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The Geometry of Angels

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Abstract. We analyze the mathematical vision behind the wings of the angel Gabriel, in a Renaissance painting, the *Cortona Annunciation* of Fra Angelico. In the context of the Renaissance mathematics of perspective and conic sections, the flapping wings generate a family of toroids. We interpret this image as a painterly representation of the three-sphere of Dante. As such, this geometrical vision of Fra Angelico presaged seminal works of modern mathematics, such as the Hopf fibration.

CONTENTS

1. Introduction.

Dante (1265-1321) was composing the *Divine Comedy* from 1307 until his death. In it, according to the very convincing argument of (Peterson, 1979), is a precognitive vision of the three-sphere, a topological object known to the history of mathematics only since 1930 or so. Dante was geometrically sophisticated (Crosby, 1997, pp. 171-173). He was a contemporary of Giotto (1277-1337), who was an early advocate of natural perspective in paintings. Fra Angelico (1400-1455) was inspired by Giotto, and participated in the revival of geometric art which characterized the Renaissance. His was "one of the most innovative and responsive pictorial minds" of his time (Spike, 1997, p. 11).

In this paper we are going to maintain that the three-sphere is represented in paintings of Fra Angelico around 1435. Further, his representation utilizes a construction of the three-sphere from two families of tori, which is basic to another construction known to mathematicians as the *Hopf fibration*, named after its discovery by Heinz Hopf around 1930.

2. Nimbus or halo

An ellipse is a circle seen obliquely, and an early application of the theory of conics to painting is in the elliptical representation of hat brims and halos by Giotto, around 1310. The halo was seen as a golden disk attached to the heads of saints. But in Fra Angelico, the dinner-plate halo is replaced by the spherical nimbus. No matter the angle of view, the nimbus is invariably shown as a golden disk bounded by a black circle: round, not oval. As Fra Angelico was mathematically sophisticated (one of his paintings is based upon an Alberti grid) we must assume that his nimbus is a semi-transparent ball of golden light. Imagine a spherical plastic bubble, which is transparent when seen from the outside, but opaque when seen from the inside. His first spherical nimbus appeared on the *Linaiuolo Madonna* of 1433 (Murray, 1963, pp. 90, 98).

This is but a subtle indication of the spatial intelligence of Fra Angelico, an indication which is suggestive in the context of our assertion that the angels of Fra Angelico, inspired by visions seen during his meditations and prayers, fly up through dimensions. They create three-dimensional space by beating their wings: two dimensions plus time make three dimensions, as a kind of alternative to perspective as a symbolic form (Panofsky, 1997).

Perfect circles are divine, according to Plato, and so also are spheres. We understand this as a basis for the perfect shapes of circular halos and spherical nimbuses.

We may compare the elliptical halos, suggestive of golden dinner plates, in Fra Filippo Lippi's *Barbadori Alterpiece*, of 1437 (Murray, 1963, p. 91).

3. The three-sphere

One, two three, infinity, said Chuang Tzu, and this is our plan: we will explain how to construct the one-sphere, then the two-sphere, and the three-sphere. The three constructions are similar, and when you get the idea, you may go on to construct the four-sphere, and so on, to infinity. But we will be satisfied with three constructions.

The one-sphere made of two one-balls

We begin in Lineland, a one-dimensional world. See (Abbott, 1963) for more information about this world. Visualize a one-ball, that is, a line segment, say two units long. Now, double it. We have two identical intervals, one on top of the other. For the next step, we need a new dimension. Citizens of Lineland would have difficulty with this exercise, but with imagination, anything is possible. Move one of the intervals away from the other, which stays unmoved. Imagine that the moving interval is pushed away in an orthogonal direction, which we (as superior beings) recognize as belonging to Flatland, a two-dimensional world. If the two directions of the original intervals are North and South, we have moved one of them East of the other.

Now bend the bounding points of the Eastern segment Westward, and the bounding points of the Western segment Eastward, and connect the corresponding bounding points, as shown in Figure 1. The connected intervals form a loop, topological one-sphere. If we bent the loop to make it round, we would have a geometric one-sphere, or perfect circle. Done deal.

The two-sphere made of two two-balls

Now we are in Flatland, a two-dimensional world. Visualize a two-ball, that is, a disk, or filled-in circle, say one unit in diameter. Now double it. We have two identical disks, one on top of the other. For the next step, we need a new dimension. Citizens of Flatland would have difficulty with this exercise, but with imagination, anything is possible. Move one of the disks away from the other, which stays unmoved. Imagine that the moving disk is pushed away in an orthogonal direction, which we (as superior beings) recognize as belonging to Solidland, a three-dimensional world. If the four directions of the original disks are North, South, East, and West, we have moved one of them Upward of the other.

