Math 145
Chaos Theory

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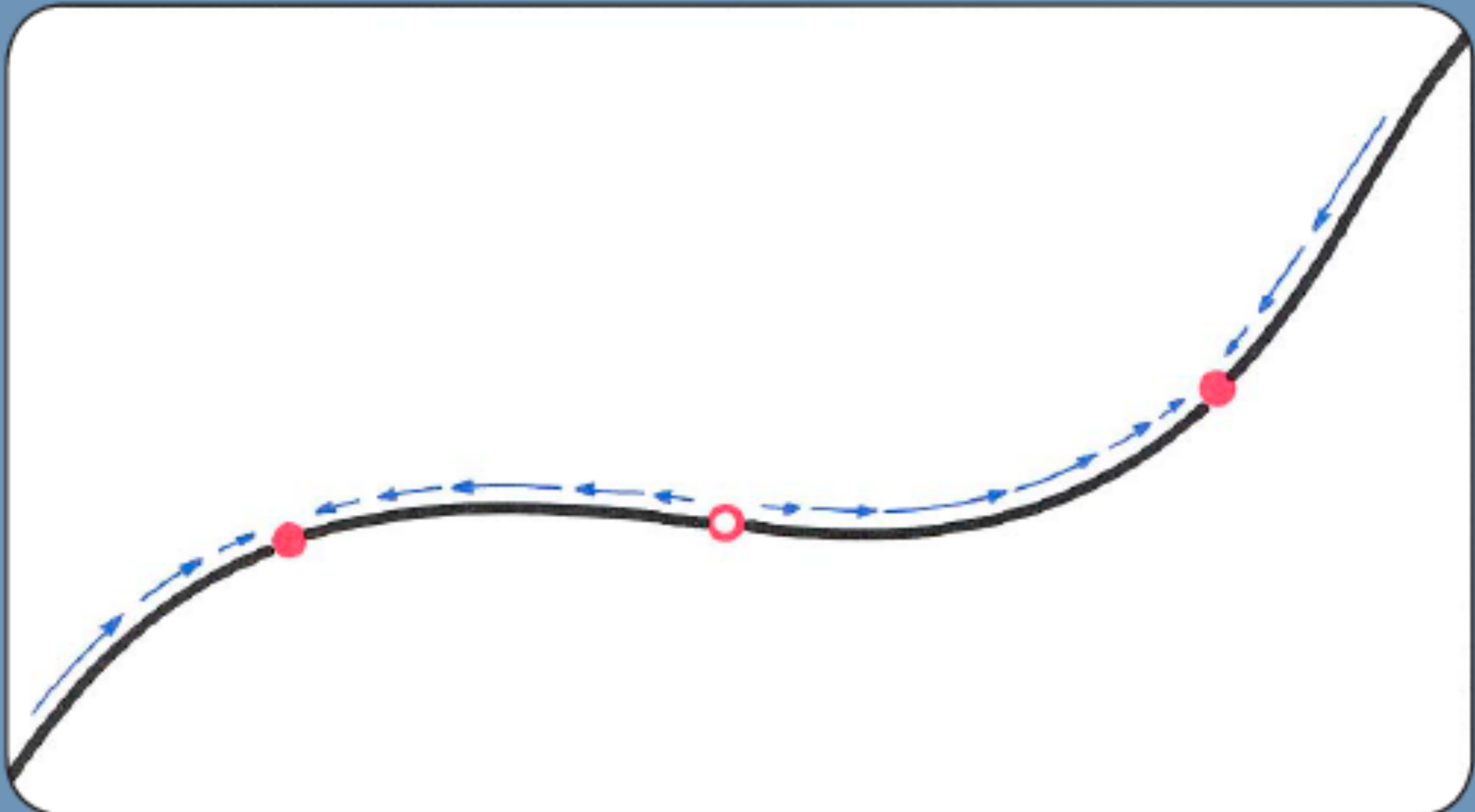
Meeting #5T, May 2

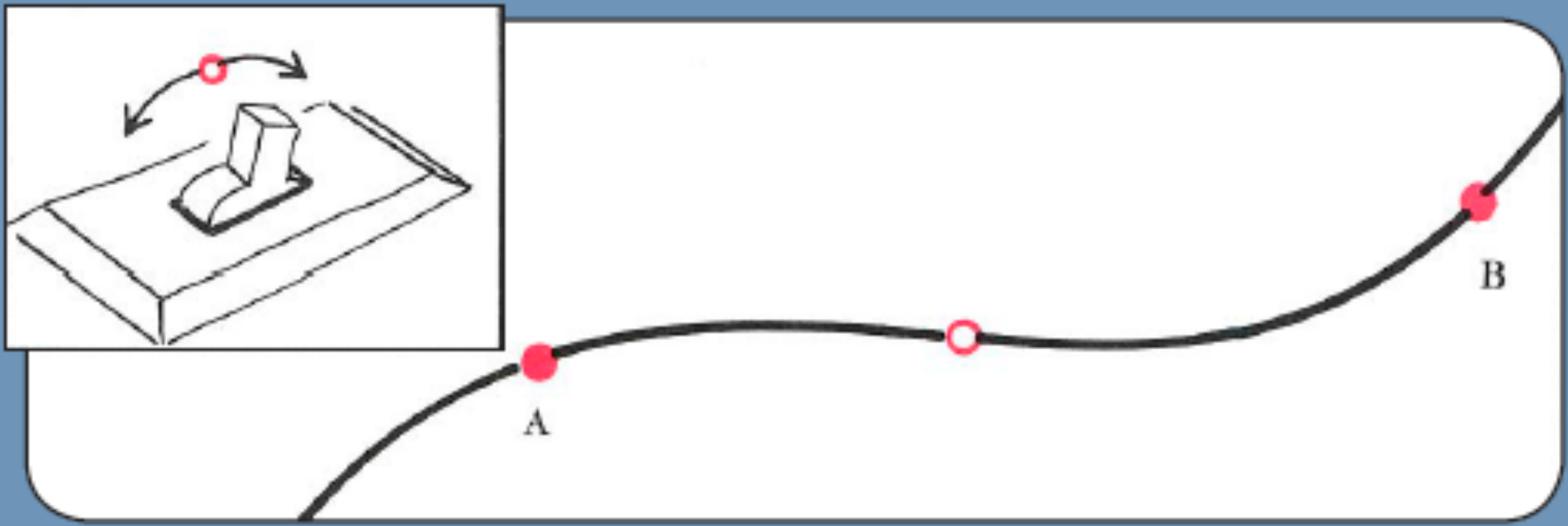
- 3D Flows, continued
  - Dynamics: The Geometry of Behavior
    - Part 3: Global Behavior
      - Chs. 10, 11, and 12
- Demos
  - Ueda
  - Lorenz
  - Rossler

Ch. 10: Phase Portraits

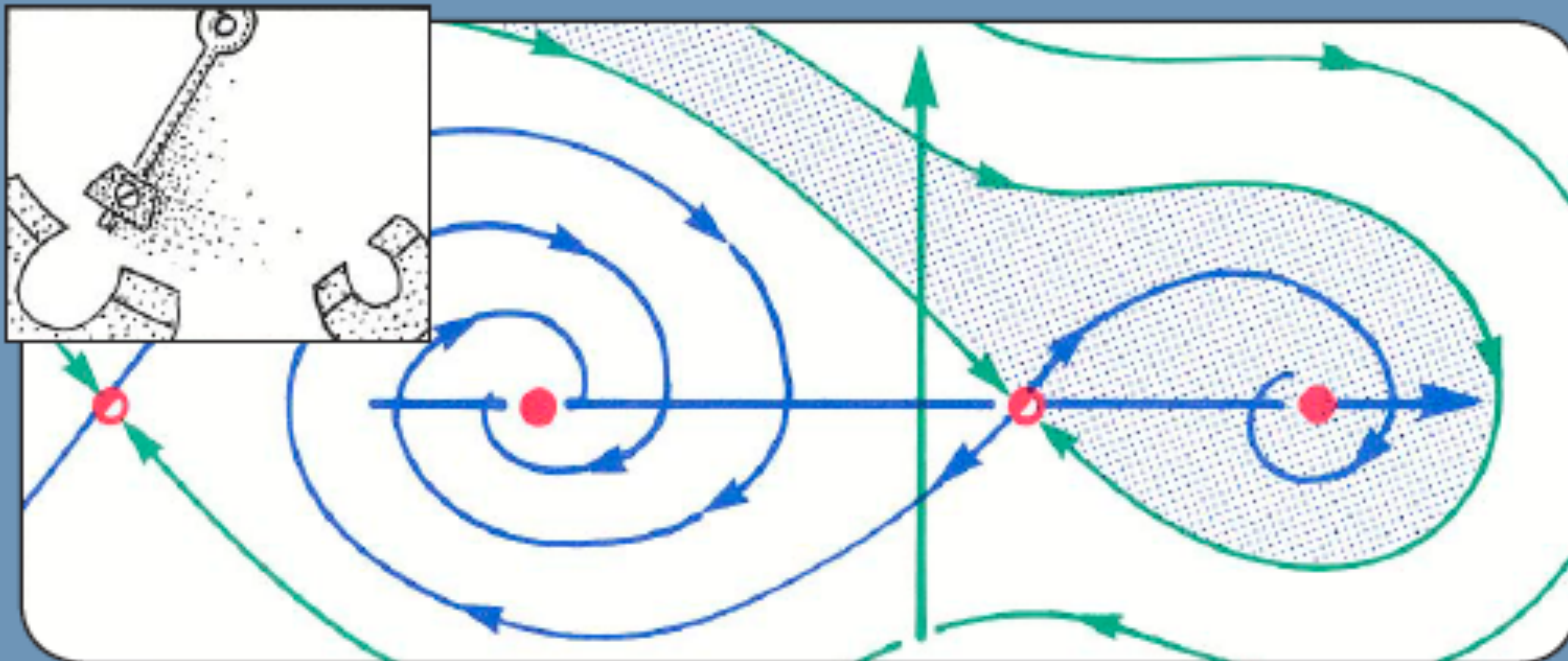
- Sec. 10.1. Multiple attractors
- Sec. 10.2. Separatrices



10.1.1.
In this example, there are *two* attractive points, each in its own basin.
The system is *bistable*, in that two distinct stable equilibria are possible.

