Math 145
Chaos Theory

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Meeting #2Th, April 13

- 2D Iterations, Applications
  - 1. Logistic Delay Scheme
  - 2. Business Cycles of Kaldor
  - 3. Business Cycles of von Neumann
  - 4. Julia sets

Application 1
Delay Schemes

Following H. A. Laurerier, 1986

Delay Schemes

The one-dimensional delay scheme

$$x_{n+1} = F(x_n, x_{n-1})$$

is equivalent to the two-dimensional iteration

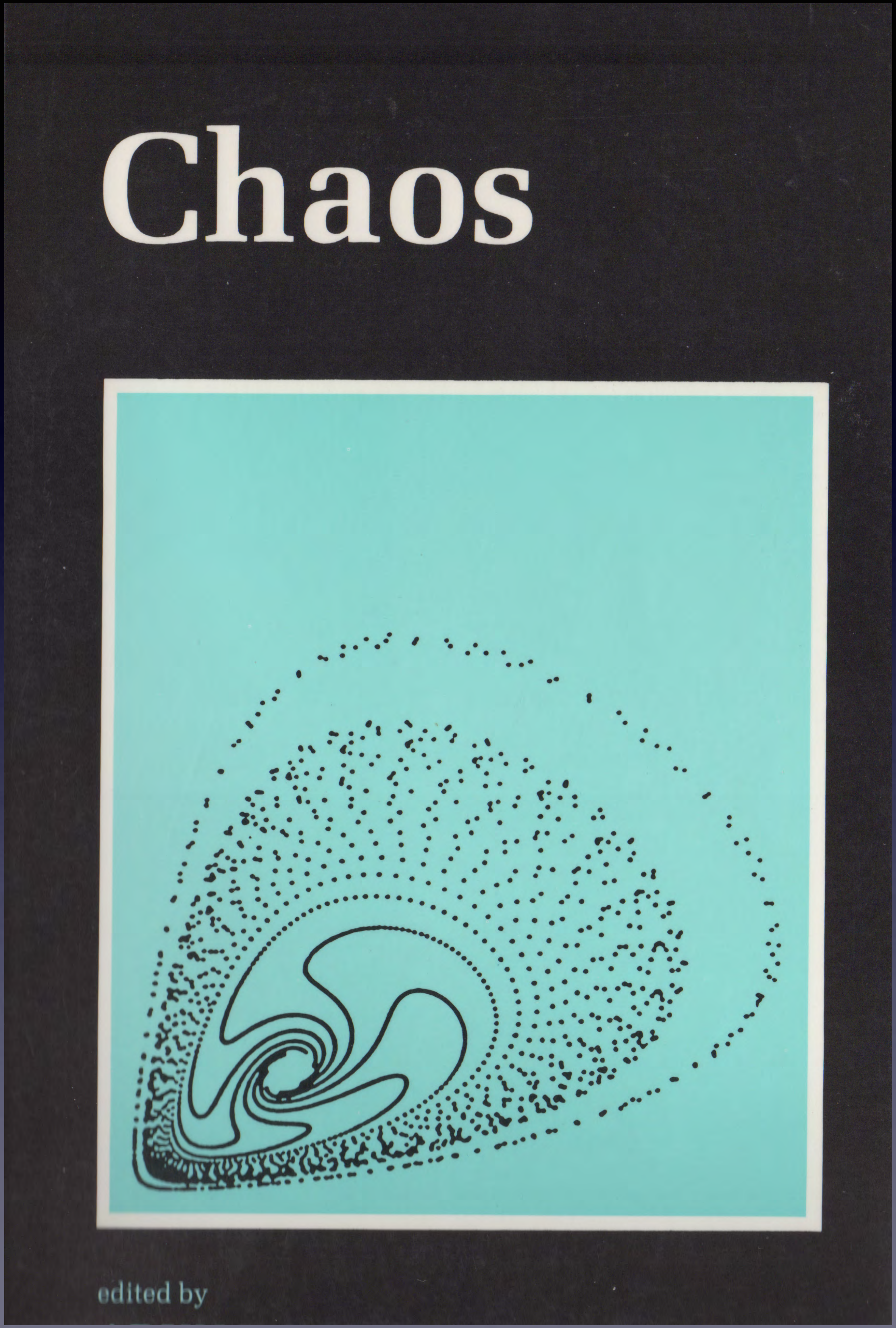
$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= F(x_n, y_n) \end{aligned}$$