Now bend the bounding circle of the Upper disk Downward, and the bounding circle of the Lower disk Upward, and connect the bounding circles, as shown in Figure 2. The connected disks form a surface, a topological two-sphere. If we bent the surface to make it round, we would have a geometric two-sphere, or perfect sphere. Done deal.

The three-sphere made of two three-balls

Now we are in Solidland, a three-dimensional world. Visualize a three-ball, that is, a ball, or filled-in sphere, say one unit in diameter. Now double it. We have two identical balls, one on top of the other. For the next step, we need a new dimension. Citizens of Solidland would have difficulty with this exercise, but with imagination, anything is possible. Move one of the balls away from the other, which stays unmoved. Imagine that the moving ball is pushed away in an orthogonal direction, which we (as superior beings) recognize as belonging to Hyperland, a four-dimensional world. If the six directions of the original balls are North, South, East, West, Up, and Down, we have moved one of them Heavenward of the other.

Now bend the bounding sphere of the Heaven disk Earthward, and the bounding sphere of the Earth disk Heavenward, and connect the bounding spheres. The connected balls form a solid, a topological three-sphere. If we bent the solid to make it round, we would have a geometric three-sphere, or perfect hypersphere. Done deal. (No figure for this one!)

4. Dante's cosmology

In the European tradition, three poets are regarded consummate: Homer, Dante, and Shakespeare. Of the three, Dante has probably enjoyed the greatest appreciation and scholarly attention over the years. A contemporary and friend of Giotto, who painted his portrait, Dante (1265-1321) has been regarded as marking the end of the Middle Ages. He was followed soon by Petrarch (1304-1374), who triggered the Renaissance, and by Boccaccio (1313-1375), the early Humanist. All were influenced by the troubadours of Provence. See (Smith, 1973, p. 18).

Dante was the author of numerous works, of which only one is widely known: *The Divine Comedy.* This, perhaps "the greatest poem of our tradition" (Bergin, 1965, p. 213), features three parts: *Inferno, Purgatorio* and *Paradiso*. Purgatory was apparently an invention of the 12th century associated with Saint Patrick, and is connected to the spheres of Hell and Paradise by a tunnel, a kind of umbilicus. See Yolande de Pontfarcy (pp. 93-116) and Jean-Michel Picard (pp. 271-286) in (Barnes, 1995).

Dante's cosmology may have been inspired by Pseudo-Dionysius the Areopagite, who wrote of the heavenly realms of angels around 200 AD in Alexandria. In Dante's cosmology, two three-balls of the same size, bounded by two-spheres, are connected The Earth ball is visualized as an onion, that is, as a family of concentric, spherical, shells. The shells are the solid Earth, the Atmosphere, and the Lunar sphere. The bounding sphere is the celestial sphere, home of the planets and stars.

The Heaven ball is also visualized as an onion, the concentric shells are the heavenly spheres, or homes of the angles. The bounding two-sphere is the celestial sphere. The center is God.

As the two three-balls are connected by identifying (gluing) the bounding two-spheres, we recognize the three-sphere constructed in the usual way. Other than the two exceptional points, Here and God, the whole universe is an onion-like affair, a family of two-spheres.

5. The three-sphere full of tori

To progress from the vision of Dante to that of Fra Angelico, we must put the two three-balls together by gluing their bounding celestial spheres as before. But first, we must replace the concentric spheres of the onion structures with concentric tubes, like rigatoni. Imagine the North and South poles of each three-ball connected by a straight line segment, its *pole*. Around each of the three-balls, within their bounding (celestial) spheres, draw an equatorial circle, its *equator*. Each of the two balls now has a special line segment, its pole, and a special circle, its equator. Other than these two one-dimensional features, we may imagine each three-ball to be made of concentric tubes, like rigatoni, as shown in Figure 3.

As we glue the two bounding (celestial) spheres together, we must make sure that their North poles are glued together, their South poles are glued together, and their equators are glued together. Then the two poles are glued together into a one-sphere or loop, and the two equators are glued together into a loop. The rigatoni-like tubes of the Heaven ball are glued to similar tubes of the Earth ball to make tori, like tortelini, or inner tubes, as shown in Figure 4. Therefore Dante's cosmos is made up entirely of concentric inner tubes, except for the two exceptional loops.

This vision of the three-sphere was fundamental to the creative work of the mathematician Heinz Hopf, who was one of the great pioneers of algebraic topology in the 1930s. See (Dieudonne, 1989) for the somewhat technical details of this story.