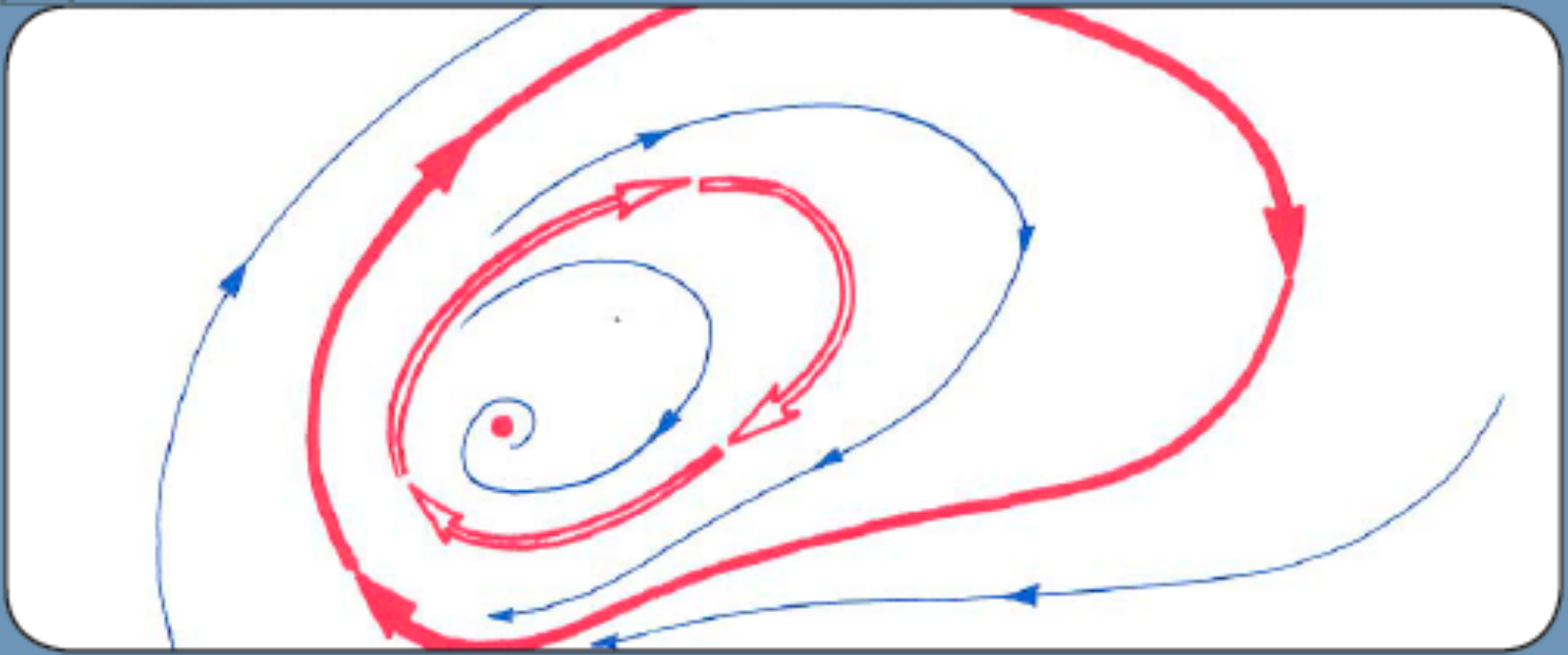


10.1.2.
Notice that in this example, the two basins are separated by the point repellor. Initial points slightly to the left of the repellor tend to attractor A, while those slightly to the right tend to attractor B. This behavior is roughly like a mechanical toggle switch.



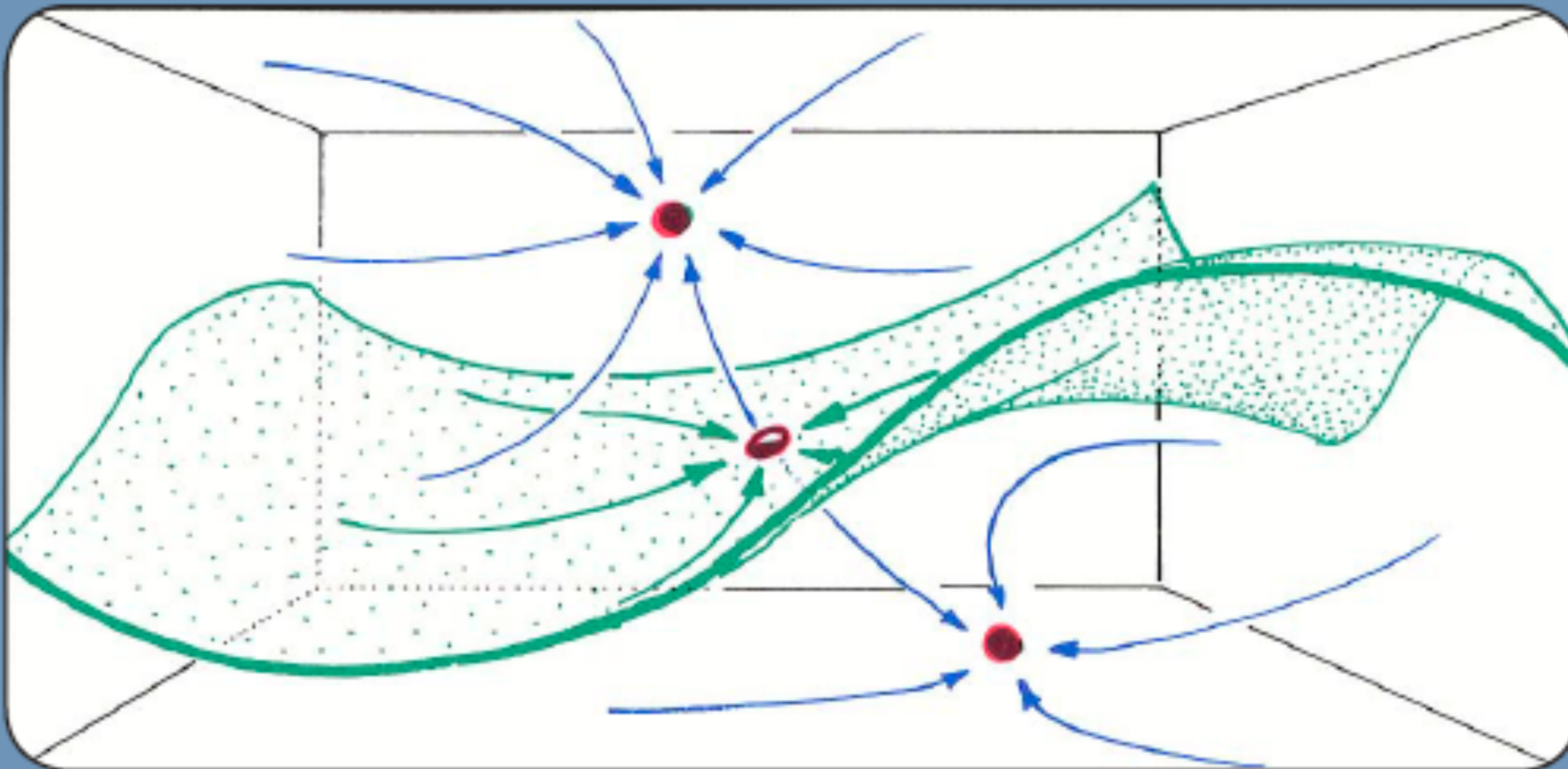
10.1.3.

Remember the pendulum? Here is the magnetic bob from Figure 2.1.22. This is also a bistable system. But the two basins are two-dimensional, so the separatrix between them is a curve. This curve is *repelling*, yet not a *repellor*. In fact, it consists of the *inset* of the saddle point between the point attractors. This saddle point represents an *unstable equilibrium* of the bob, balanced between the forces of the two magnets. And its inset represents those improbable initial states which tend to this unstable equilibrium and balance there.



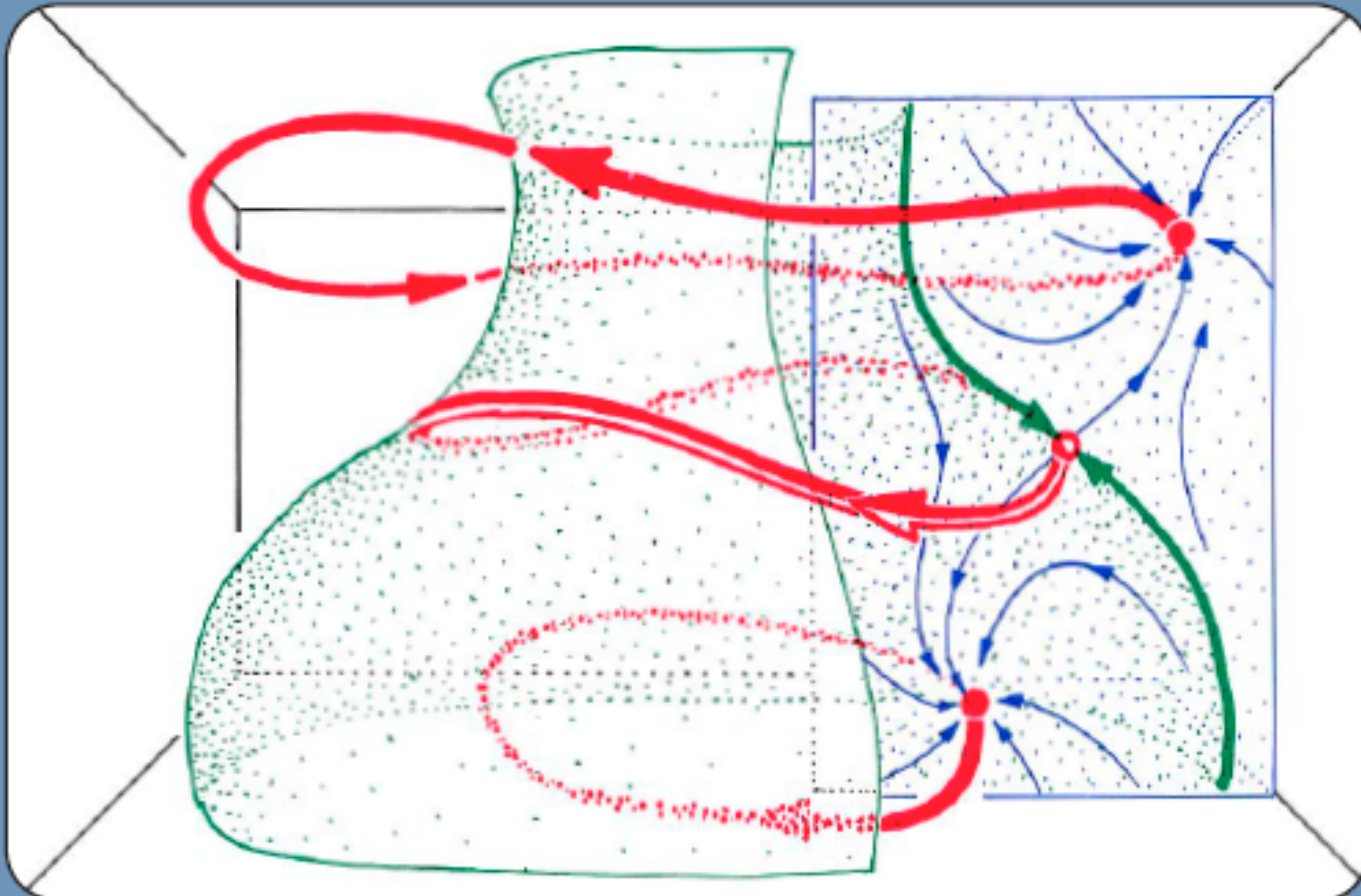
10.1.4.

Recall this portrait, from Figure 1.5.8. Here, the periodic repellor bounds the two-dimensional basin of an attractive point. It is a separatrix.

