Endosingularities

Ralph Herman Abraham

Abstract. We review the history and basic concepts of singularity theory, and contrast with the problems of iteration theory and the method of critical curves, in the two-dimensional case. We call the combination of these two theories *endosingularity theory*.

1. Introduction.

The theory of singularities of smooth mappings has been unfolding from the nascent works of Morse (1931), Tucker (1936), Whitney (1943, 1944), and Wolfsohn (1952). In his seminal work of 1955, Hassler Whitney introduced the basic concepts of folds, cusps, and other generic singularities into the literature of mathematics, and modern singularity theory was underway.

Very soon afterwards, these ideas were radically extended and reformulated by René Thom, utilizing the new ideas of jets and transversality. Thom's lectures in Bonn in 1959, recorded in a splendid sets of notes by Harold Levine, broadcast the fundamentals of a new branch of mathematics to an eager group of young geometers. Arriving in Berkeley in the Fall of 1960 for my first university post, I was fortunate to be able to attend a series of lectures by Thom himself on the new theories and problems.

In the next few years, at Berkeley, Columbia, and Princeton, I continued to work on the classification of generic singularities, and the technology of transversal intersections. (Abraham, 1963, 1967) Some of these problems were solved during this time by John Mather, then a young graduate student at Princeton. From this development in global analysis there evolved two important derivatives: *catastrophe theory* and the theory of *bifurcations*.

2. Catastrophes.

In the catastrophe theory of René Thom, the idea of *structural stability* was elevated into a new paradigm for applied mathematics, and the concepts of *attractor* and *chreode* of the theoretical biologist Conrad Waddington were introduced into the literature of *dynamical systems theory*, now also known as *chaos theory*. In 1965 and 1966 I was receiving handwritten chapters of catastrophe theory from Thom in Paris. Against a growing resistance by the mathematical community, I persuaded my own publisher to bring these out as a book, which appeared belatedly in French in 1973, and in English in 1975. These publications brought the new paradigm to the attention of a wide scientific audience, and after

Zeeman's splendid collection of articles appeared in 1977, to a popular audience as well.

Unfortunately for the history of mathematics and its applications to the sciences, catastrophe theory became the target of hostile attacks by highly respected mathematicians, most notably in a critical review of Zeeman's book by Stephen Smale. Smale and Zeeman had been competitors in the race to prove Poincaré's conjecture, for which Smale had been awarded the coveted Fields medal of the International Congress of Mathematicians, in 1960. This backlash movement successfully disposed of catastrophe theory, which is little known today in spite of its great power in crafting dynamical models for natural phenomena, and this backlash has seriously impeded the growth of chaos theory as well.

3. Bifurcations.

After catastrophe theory, the second spin-off from Thom's lectures (and Levine's notes) on the theory of singularities of smooth mappings was the theory of structural stability and bifurcation of continuous-time dynamical systems. In addition to the 19th century ideas on bifurcation due to Jacobi, Poincaré, Andronov, and Hopf, the new bifurcation theory also utilized Pontriagin's idea of structural stability, 1934, Whitney's notion of stable singularity, and Thom's technology of jets and transversal intersection. All this led to a new approach to bifurcations, formulated by Sotomayor in the 1960s. (See Abraham, 1977, for more historical details.) Since this pioneering work, bifurcation theory has produced an enormous literature, especially dealing with the onset of chaos.

4. Critical curves.

Discrete-time dynamical systems generated by a diffeomorphism (invertible map) were introduced by Poincaré in his approach to the Oscar prize problem on the stability of the solar system in the 1880s. The theory of their qualitative behavior has been developed ever since in parallel with continuous-time dynamical systems theory. However, discrete-time dynamical systems generated by a *noninvertible* map were largely ignored by the pure math community. With the advent of computers and computer graphics, workers in the physical sciences, engineering, and other applied areas began to study these systems, particularly their chaotic behavior, and to make significant mathematical discoveries. Most notable here was the work Myrberg on the one-dimensional case in the 1950s, the work of Gumowski and Mira in the 1960s on critical curves in the two-dimensional case, and the discovery of fractal geometry by Mandelbrot in the 1970s.