6. Fra Angelico's angels

If indeed a time traveler suddenly appeared, we would be shocked. As hard as it would be for us to understand the culture and iconography of the future, it is equally hard to grok those of the past. A case in point: angels. See (Fox and Sheldrake, 1996) for a gallant effort.

Fra Angelico was an important friar, and also one of the great painters, of the Early Renaissance. As a person of great integrity, we must take him seriously when he includes angels in his paintings. Probably he was also influenced by Pseudo-Dionysius; see (Spike, 1997, p. 64). All his representations of Biblical scenes give the impression of total religious belief, without hypocrisy.

So in his paintings of the Annunciation, that is, the announcement to Mary by the Angel Gabriel that she is to give birth to the son of God, the representation of the angelic form as a humanoid with wings must be taken seriously. We are going to interpret this graphical code as a geometrical statement, or symbolic form, in the spirit of Dante's cosmology.

First of all, Fra Angelico was geometrically sophisticated for his time. This is established by his use of a formal grid on one painting, which shows total mastery of linear perspective, newly introduced by Brunelleschi and Alberti. In fact, it is likely that Alberti himself contributed the grid to this painting. (Pope-Hennessy, 1981) However, Fra Angelico did not use linear perspective in many other paintings; he preferred a more natural style for the representation of three-dimensional space. See several geometric analyses, in (Morichiello, 1996). See also (Spike, 1997, pp. 27, 52, 53). On this basis we have argued above that the nimbus he puts around the heads of his saints and angels is a three-ball bounded by a two-sphere. The perfect sphere is, after all, a divine shape in the idealogy of Plato, and is most appropriate as a symbol of the divinity of these beings: they belong to the Heavenly Ball of Dante's three-spherical world.

Now look at the shape of the wings of the angel in the *Cortona Annunciation*, Figure 5. (See also the rather similar Annunciation in the Convent of San Marco, Florence.)The probable date for this painting is 1438. (See Spike, 1997, p. 47.) They are drawn of elliptical arcs. Recall that an ellipse is easily drawn with a loop of string and two pins. And in fact, the arc of the frontal edge of the angel's right wing matches an ellipse drawn with two foci within the frame of the picture, one of which is marked by a small nimbus! We must consider the fact that, like a halo, the boundary curves of the angel winds, seen head-on, are perfect circles.

This indicates a geometrical vision as follows. Each wing represents a circle. By stretching out the wings and flapping them, the angel carves out the tori of the cosmic three-sphere. As two spatial dimensions plus movement make three dimensions, the flight of angels may be seen as a construction of the world, and a means of navigating between the Ball of Heaven and the Ball of Earth.

This prevision of the topology of the 20th Century on the part of Fra Angelico is no more extraordinary than the prevision of Dante: both were informed by extensive meditation, prayer, and other spiritual exercise.

7. Conclusion.

We may include the vision of Fra Angelico in a broader context of major social transformation, following the domino theory of Flinders-Petrie. (Abraham, 1994.) This transformation consists of a 600 year sequence of paradigm shifts, each one a trigger for the next like dominoes, including:

- the troubadours, around 1100, a partnership resurgence, see (Eisler, 1989)
- a shift in mathematics brought about by Leonardo of Pisa around 1200
- perspective, a shift in painting (Giotto)
- humanism in literature: Dante, Petrarch, Boccaccio
- the Italian Renaissance, including Fra Angelico
- another shift in mathematics: the dynamic mentality, Newton, Leibniz, 1700

and many more.

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Fig. 1a.

- We begin with a unit 1-ball, that is, a line segment two units long.
- Double the 1-ball, so there are two of them, one on top of the other. Then move one up and the other down.

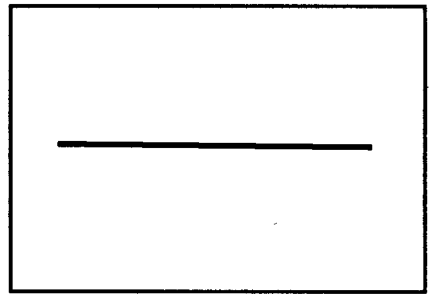

Fig. 1b.

- Now we have two 1-balls, one about the other.
- Next, bend the boundary (two ends) of the upper 1-ball down, and the boundary of the lower 1-ball up.

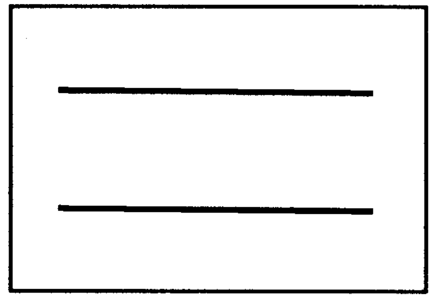

Fig. 1c.