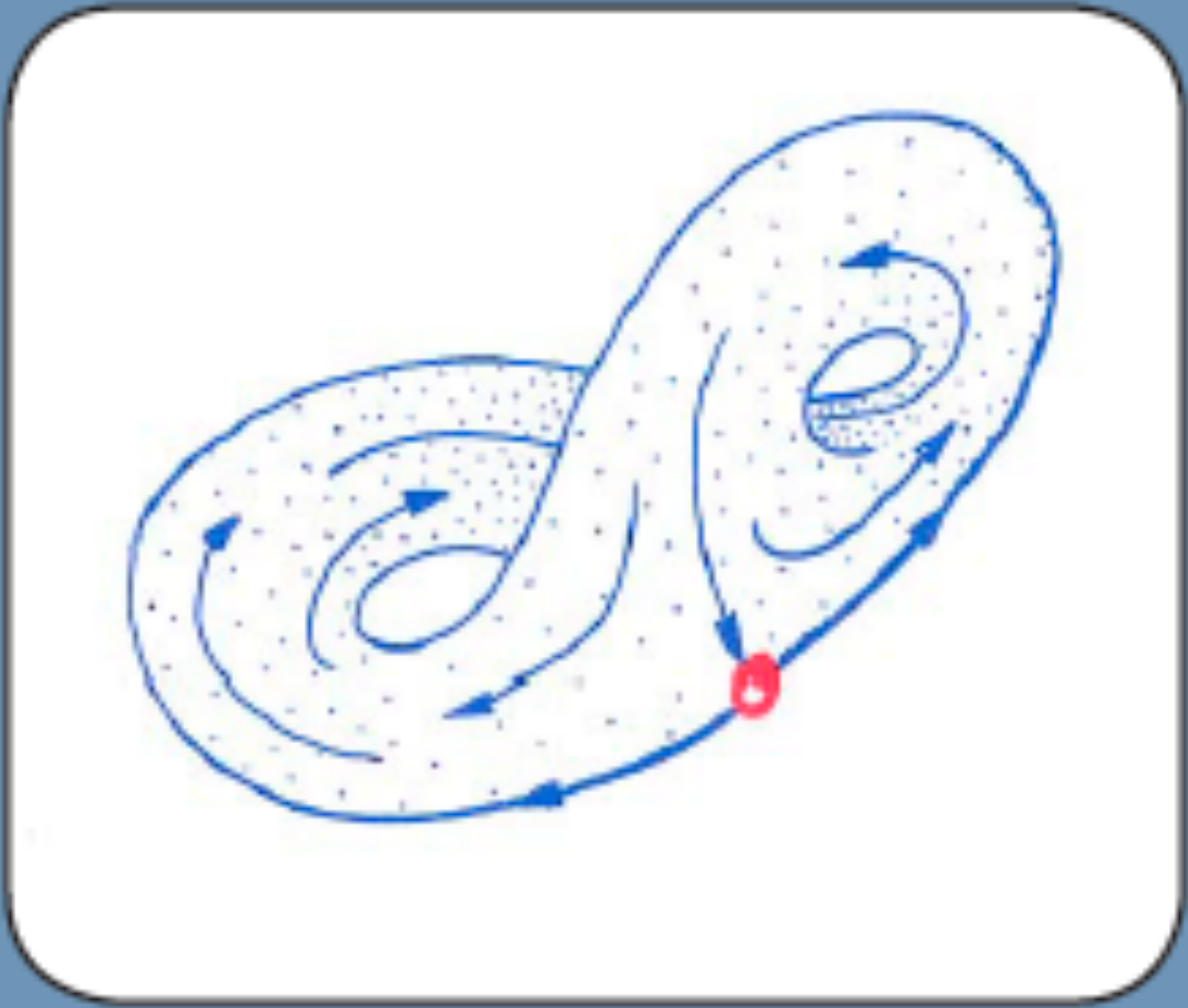


10.1.5.

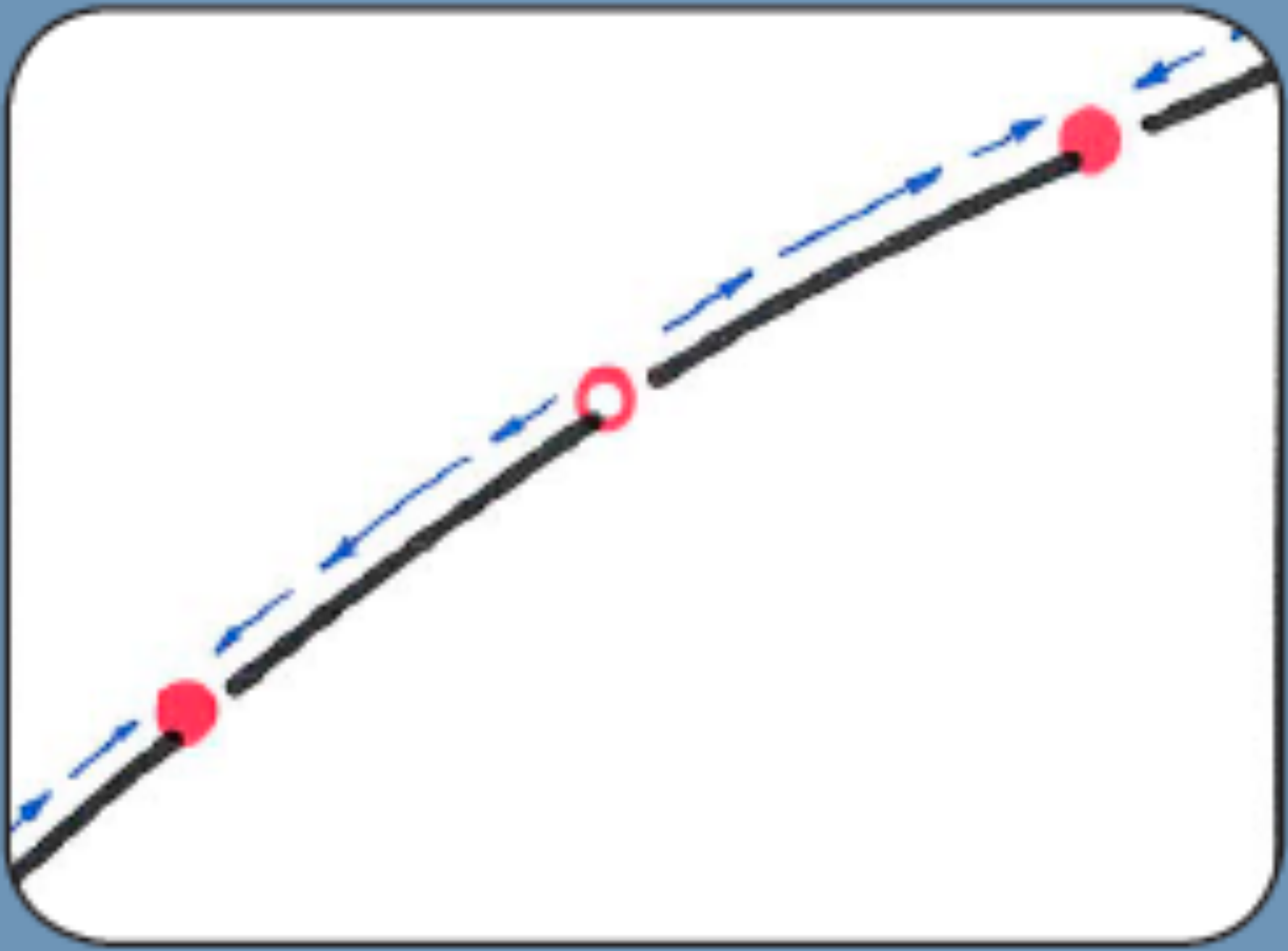
In this portrait of a simple bistable system in 3D, there are again two attractors. Both are rest points. Their basins are three-dimensional, and are bounded by a surface. This surface, the separatrix in this example, is the inset of a saddle point of index 1.



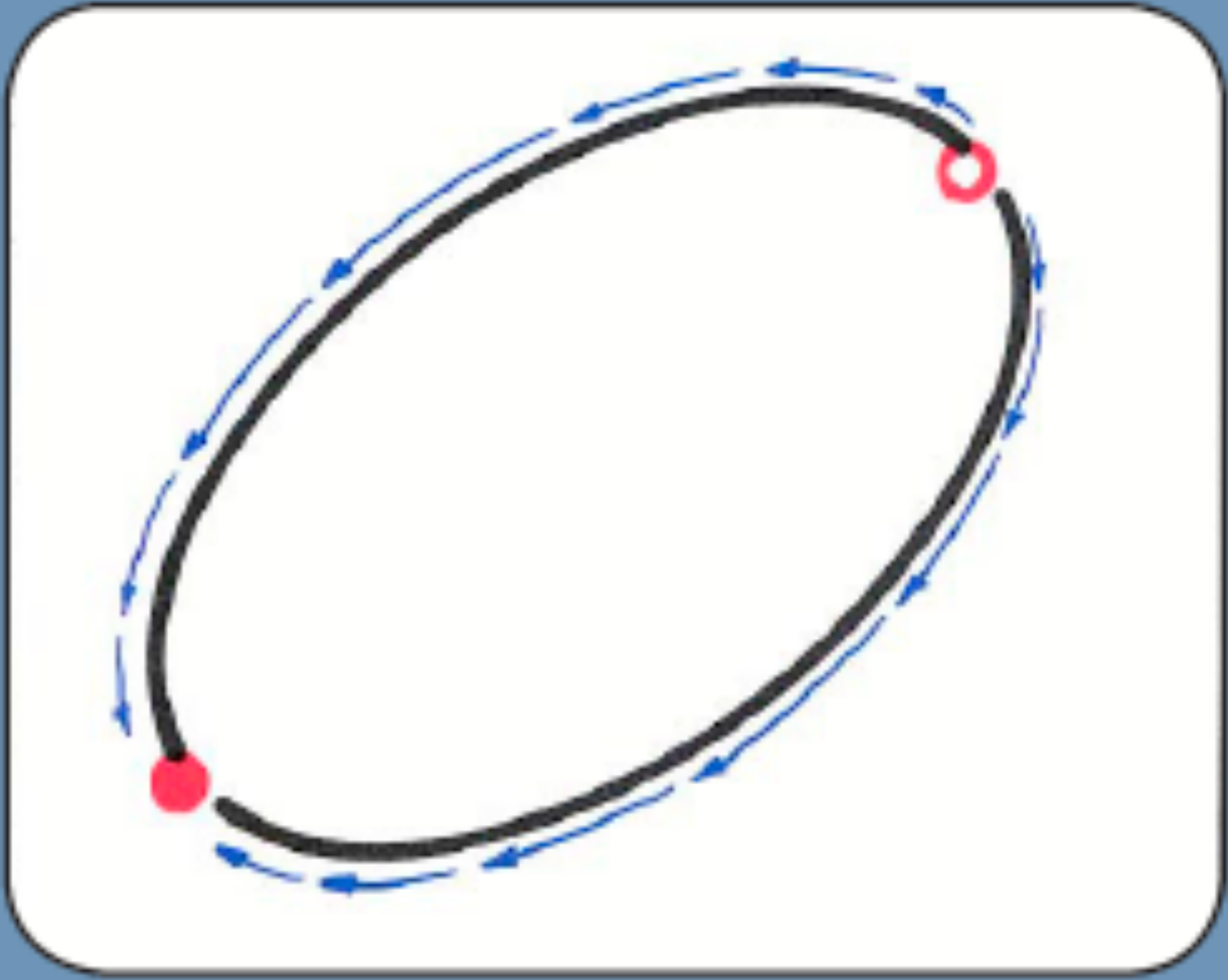
10.1.6.
Here, for example, is a bistable system with two periodic attractors. Their basins are bounded by a cylindrical surface, the separatrix. It is the inset of a periodic saddle.



