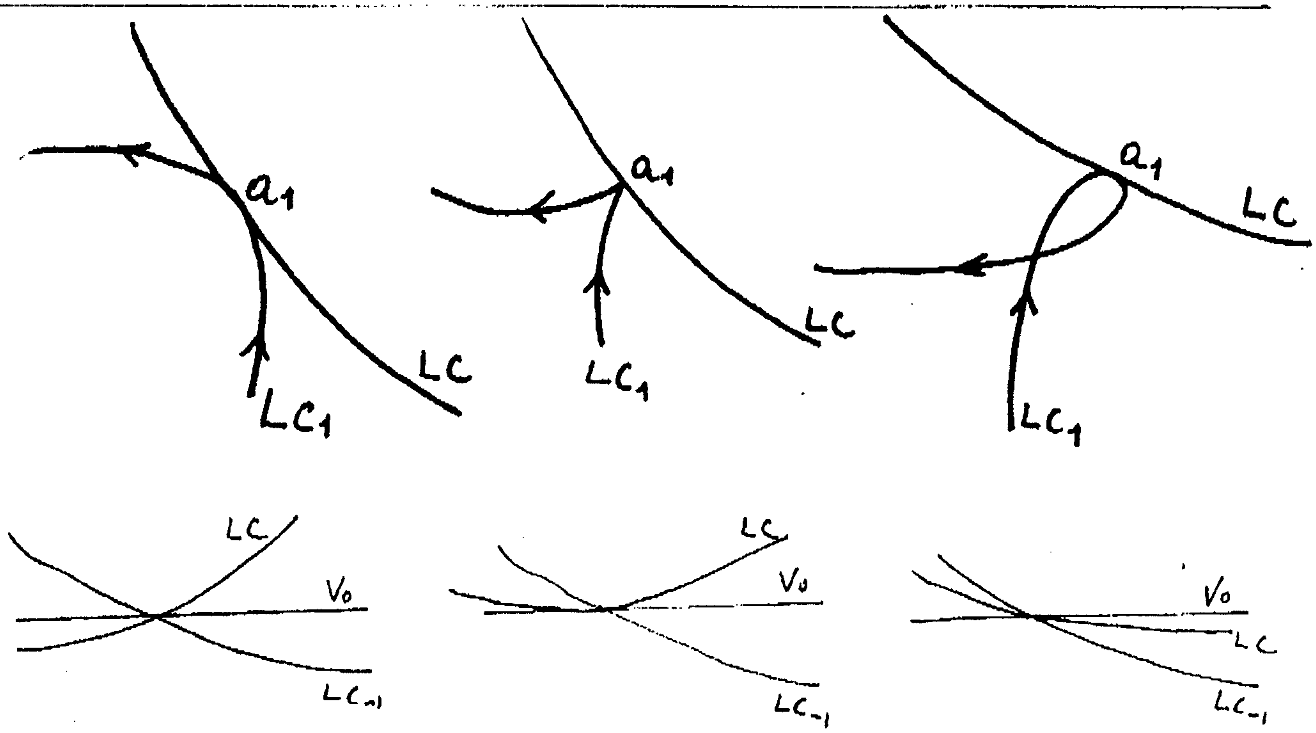
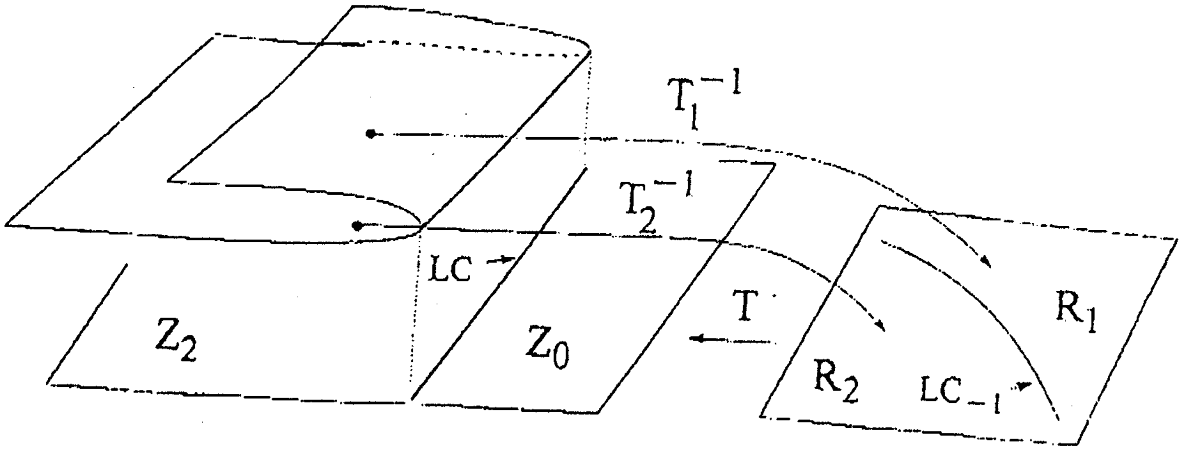
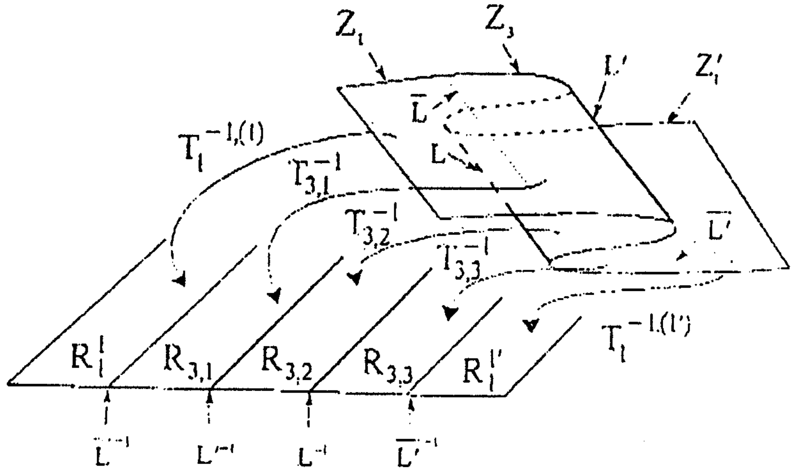