See (Abraham, 1997) for extensive historical material on discrete dynamics, written by Mira, as well as an elementary introduction to critical curves. See (Mira, 1996) for a more advanced treatment of the subject. The main steps of these three parallel evolution paths are shown in Table 1.

Date	Continuous-time	Discrete-time	Singularities
1880	Poincaré		
1890			
1900			
1910	Birkhoff	Julia	
1920		Fatou	Morse
1930	Andronov		Tucker
1940	Hopf		Whitney
1950		Myrberg	Thom
1960	Sotomayor	Gumowski, Mira	Mather
1970		Mandelbrot	Zeeman

Table 1. Chronograph of three theories.

5. The 1-jets of morphisms.

Here we will recall the simplest level of singularity theory, which is simply linear algebra, but expressly strangely. Let M and N be smooth manifolds, $f, g: M \to N$ smooth maps from M to N, and $p \in M$ a point. Let $T_p f \in L(T_p M, T_{f(p)} N)$ denote the tangent (the first derivative as a linear transformation) of f at p.
Then f and g are said to be 1-*equivalent* at p iff $f(p) = g(p)$, and $T_p f = T_p g$. A 1-equivalence class is a 1-*jet*, and the set of all 1-jets, $J(M, N)$, has a natural vector bundle structure, essentially identical to the linear map bundle $L(TM, TN)$ over MxN. See (Abraham, 1978) for definitions of these. The map

$$j^1 f: M \to J^1(M, N)$$

which assigns to every point $p \in M$ its 1-jet is called the 1-*jet extension* of f. Keep in mind that the 1-jet is essentially the first derivative of f at p, a linear map. But formally, it consists of p, q, and an equivalence class of maps, all of which map p to q, and have the same first derivative at p.

Thus, given a smooth map, $f: M \to N$, we may indicate the 1-jet of f at a point $p \in M$, if $q = f(p)$, as $j^1_p f = (p, q, T_p f)$, and thus obtain the two essentially identical diagrams

Figure 1

shown in Figure 1.

Fixing points *(p, q)* in *MxN*, the fiber $J^1_{(p,q)}(M, N)$ may be identified with the space of linear transformations $L(T_pM, T_qN)$. Supposing dimensions *m* of *M* and *n* of *N*, and choosing coordinate charts at the points $p \in M$ and $q \in N$, the 1-jet space may further be identified with the space of linear maps $L(R^m, R^n)$, and thus with real matrices of size *n* by *m*. Choosing different coordinate charts, we obtain a different isomorphism from 1-jets to matrices. This is just linear algebra.

The 2-jet bundle is obtained by iterating this procedure, that is, in terms of the 1-jet of the 1-jet of a map. This leads to multilinear algebra. But we continue now with 1-jets.

The relationship between the matrix representatives of 1-jets induced by different pairs of coordinate charts is shown in Figure 2, the *tent of equivalence* of linear algebra. Here a linear transformation *T* from a real vector space *V* of dimension *m* to a real vector space *W* of dimensions *n* is represented by two real matrices of size *n* by *m*, say P_1 and P_2, which are equivalent, $P = B^{-1}P_1A$, if *A* and *B* are the matrices for the changes of bases.

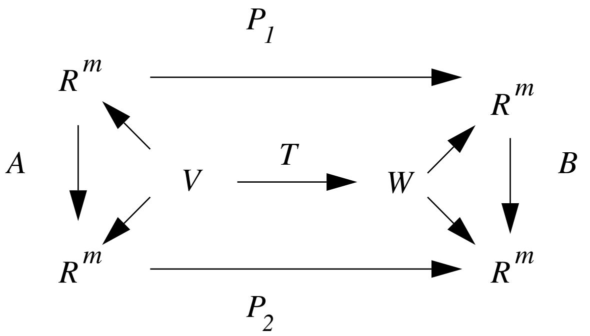

Figure 2

6. Singularities.

As *rank* is an equivalence invariant for linear transformations and their matrices, the sets $S_k \subset J^1(M, N)$ of 1-jets of *corank k* (that is, of *rank k* less than the maximum rank) for *k* = 0, 1, 2,..., *min(m, n)* – 1, may be defined in terms of the matrices representing 1-jets in coordinate charts. A point $p \in M$ is a *singular point* of a map, $f: M \rightarrow N$, iff the 1-jet of *f* at *p* belongs to $S_k \subset J^1(M, N)$ for $k > 0$.