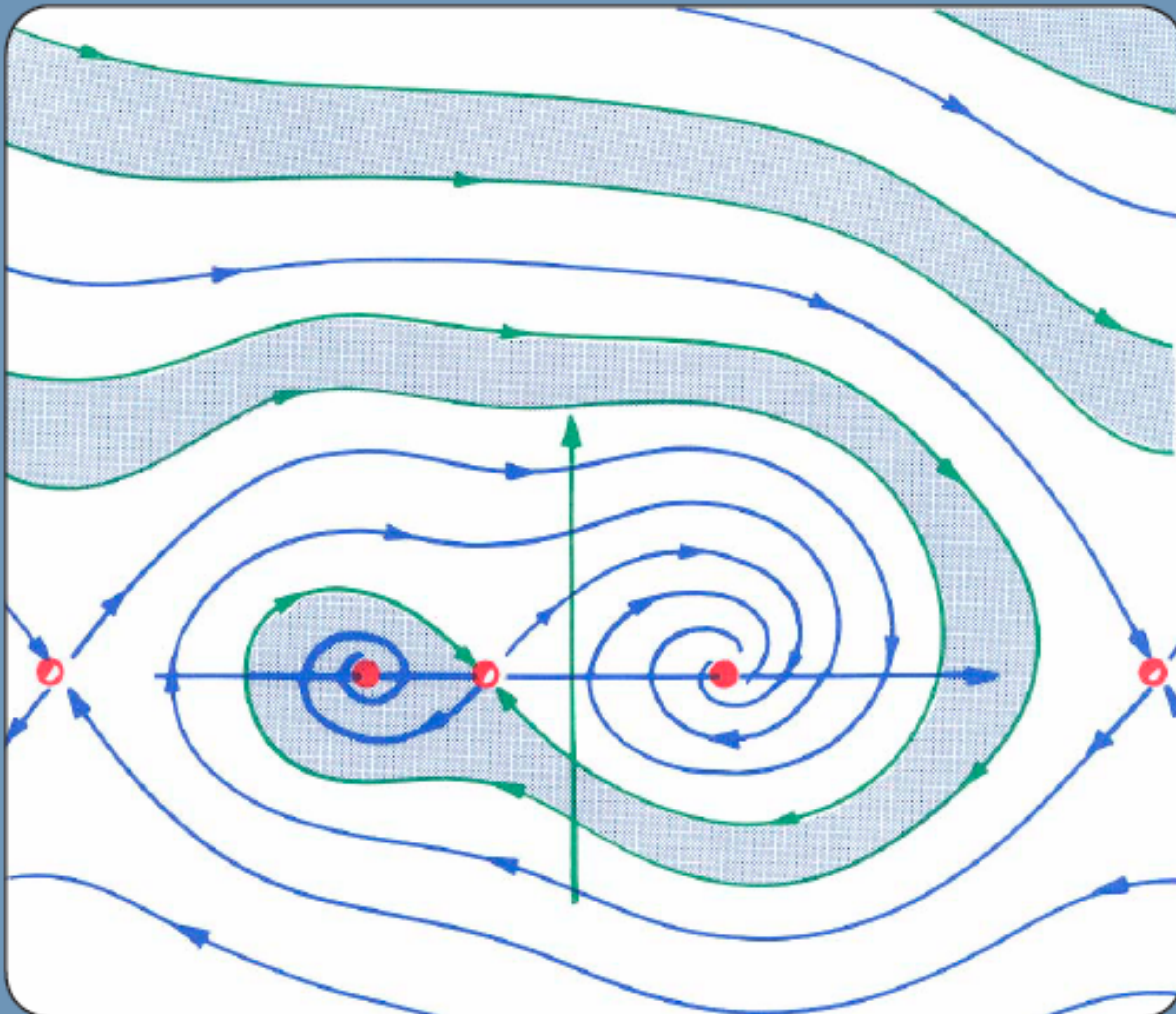
10.1.7.
This is one of the most famous chaotic attractors.



10.2.1.
As we have seen in the preceding section, point repellors may separate basins in one-dimensional state spaces.



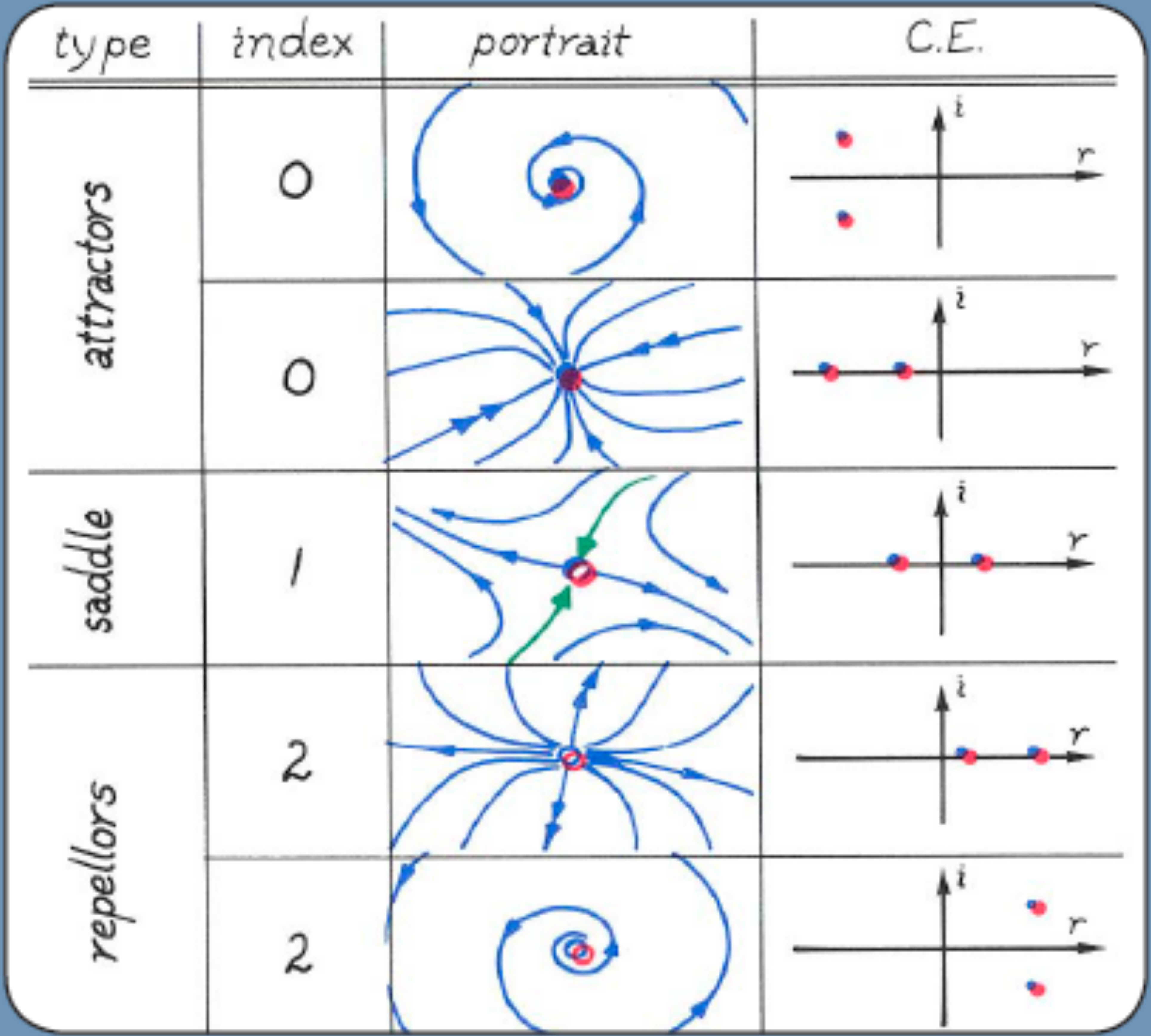
10.2.2.
But if we connect the ends of the curve, we have a unistable system! There is only one basin. The separatrix (a single point repellor) bounds it, but does not separate anything. It is a *virtual separatrix.*



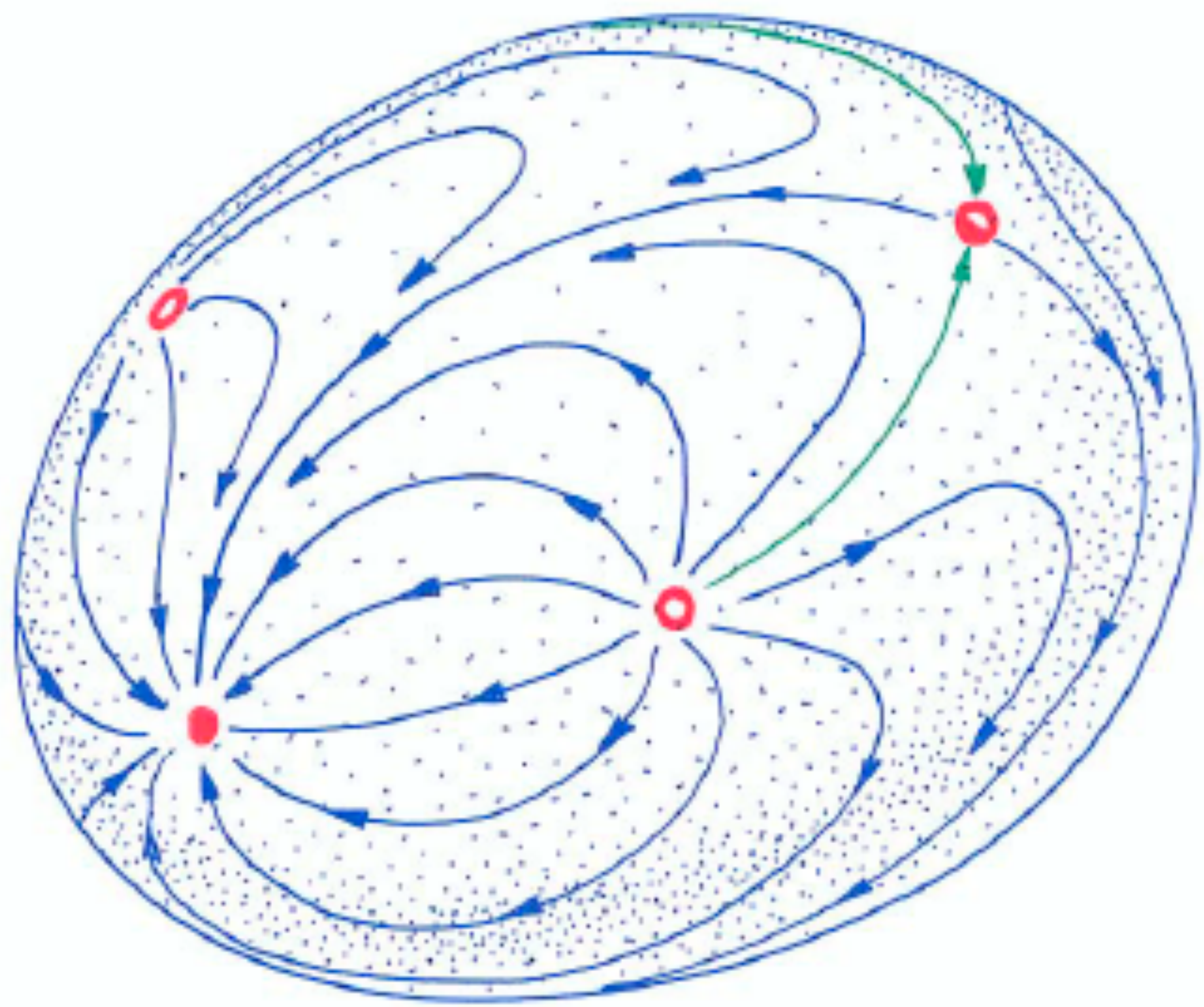
10.2.3.
The saddle point at the top of the swing represents the watershed between falling to the right and falling to the left. Its inset consists of those improbable initial states that tend to balance at the top of the swing. As shown here, the initial states close to this inset, to either side, belong to the same (unshaded) basin. Thus, this inset curve is a *virtual separatrix*.