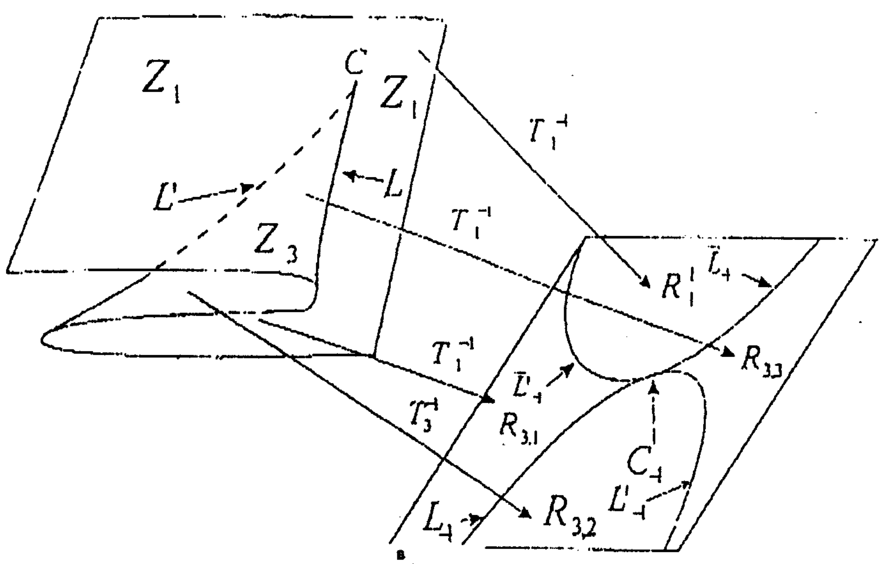
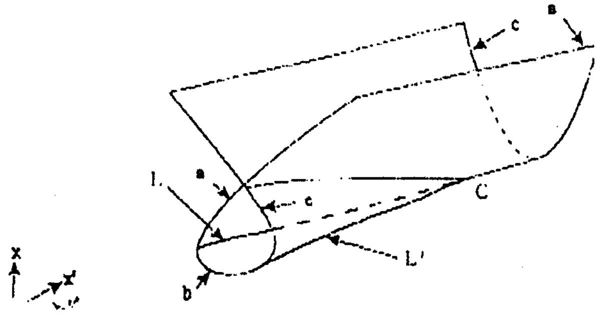