Basic to the approach of René Thom in the 1960s was his observation that these invariant sets, the singularities $S_k \subset J^1(M, N)$ with $k > 0$, were finite unions of smooth submanifolds. He then defined genericity for a map $f: M \rightarrow N$ in terms of transversal intersection of the jet of f and the singularity submanifolds in the jet bundle. A similar situation occurs for the higher jets, which are essentially (that is, in local representation by coordinate charts) Taylor series polynomials of higher degree. By means of his jet transversality approximation theorem, Thom showed that any map could be approximated by one which met these singularity submanifolds transversally. Thus the dimensions of these intersections may be counted. For example, the corank-1 singularity occurs generically at isolated points for maps from the line to itself (real-valued functions of a single real variable), and on curves for maps from the plane to itself. Whitney called these *fold curves*, and they coincide approximately with the critical curves LC_{-1} of Gumowski and Mira. If a plane endomorphism is restricted to a critical curve in this sense, the inverse image of the 1-jet by the 1-jet extension of the endomorphism, the restricted map may have a generic corank-1 singularity at isolated points on the fold curves. These singular points were called *cusp points* by Whitney. The corank-2 singularity does not appear generically in these low dimensions.

These generic singularities of plane endomorphisms comprised the main results of Whitney in 1955, which were extended significantly by Thom in 1959. But Whitney also computed normal forms for plane endomorphisms exhibiting the fold and cusp singularities. He showed that, by separate choices of special coordinate charts (x, y) at a cusp point and (u, v) at its image in the plane, the local representative of the map could be put in the normal form,

$$u = xy - x^3$$
$$v = y$$

For a clear and elementary explanation and computer graphic study of this map, see the paper of Gardini, Bischi, and Abraham in this volume.

7. The 1-jets of endomorphisms.

The normal forms of singularity theory, such as those of Whitney seen above, require careful choices of separate coordinate charts at the source and the target of the map. This is like the theory of equivalence in linear algebra and matrix theory. But for the endomorphisms of iteration theory, we must be satisfied with a single choice of coordinate chart at the source and target of a map. For this reason, we usually express a map in local representation in the form

(x, y) maps into (x', y'),

rather than

(x, y) maps to (u, v)

of singularity theory. As singularity theory is to equivalence of matrices, iteration theory is to similarity of matrices. As the normal forms of singularity theory are to the diagonal matrix with ones and zeros on the diagonal, the normal forms of iteration are to the real canonical matrix, with blocks on the diagonal corresponding to eigenvalues. The equivalence relation for iteration theory corresponding to the similarity of matrices is usually called *conjugacy*. That is, two endomorphisms from *M* to itself, *f*, *g*, are *conjugate* if their exists a diffeomorphism *h* of *M* to itself such that $g = h \circ f \circ h^{-1}$.

As in linear algebra, the reduced freedom of choice in choosing smart coordinate systems results in an increase in the number of equivalence classes, and thus in normal forms. We may reconsider the Whitney fold and cusp maps now as examples.

8. The Whitney families.

If there is a theory of normal forms for singularities of a map under conjugacy, we do not know of it. So we can only provide some guesses here.

Consider first the one-dimensional fold map, $x' = x^2$ Any line endomorphism, at a fold, can be transformed into this representation by suitable changes of coordinate chart at the source and target. But restricted to a single coordinate chart at both the source and target, we may not improve on the form, $x' = a + x^2$. Thus we have a one-parameter family of normal forms in the context of iteration theory, all exhibiting the fold singularity generically. This is the main point of this paper.