- Now we are ready for the glueing step.
- Pull the boundary (both ends) together and glue them to join smoothly.
- Now we have a loop.

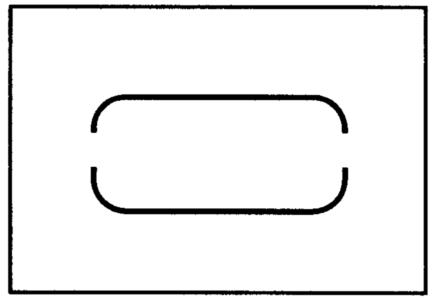

Fig. 1d.

- Here is the loop.
- It is a topological 1-sphere, that is, it can be deformed into a geometrical 1-sphere.
- Go ahead. Deform it into a perfect circle.

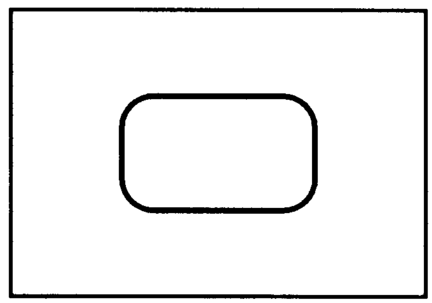

Fig. 1e.

- Here is the geometrical 1-sphere, a perfect circle.
- It is one-dimensional, that is, the inside is empty.

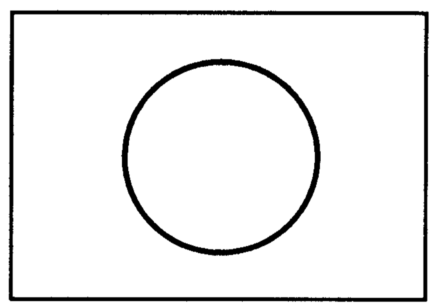

FIGURE 1.

A 1-sphere made by glueing together two 1-balls.

Fig. 2a.

- We begin with a unit 2-ball, that is, a 2-dimensional disk, or perfect circle filled-in with a disk of flat 2-dimensional space.
- Double the 2-ball, so there are two of them, one on top of the other. Then move one up and the other down.

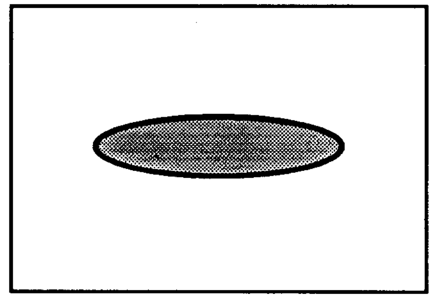

Fig. 2b.

- Now we have two 2-balls, one about the other.
- Next, bend the boundary of the upper 2-ball down, and the boundary of the lower 2-ball up.

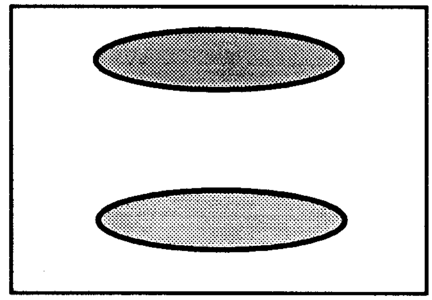

Fig. 2c.

- Now we are ready for the glueing step.
- Pull the boundaries together and glue them to join smoothly.
- Now we have a cylindrical surface, like a tin can.

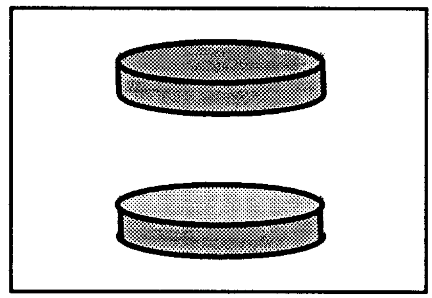

Fig. 2d.

- Here is the surface.
- It is a topological 2-sphere, that is, it can be deformed into a geometrical 2-sphere.
- Go ahead. Deform it.

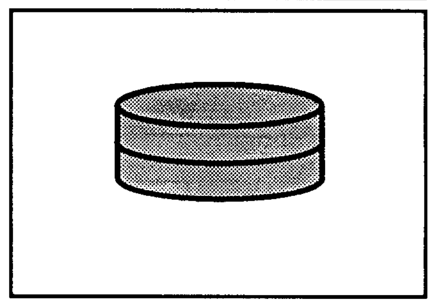

Fig. 2e.

