Math 145, Winter 1999: Chaos Theory

Part I: A Survey of Chaos Theory

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Compiled from the website, www.vismath.org -> Chaos Course

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1. Lecture 1F, 08 Jan 1999: 1D flows

- State space and tangent space
- The tangent bundle
- Trajectories in space-time

1.1 The state space and its tangent spaces

This is an exemplary one-dimensional state space, S. The point, p, represents a virtual state of some system. [Figure 1.1.1]

This is a secant line. Think of the right end point as a sliding bead, and the left endpoint, p, as a rotating hinge. [Figure 1.1.2]

As the sliding bead moves to the rotating hinge, the secant line becomes tangent to the state space. This tangent line is the tangent space to S at the point p. [Figure 1.1.3]

1.2 The tangent bundle

Here we flatten the state space, S, into a straight line, and rotate the tangent space at the point, p, to vertical. The two-dimensional space thus constructed is the tangent bundle of S. [Figure 1.2.1]

This graph of a function in the tangent bundle assigns a tangent vector to every point of the state space. This is a vectorfield on S. [Figure 1.2.2]

1.3 Trajectories of a vectorfield

Assume a one-dimensional state space, a straight line segment, and a vectorfield on it. We visualize the vectors lying right on top of the state space. Each vector is now going to be interpreted as a velocity vector. Suppose that units have been agreed for measuring both space and time, say feet and seconds, and that speeds are given by the velocity vectors accordingly in feet per second. For example, let the vector at the point p indicate a forward velocity of 15 fps. Orient the state space (blue) and the velocity vectors (green) vertically. Extend a new dimension (yellow) horizontally corresponding to time. Finally, we choose a time, t, for this construction.

Here is the space-time plane, showing the agreed units and the chosen point, (t, p). [Figure 1.3.1]

Attach a small red rectangle centered at the point (t, p.) Choose its size so that:

- it extends one second both to left and to right of its center,
- its height is twice the green vector at p, here shown pointing up.

Draw a red diagonal line through the red box, from the lower left corner to the upper right corner. This diagonal line segment is the direction element determined by the green vector at p.
NOTE: if the vector at p had pointed down, we would have drawn this diagonal from the upper left corner to the lower right.

Here is the direction element at the chosen point. [Figure 1.3.2]

This construction is repeated for all points (t, p) on the horizontal line through the fixed p as t varies. So all direction elements on this horizontal line are the same. Then this is repeated for a new point p, and so on. To illustrate the process, we choose the open interval (0, 2) as S, and the logistic function f(x) = (3.6)x(1-x) as the vectorfield.

Here is a view of this vectorfield, shown in the tangent bundle representation. [Figure 1.3.3]

And here is a plot of the direction field of this vectorfield, with trajectories superimposed. [Figure 1.3.4]

Trajectories are curves, that is, graphs of functions in the space-time plane, which are tangent to the direction elements at all points.

2. Lecture 2M, 11 Jan 1999: 1D flows, cont.

Attractors: Deduce from slope of the vectorfield at a zero crossing that the rest point as attractive or repulsive

Basins: Examine all points in the state space to see where they go in the long run

Separatrices: The boundary between basins is the separatrix

*Fold bifurcation*s: Move the graph of the vectorfield up and down and observe the effect on the rest points

Coming next: 2D attractors, basins, and separatrices

3. Lecture 2W, 13 Jan 1999: 2D flows

Where do 2D flows come from?

- Coupling two 1D flows in Stella
- A second order ODE in Stella

State spaces: plane, cylinder, torus, sphere,...

Rest points

- Attractor (focus, node), saddle, repellor
- The Characteristic Exponents (CEs)

Attractors

- Points
- Cycles

Separatrices

- Insets of saddles, repelling cycles

Poincare section of a limit cycle

- Construction of the section
- Construction of the first return map
- The slope of the graph of the map

Coming next: 2D bifurcations, 3D flows

4. Lecture 2F, 15 Jan 1999

- 2D flows, cont.
- 3D flows
- What's up next?

2d Flows, cont.

- the static fold bifurcation in 2D flows
- the periodic fold bifurcation in 2D flows
- limit cycles in 2D flows
- characteristic multipliers (CMs)
- periodic attractors and repellors
- the Van der Pol story and dynamical scheme (See DGB2, Strogatz p. 198)
- the Hopf bifurcation
- three kinds of bifurcation (subtle, catastrophic, explosive)

3D flows

- where do 3D flows come from?
  3=1+1+1: bill smith's model for puberty
  3=2+1: the ring model for a forced pendulum

- rest points in 3D and their CEs
- limit cycles in 2D
- Poincare section and first return map
- fixed points of a 2D cascade
- CMs of a fixed point
- periodic points of a 2D cascade

Coming in week 03:

- homoclinic tangles in 2D cascades
- homoclinic tangles in 3D flows
- chaos in forced oscilators (the ueda attractor)
- bifurcations of 3D flows
- complex dynamical systems
- cellular dynamical systems

and then in week 04:

- begin part II (iteration theory)

5. Lecture 3W, 20 Jan 1999

Review of the 2D Poincare section for

- the 3D ring model for a forced oscillator (eg, Duffing, Van der Pol)
- Homoclinic tangle for a fixed point of a 2D cascade

References for this material:

- Abraham and Shaw
- Thompson and Stewart
- Guckenheimer and Holmes

6. Lecture 3F, 22 Jan 1999

The Ueda attractor and the Japanese attractor

- the forced Duffing system
- translation to the ring model
- the 2D control space of (B, k)
- the outset of the fixed saddle 1D1

References for this material:

- Abraham and Shaw
- Thompson and Stewart
- Ueda, esp. pp. 158-174

The End of Part I, Survey of Chaos Theory