Ch. 11: Generic Properties

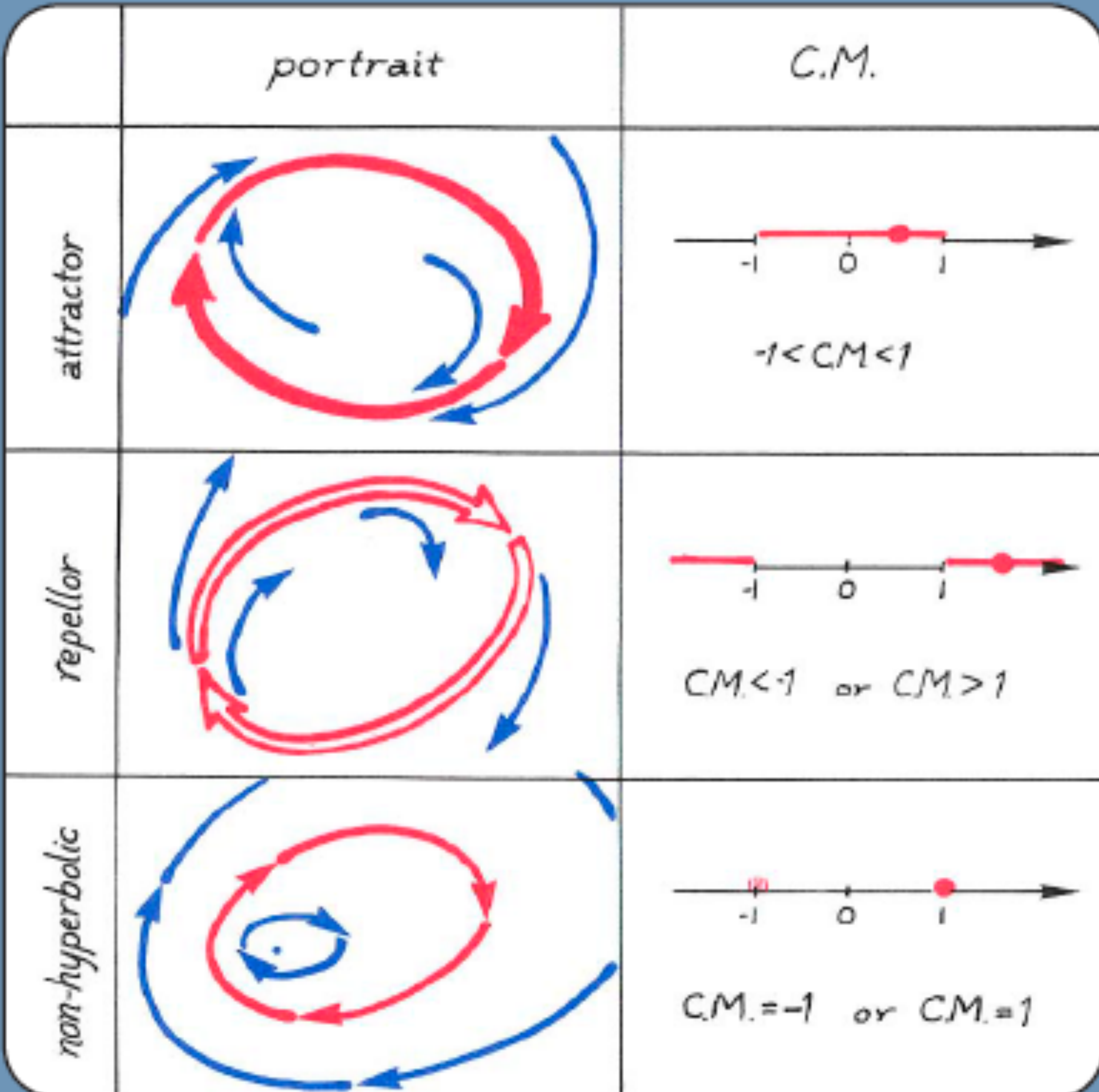
- Sec. 11.1. G1 for critical points
- Sec. 11.2. G2 for closed orbits
- Sec. 11.3. G3 for saddle connections
- Sec. 11.4. G4 and F



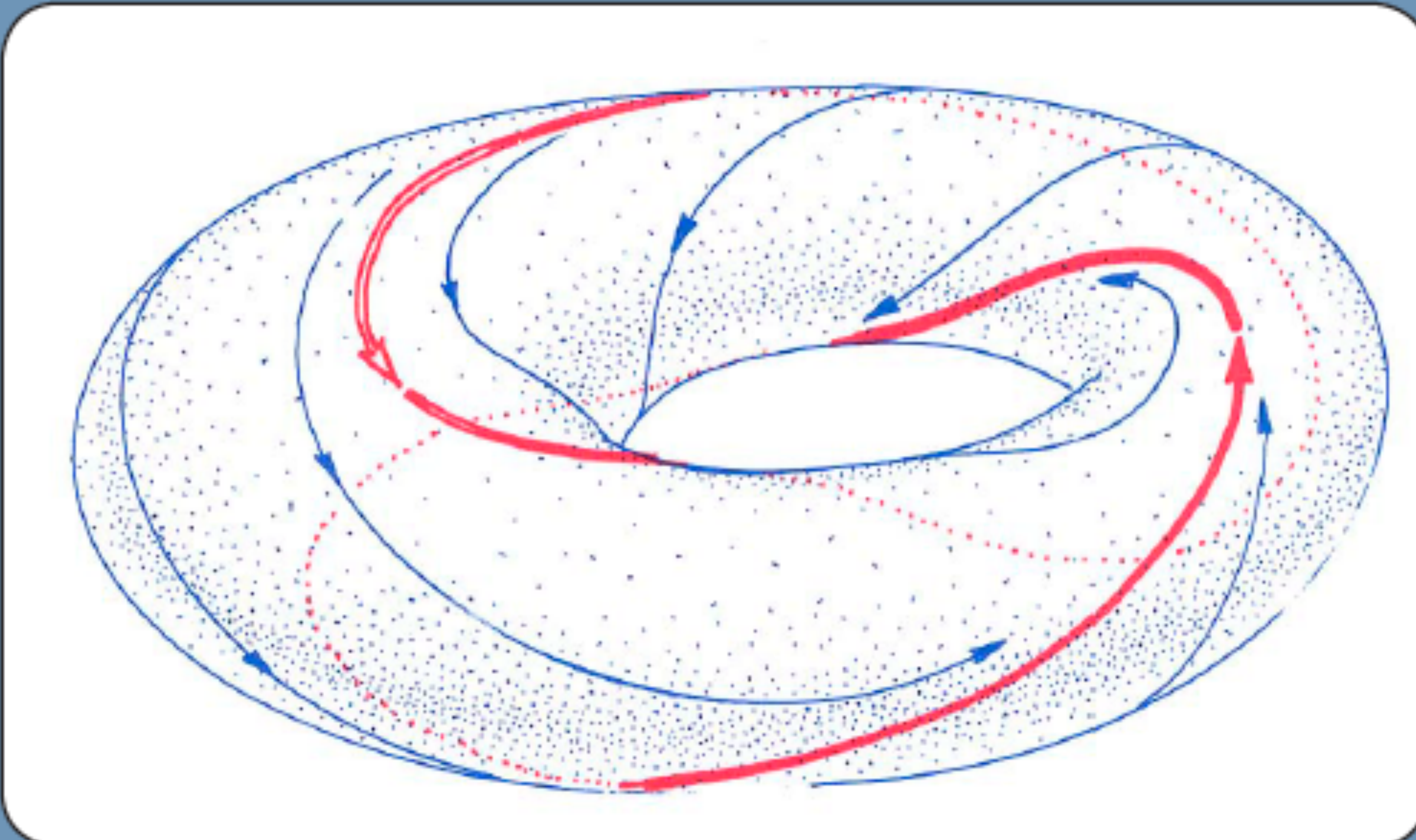
11.1.1.
This is Figure 6.4.8, showing the five elementary critical points in 2D. There are seven hyperbolic critical points, namely, these five together with the radial attractor and the radial repellor.



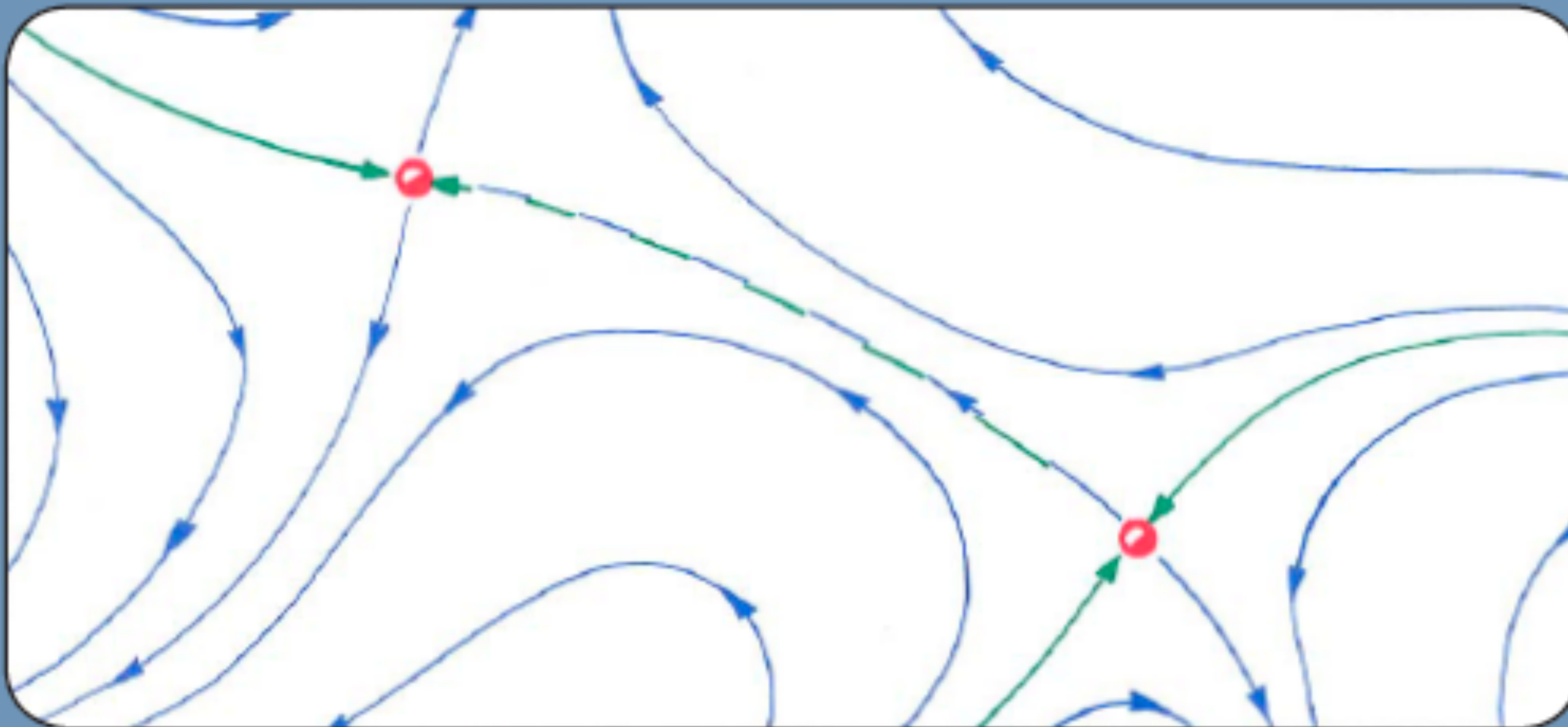
11.1.5.
A dynamical system has property G1 if all of its critical points are elementary. In this example, each and every critical point is elementary.



11.2.1.
In 2D, a limit cycle has only one characteristic multiplier (CM), which is *real*. These are the only hyperbolic limit cycles in 2D. The absolute value of the CM is smaller than 1 (periodic attractor) or greater than 1 (periodic repellor).