- Here is the geometrical 2-sphere, a perfectly spherical surface.
- It is two-dimensional, that is, the inside is empty.

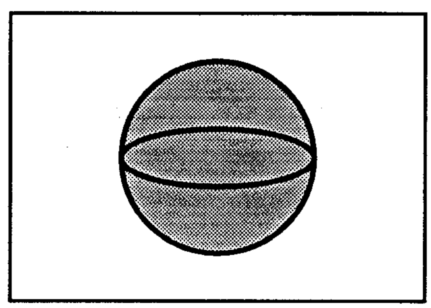

FIGURE 2.

A 2-sphere made by glueing together two 2-balls.

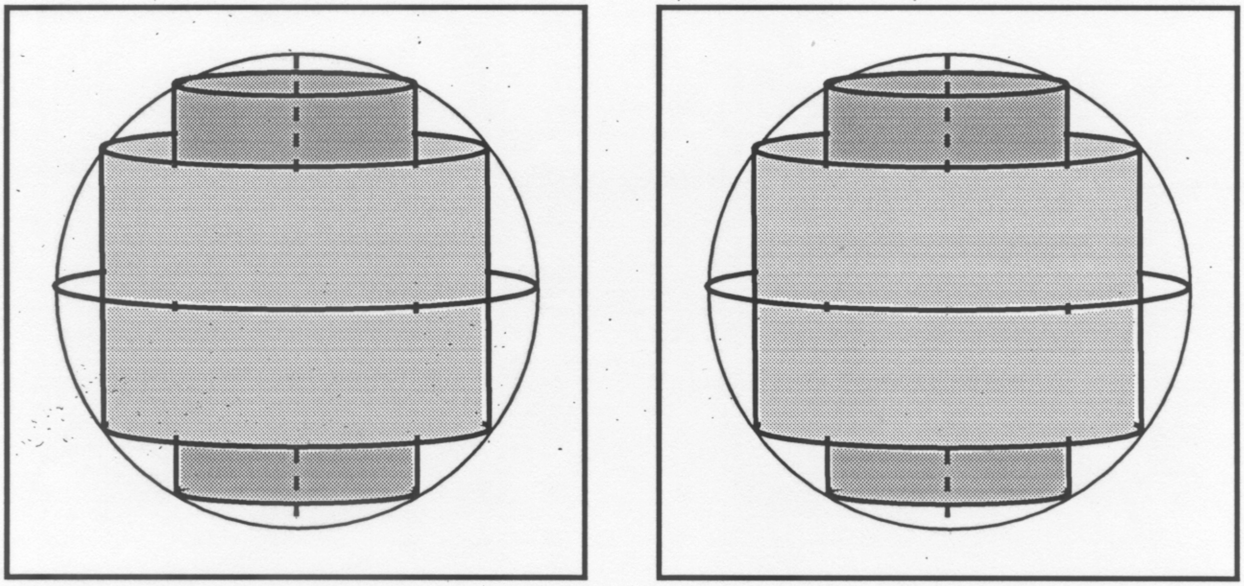

FIGURE 3. The two 3-balls, each filled with rigatoni, only two tubes of which are shown.

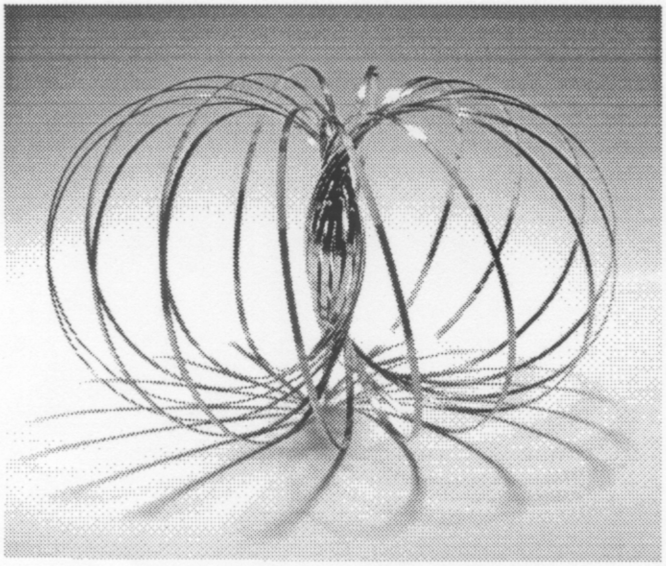

FIGURE 4. Two pieces of rigatoni joined to make a 2-torus, wrapped with wire for visibility.

FIGURE 5.

The *Cortona Annunciation* of Fra Angelico, 1438 AD, from (Lloyd, 1979).