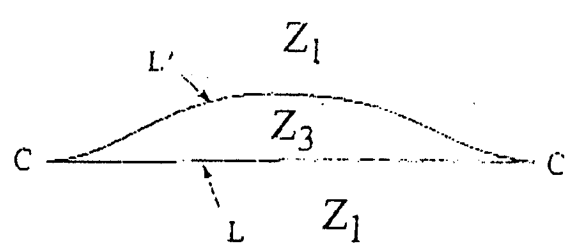
Fig. 6


T1^-1
T2^-1
LC
T
Z2
Z0
R1
R2
LC-1


Z1
Z3
L'
Z'1
L
T1^-1,(1)
T3,1^-1
T3,2^-1
T3,3^-1
L'
T1^-1,(1')
R1^1
R3,1
R3,2
R3,3
R1^1'


Z1
C
Z1
T1^-1
L'
L
Z3
T1^-1
R1^1
L-1
R3,3
T1^-1
L-1
T3^-1
R3,1
C-1
L'-1
L-1
R3,2


c
a
L
a
c
C
L'
b


Z1
L'
Z3
C
C'
L
Z1

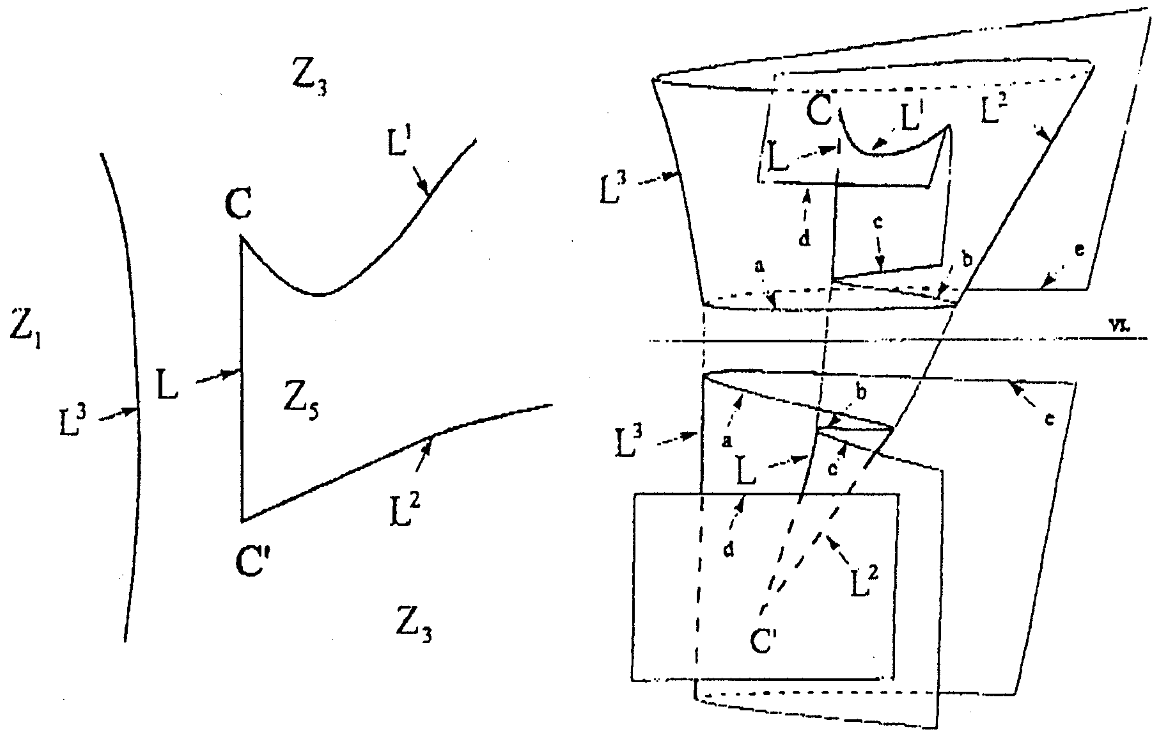

Z_3
L^1
C
Z_1
L^3
L
Z_5
L^2
C'
Z_3
C
L^1
L^2
L
L^3
d
c
a
b
e
VL
L^3
b
e
a
L
c
d
L^2
C'