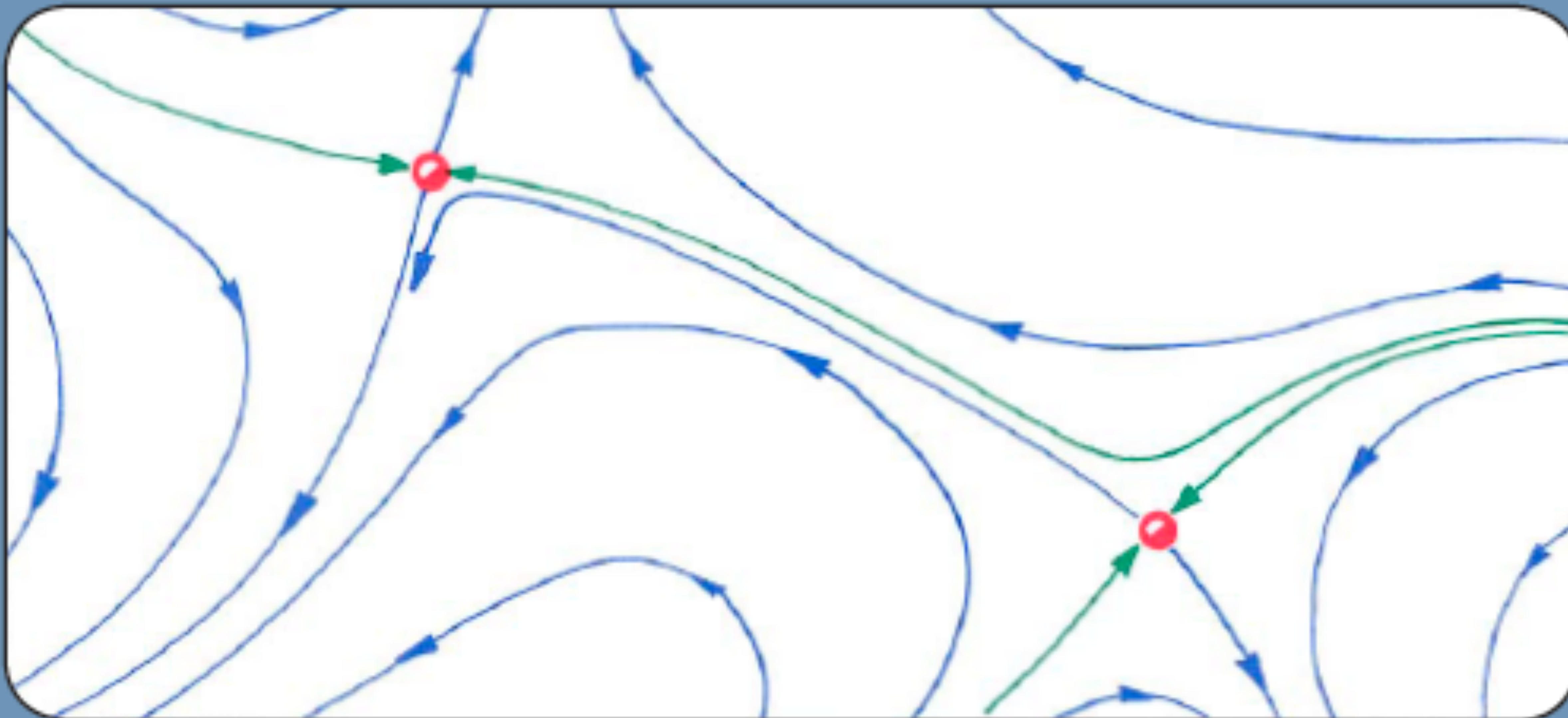


11.2.4.
A dynamical system satisfies property G2 if each and every one of its limit cycles is elementary. In this example on the two-dimensional torus, there are several limit cycles in a braid, and each is elementary.



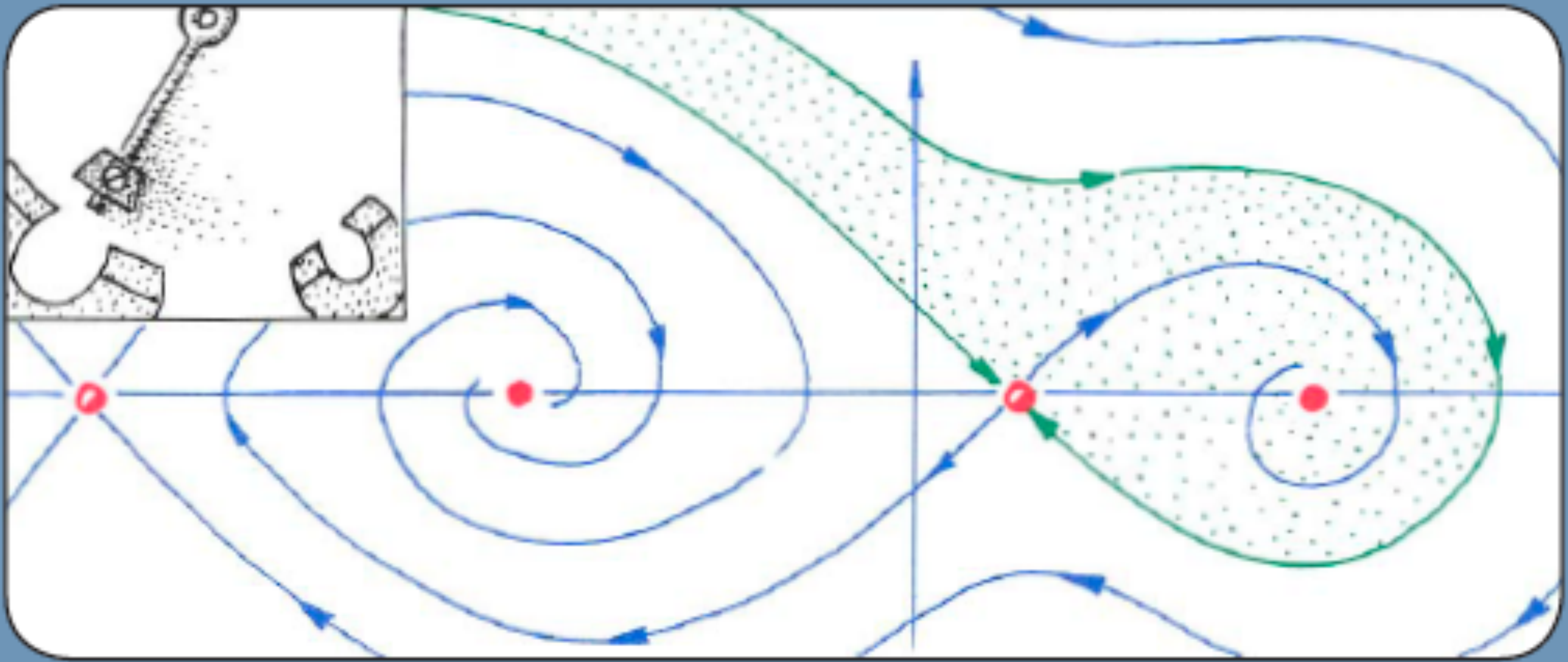
11.3.1.

This is a saddle connection in 2D. The dashed trajectory comprises half of the outset of the hyperbolic saddle point on the left, its donor. Simultaneously, it is half of the inset of the hyperbolic saddle point on the right, its receptor. As this system contains a saddle connection, it does not satisfy G3.



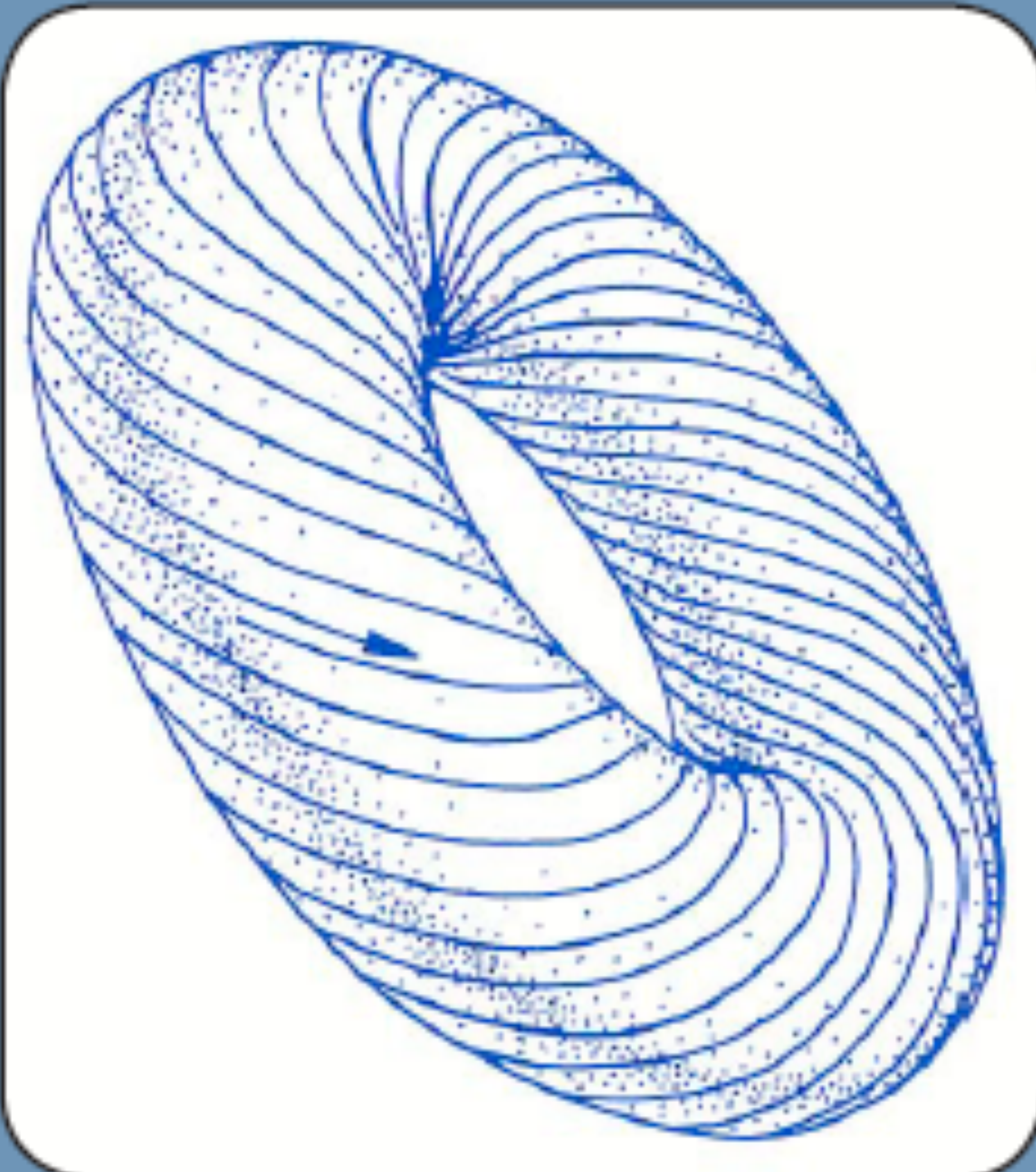
11.3.2.

This system has no saddle connection. The outset of the saddle points on the left consists of two trajectories, which go to attractors (not shown). The inset of the saddle point on the right consists of two trajectories, which come from repellors (not shown). One of the trajectories leaving the left saddle narrowly misses one of the trajectories approaching the saddle on the right. This portrait is obtained from the preceding one by a slight perturbation.