Logistic Delay Scheme

$$\begin{aligned} u &= y \\ v &= ay(1-x) \end{aligned}$$

Typically a = 3.5

Logistic
Delay
Chaos



Application 2
Business Cycles
Kaldor

Following H-W Lorenz, 1993

Kaldor Model, 1940

As difference equations,
Y = income, K = capital stock

$$
\begin{aligned}
Y^+ &= Y + \Delta Y \\
K^+ &= K + \Delta K
\end{aligned}
$$

Business Cycles
Kaldor Model, 1940

General Kaldor model,
I = investment, S = savings

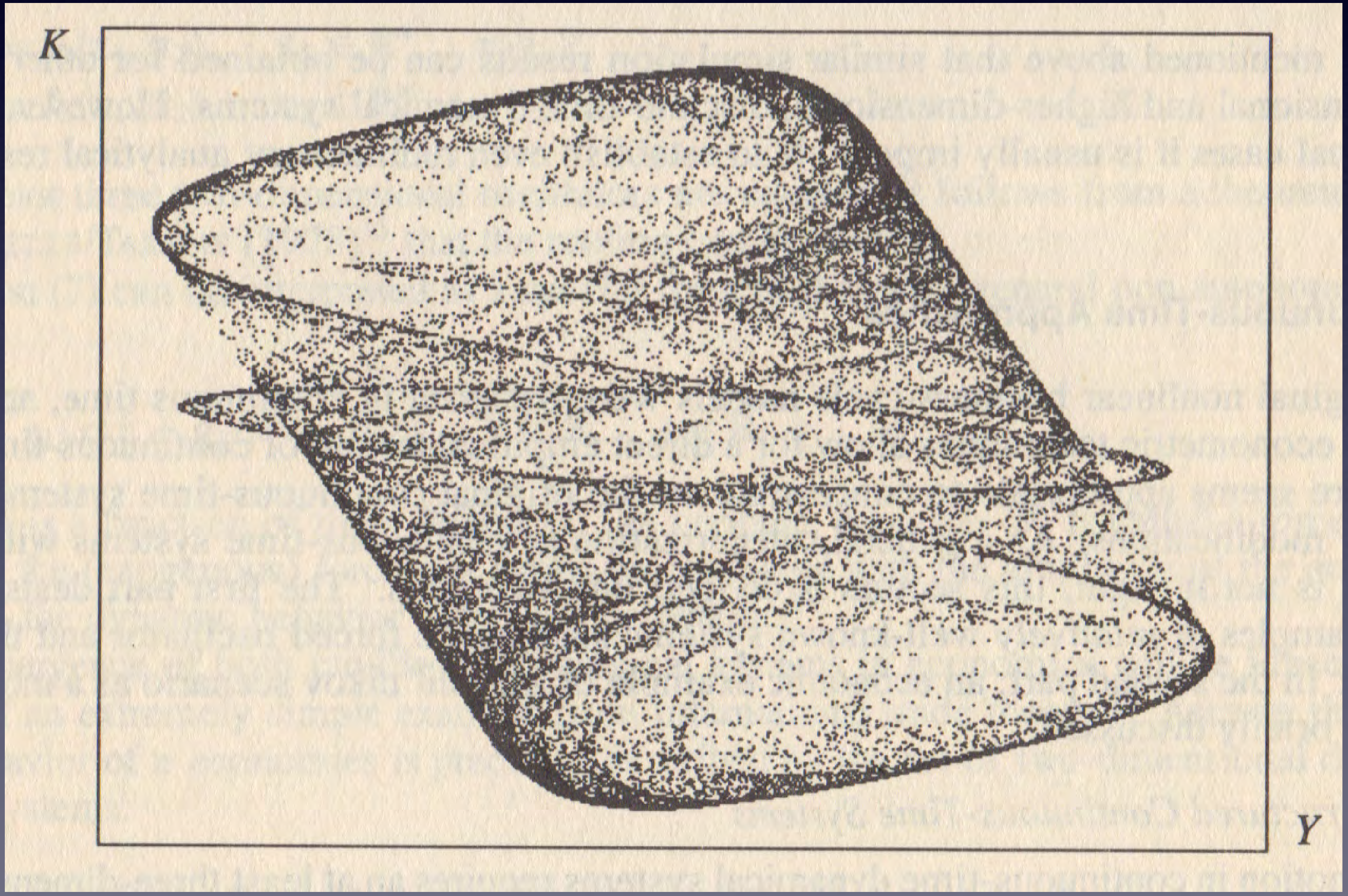
$$\begin{aligned} \Delta Y &= \alpha(I - S) \\ \Delta K &= I - \delta K \end{aligned}$$

Business Cycles
Kaldor Model, 1940

Herrmann form, 1985

$$\begin{aligned}
\Delta Y &= \alpha(\beta(kY - K) + \delta K + C(Y) - Y) \\
\Delta K &= \beta(kY - K)
\end{aligned}$$