fig 20

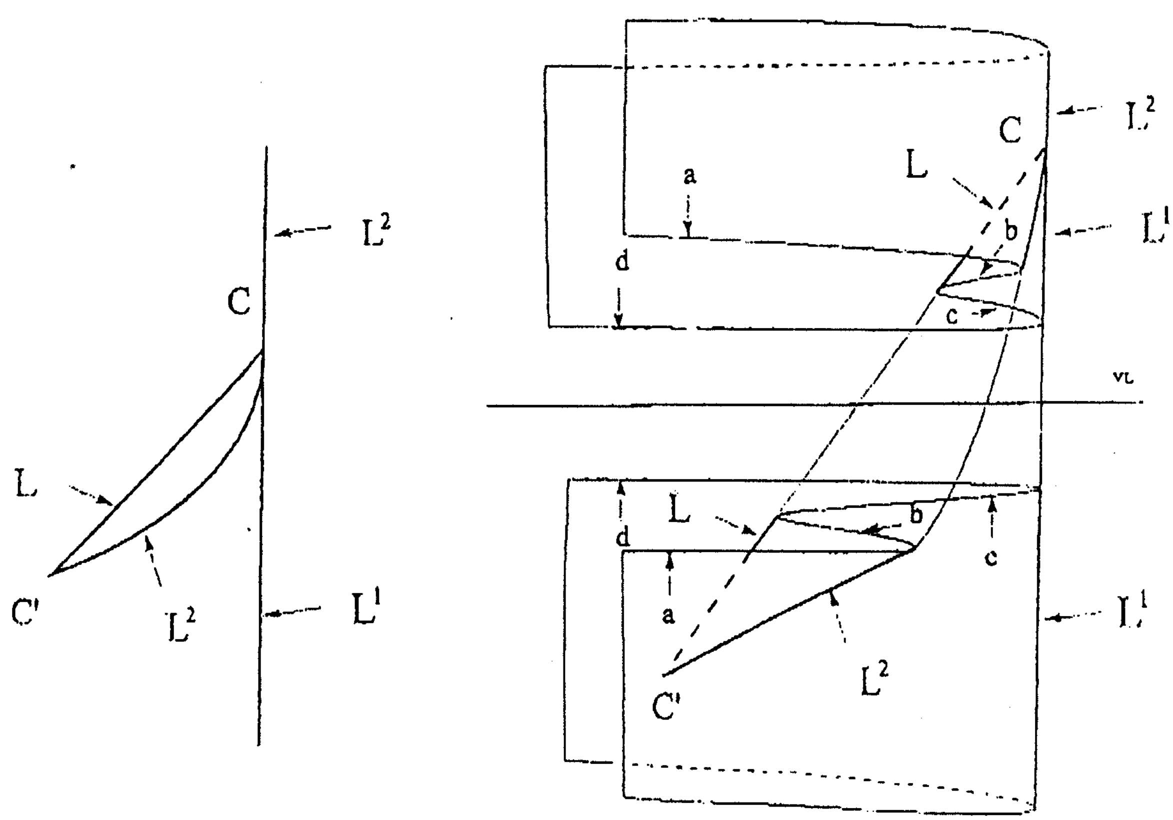

L^2
C
L
C'
L^2
L^1
L^2
C
L
a
b
L^1
d
c
VL
L
b
d
c
a
L^1
L^2
C'

fig 21

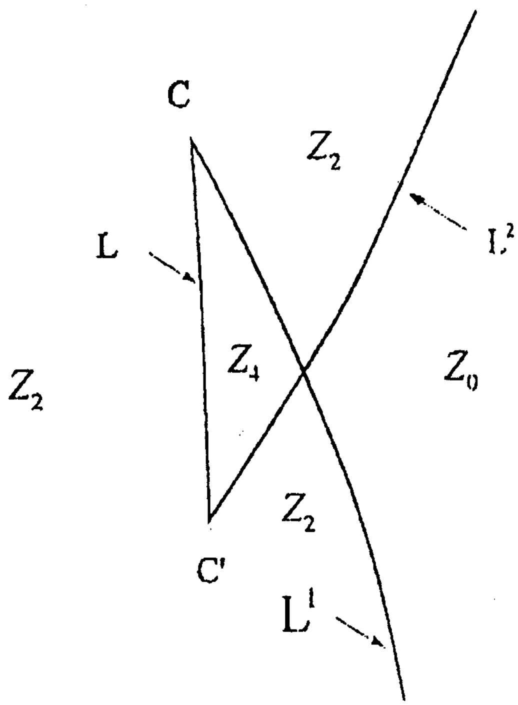
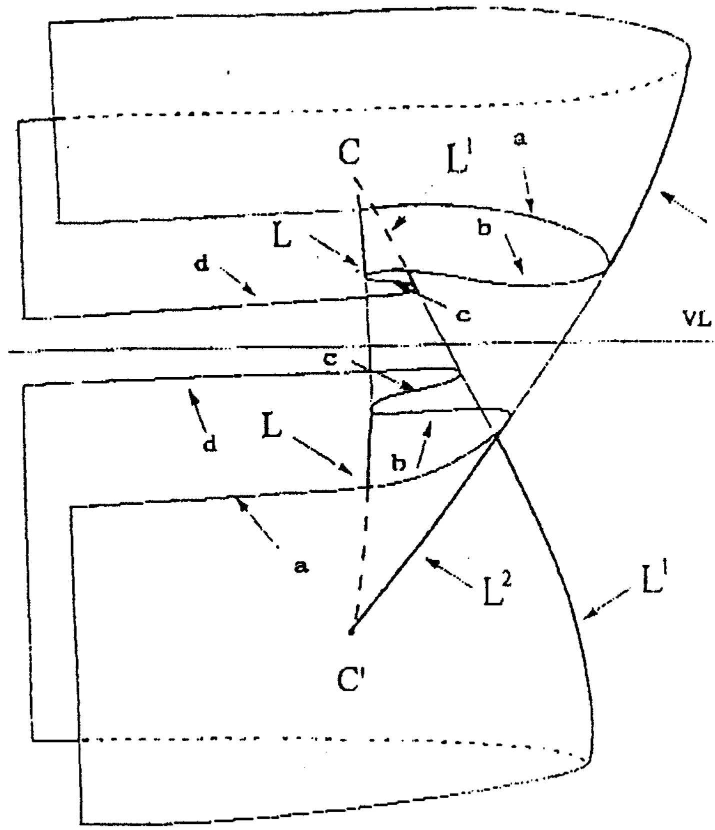