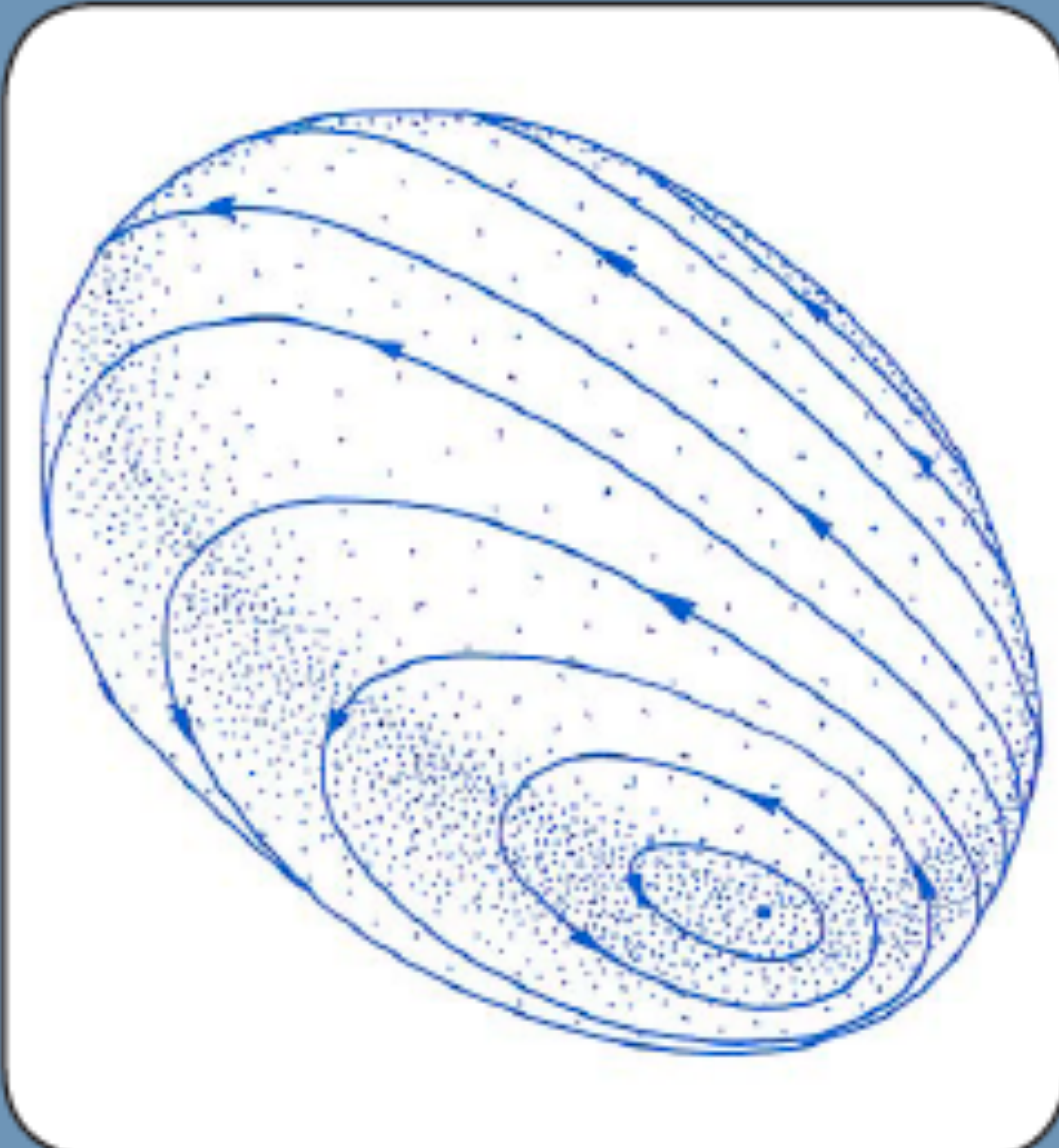


11.3.3.

The magnetic pendulum is a global system satisfying property G3. All four saddle outset trajectories successfuly avoid all four saddle inset trajectories. (See Figure 2.1.22.)



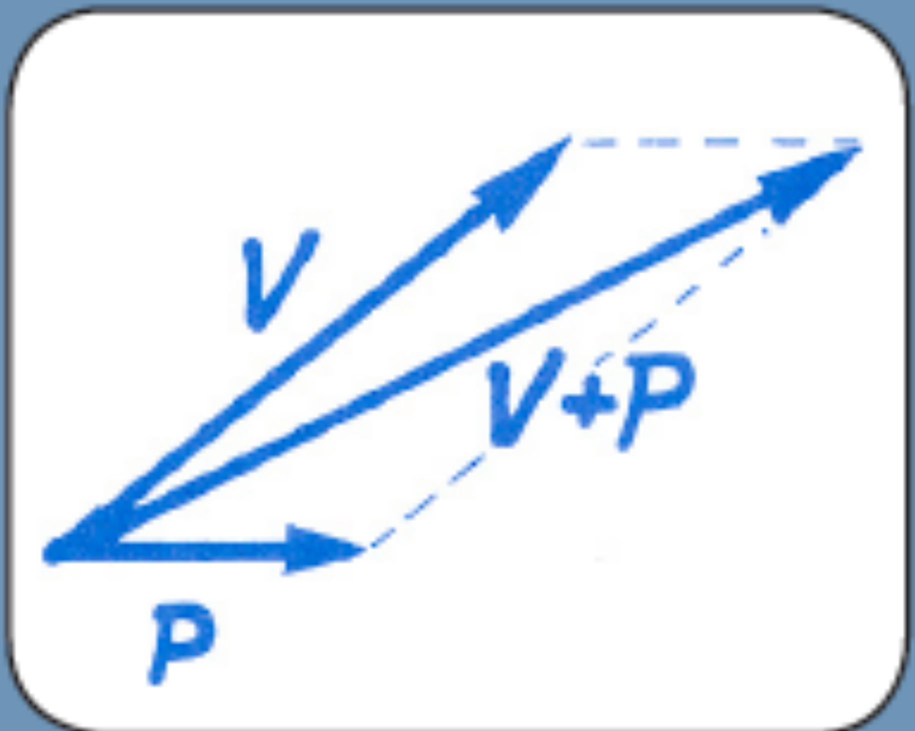
11.4.1.
Recall this solenoid, from Figures 1.4.11, 4.4.21, and 4.4.22. All trajectories on this torus are recurrent in the sense that their omega (and alpha) limit sets are the entire torus. Thus, if we choose any little disk in the torus, each trajectory recurs, or passes through that disk again and again in its future (and past). We call such a system a *limit torus*.



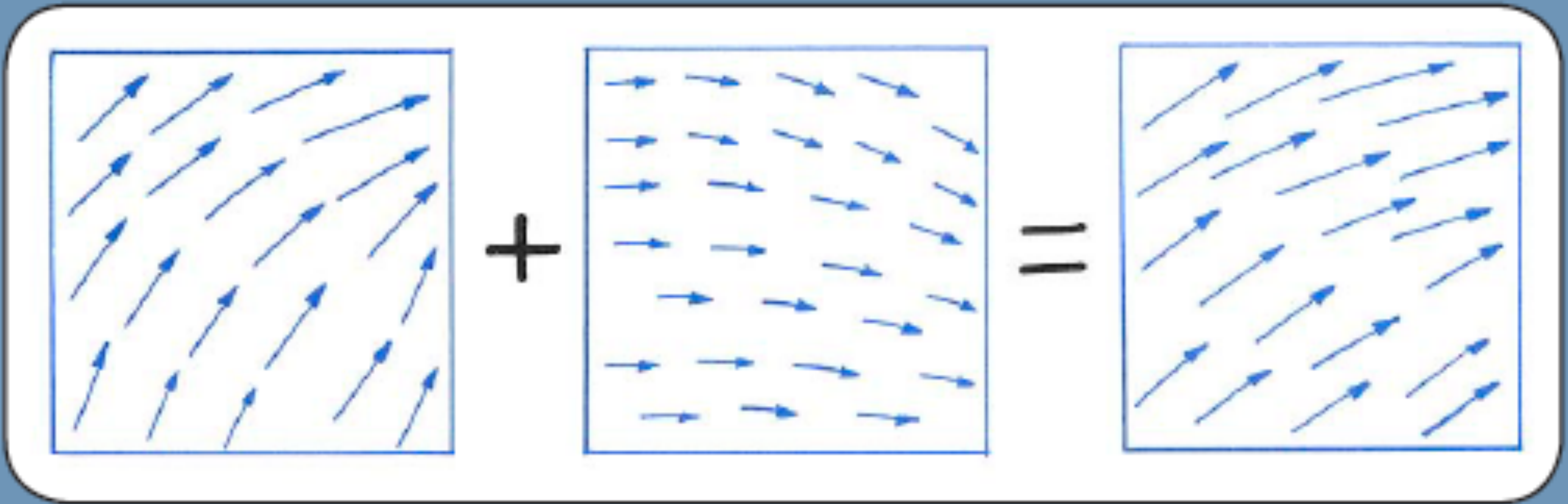
11.4.2.
Here is a 2D system violating property F. It has a center: an infinite number of limit cycles, arranged as concentric cycles around a limit point. See Figures 2.1.18, 2.2.3, and 2.2.5 for examples.

Ch. 12: Structural Stability

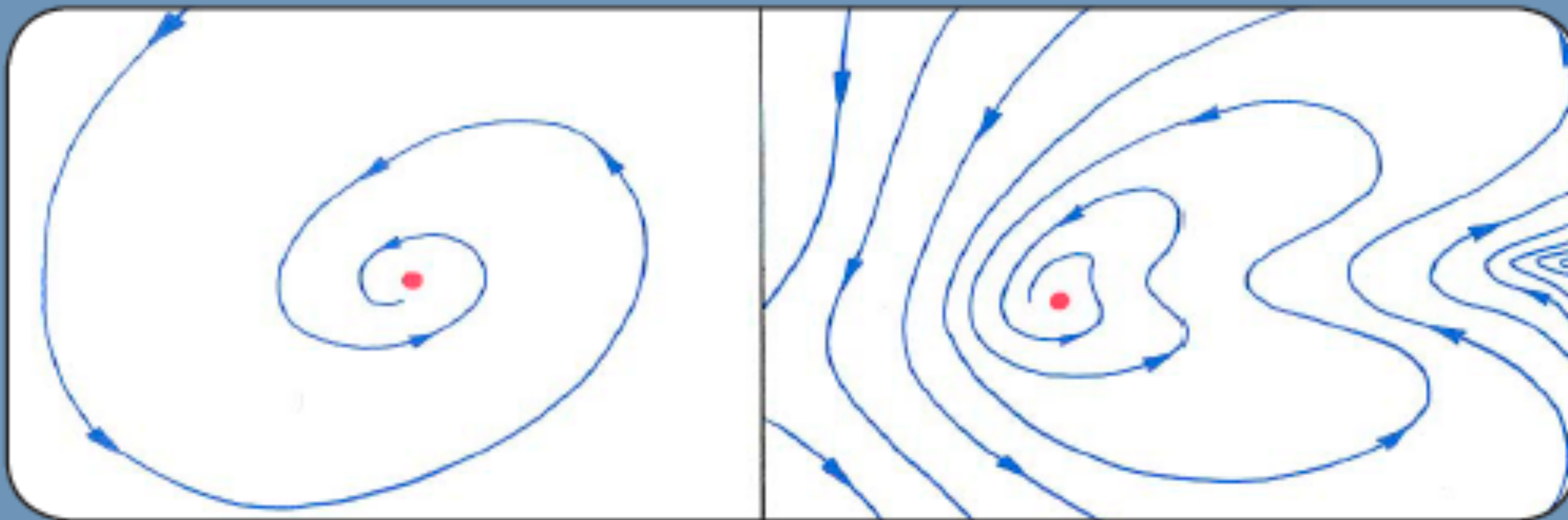
- Sec. 12.1. Stability concepts
- Sec. 12.2. Peixoto’s theorem



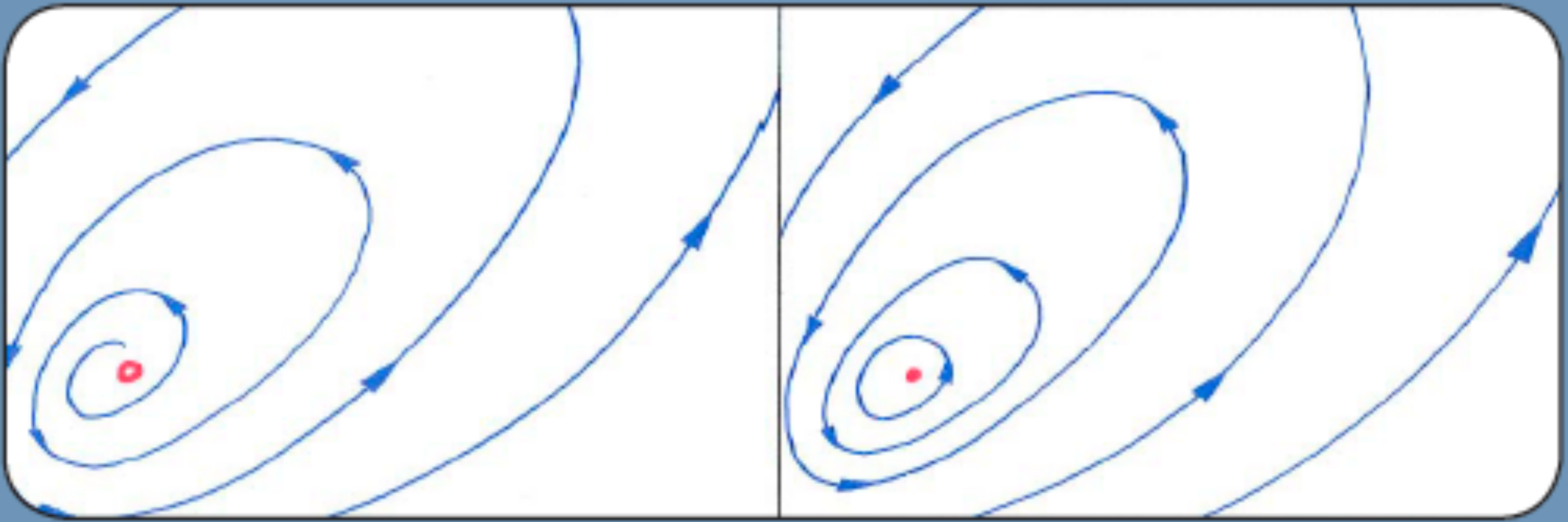
12.1.1.
The criteria for structural stability rely upon two supplementary notions: perturbation and topological equivalence. A *perturbation* of a vectorfield means the addition to it of a relatively small vectorfield, frequently unspecified. Here we show the effect of a perturbation, at a single point in the state space.