Business Chaos
Herrmann Model, 1985



Application 3
Business Cycles
von Neumann, 1932

Following Goodwin, 1990

von Neumann, 1932

As 2D map,
v = production, u = wages = capital stock

$$
\begin{aligned}
u^+ &= bu^2v \\
v^+ &= av^2(1-v) - uv
\end{aligned}
$$

Bifurcation Sequence

As a increases from 2.0 to 4.0,
an attractive point expands
to a chaotic attractor

a = 3.6

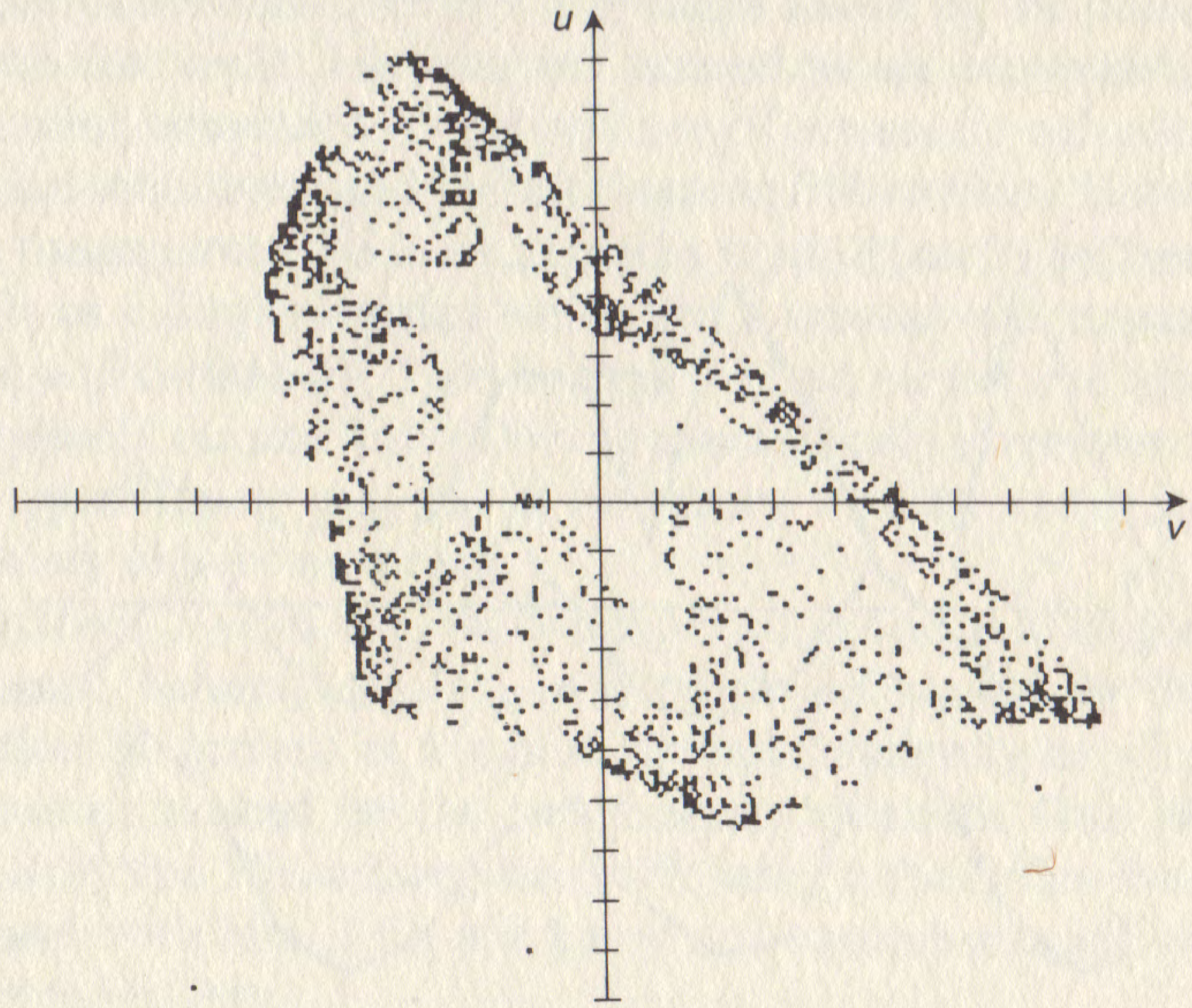


FIG. 3.10

Application 4
Julia sets
Mandelbrot, 1975

Consider a complex variable,

$z = u + vi$,

and a complex constant,

$c = a + bi$.

Then the map $z \longrightarrow z^2 + c$ represents a 2D iteration:

$$u_+ = u^2 - v^2 + a$$
$$v_+ = 2uv + b$$

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On to 2D Experiments