C
Z_2
L
L^2
Z_4
Z_0
Z_2
Z_2
C'
L^1

fig 12


C
L^1
a
L
b
L^2
d
c
VL
c
d
L
b
a
L^2
L^1
C'

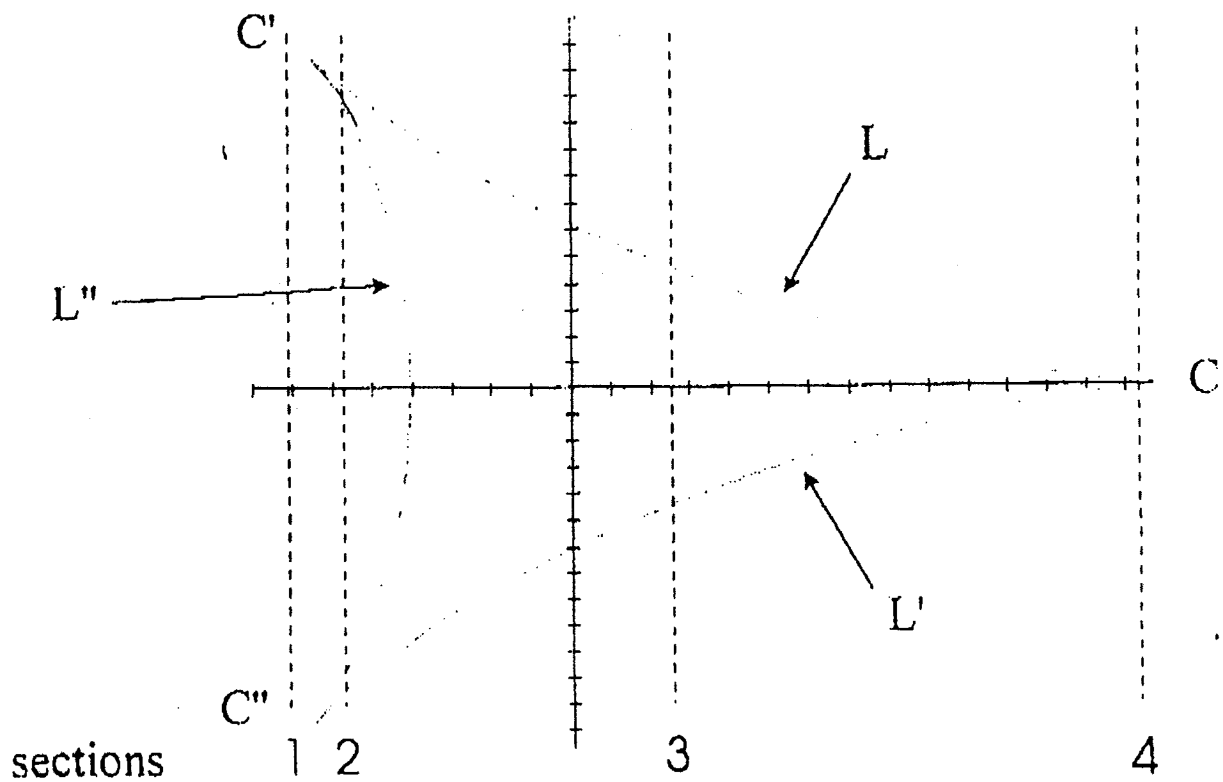