12.1.2.
Here we show the effect of a global perturbation. The perturbation is itself a vectorfield, as shown here. The effect of adding this perturbing vectorfield to the original one (on the left) is to modify it at every point in the state space.

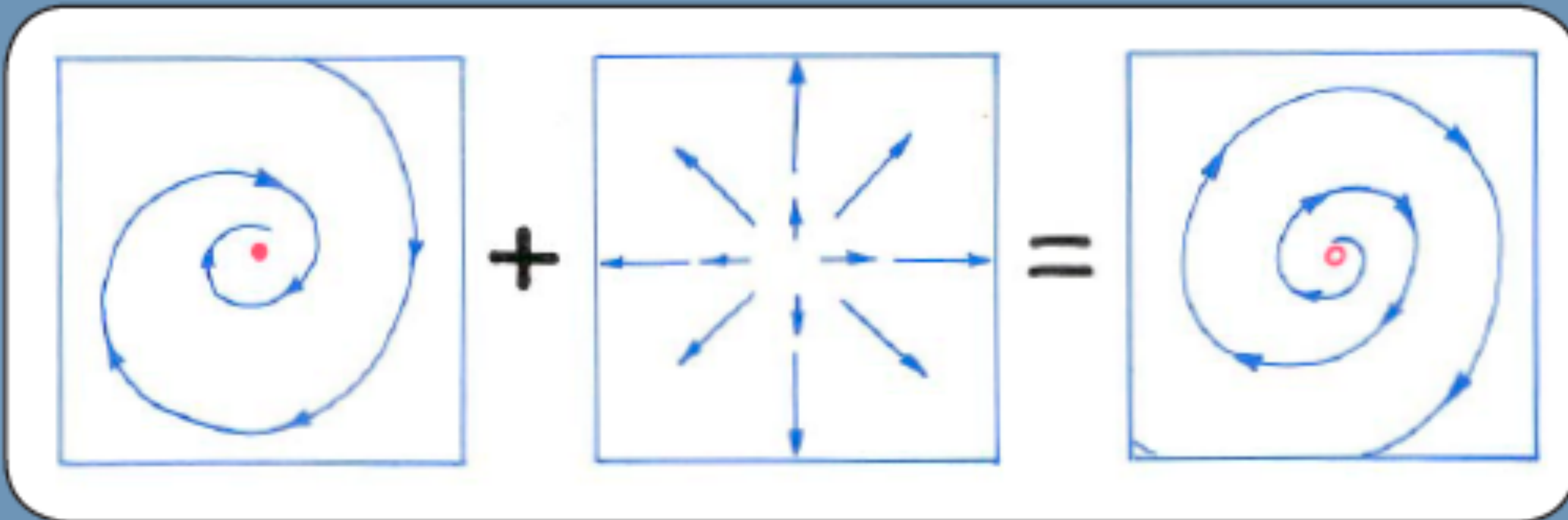


12.1.3.
These two point attractors are *topologically equivalent*. A homeomorphism can deform one into the other, preserving the integral curves.



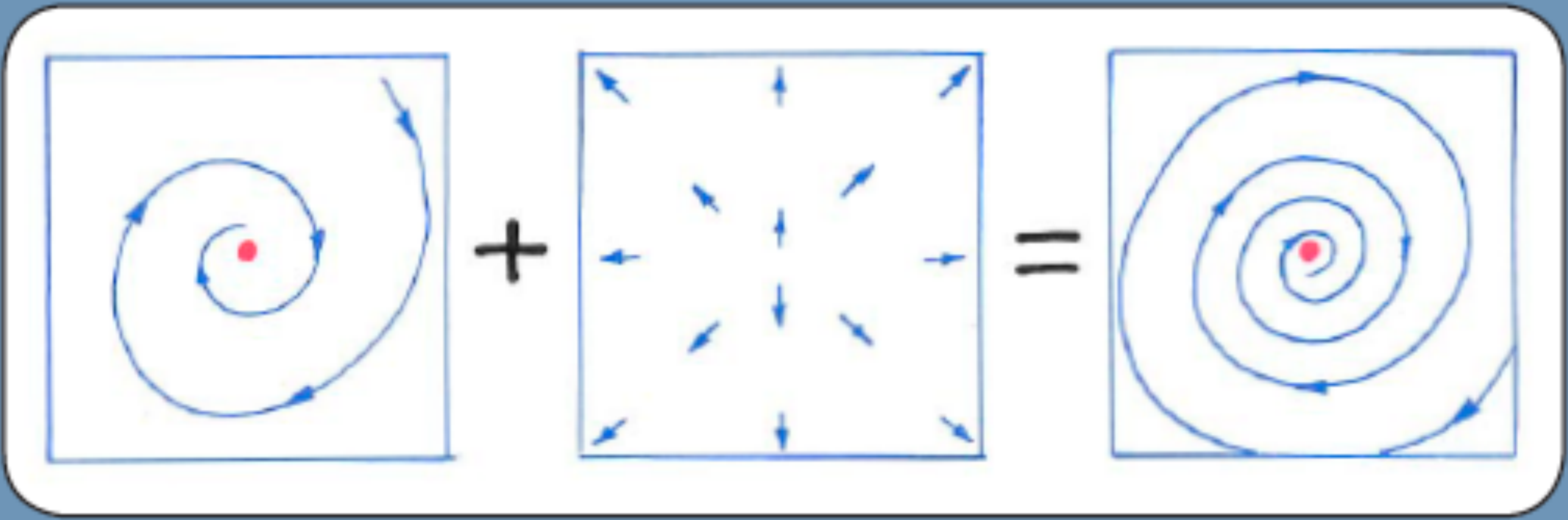
12.1.4.

But the point repellor on the left is *not topologically equivalent* to the center on the right. A homeomorphism cannot map a spiral onto a circle.

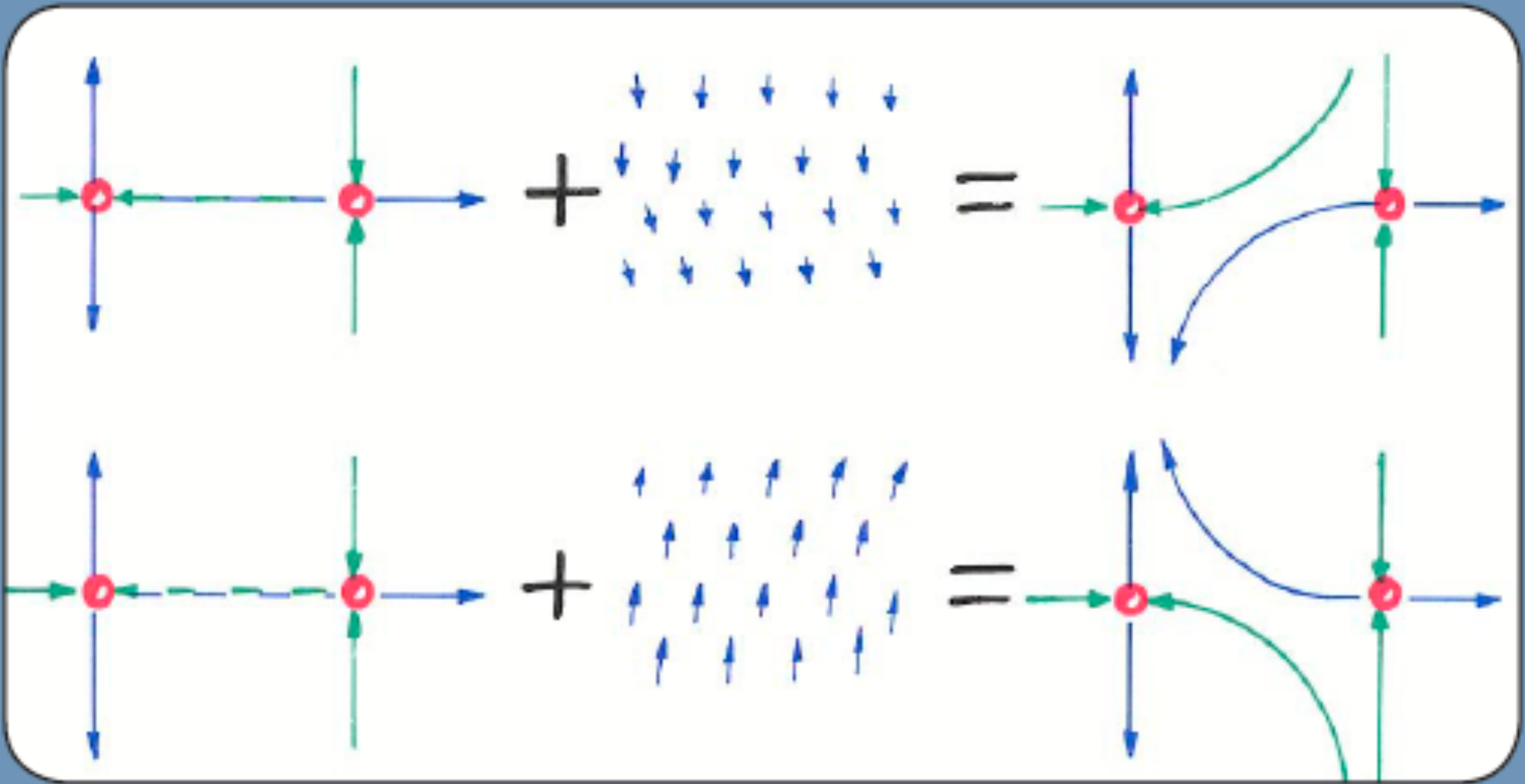


12.1.5.

Imagine a system with a spiral attractor which attracts *very weakly*. By adding a medium-sized perturbation pointing outward, we might be able to change it into a spiral repellor.



12.1.6.
But adding a delta perturbation pointing outward (sufficiently feeble) may make our attractor weaker, but it still attracts. It is topologically (in fact, epsilon) equivalent to the original system. This is an example of a *structurally stable system.*



12.1.10.

Here is another important example. Consider a system with a saddle connection, as in Figure 11.3.1. Adding a delta perturbation pointing downward (or upward), we destroy the saddle connection. The resulting phase portrait is not topologically equivalent. These two examples illustrate all basic types of structural instability in 2D.

Peixoto's theorem: among all smooth dynamical systems on a compact, orientable surface,

A. properties G1, G2, G3, G4 and F are generic,

B. property S is equivalent to these properties (A), and

C. property S is generic.

2D ONLY !!!

Math 145 Spring 2017 Meeting #5T

On to some demos