For endomorphisms in two dimensions, the generic fold may be the two-parameter family,

$$x' = a + x^2$$
$$y' = by$$

Consider now the cusp singularity. We do not have a definitive normal form family to offer, but recommend the study of the three-parameter family:

$$x' = a + bxy - x^3$$
$$y' = cy$$

We call these the *Whitney families* of endomorphisms.

9. Conclusion.

As seen in Table 1, the evolution sequences of

- continuous-time dynamics, or flows,
- discrete-time dynamics, or iteration theory of noninvertible maps, and
- singularity theory for maps

are sequentially delayed in historical time. While we know that noninvertible iterations developed from invertible iterations, and these in turn from continuous-time dynamics, the later emergence of singularity theory seems to be relatively independent. Of course, all stem from a common root in Poincaré. In this paper we have indicated the difference between singularities of maps as defined by equivalence and those of endomorphisms as defined by conjugacy. These latter we may call *endosingularities*. The normal forms for singularities and those for endosingularities are quite different. We propose thus two things: an experimental program for iteration theory, based on moduli (families) of normal forms for the fold and cusp endosingularities, and a theoretical program for global analysis aimed at deriving the endosingular moduli.

ACKNOWLEDGEMENTS

We are grateful to René Thom and Harold Levine for their lessons on singularities during the 1960s, to Laura Gardini and Christian Mira for their initiation into the arcana of critical curves during the 1990s, and to Laura Gardini and to ECIT96 for the inspiration to connect these two subjects.

BIBLIOGRAPHY

Abraham, Ralph. Transversality in manifolds of mappings. *Bull. Amer. Math. Soc.* 69, 470-474, 1963.

Abraham, Ralph. *Lectures of Smale on differential topology*. New York: Columbia University, 1963.

Abraham, Ralph. *Linear and multilinear algebra*. New York: W.A. Benjamin, 1966.

Abraham, Ralph, and Joel Robbin. *Transversal mappings and flows*. New York: W. A. Benjamin, 1967.

Abraham, Ralph, and Jerrold E. Marsden. *Foundations of mechanics*, 2d ed., Reading, Mass.: Benjamin/Cummings Pub. Co., 1978.

Abraham, Ralph, Laura Gardini, and Christian Mira. *Chaos in discrete dynamical systems*. Santa Clara, CA: Telos/Springer-Verlag, 1997.

Chern, Shiing-Shen, and Stephen Smale, editors, *Symposium in Pure Mathematics, University of California at Berkeley, 1968. Global analysis*. Providence: American Mathematical Society, 1970.

Golubitsky, Martin, and Victor Guillemin. *Stable mappings and their singularities*. New York: Springer-Verlag, 1974.

Levine, Harold I. *Singularities of differentiable mappings*. Bonn: Mathematisches Institut der Universitat, 1959.

Mira, Christian, Laura Gardini, A. Barugola, and J.C. Cathala. *Chaotic dynamical systems of two-dimensional noninvertible maps*. Singapore: World Scientific, 1996.

Smale, Stephen. *The mathematics of time: essays on dynamical systems, economic processes, and related topics*. New York: Springer-Verlag, 1980.

Thom, René. *Stabilite structurelle et morphogenese; essai d'une theorie generale des modeles*. Reading, Mass.: W. A. Benjamin, 1973.

Thom, Rene. *Structural stability and morphogenesis; an outline of a general theory of models*. Transl. D. H. Fowler. Reading, Mass.: W. A. Benjamin, 1975. Redwood City, Calif.: Addison-Wesley, Advanced Book Program, c1989.

Zeeman, E. C. *Catastrophe theory: selected papers, 1972-1977*. Reading, Mass.: Addison-Wesley Pub. Co., Advanced Book Program, 1977.

Ralph Herman Abraham
Visual Math Institute
Santa Cruz, CA, USA-95061-7920
(email: abraham@vismath.org)