C'
L
L"
C
L'
C"
sections
1
2
3
4

fig 14

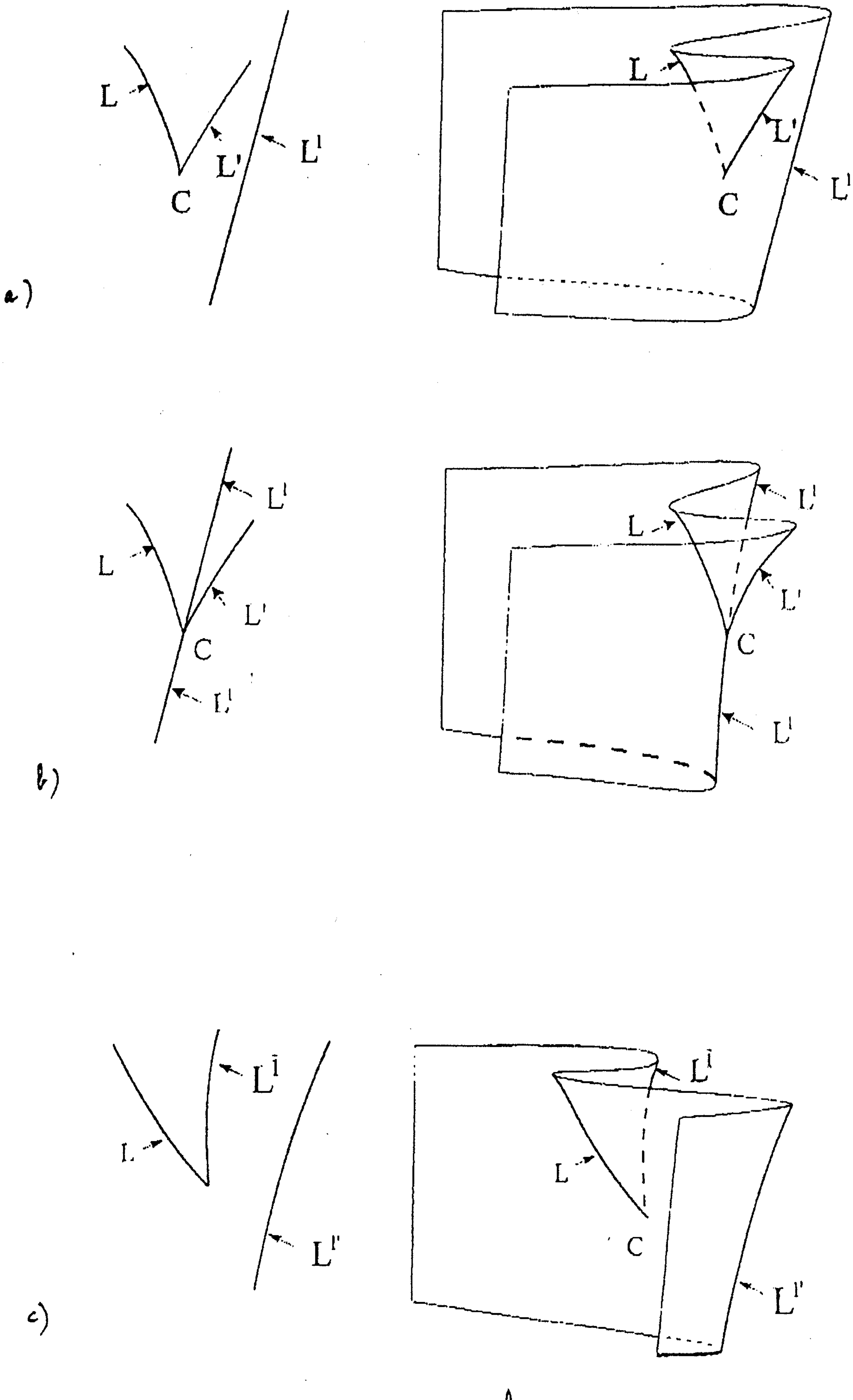


fig 18