Math 145
Chaos Theory

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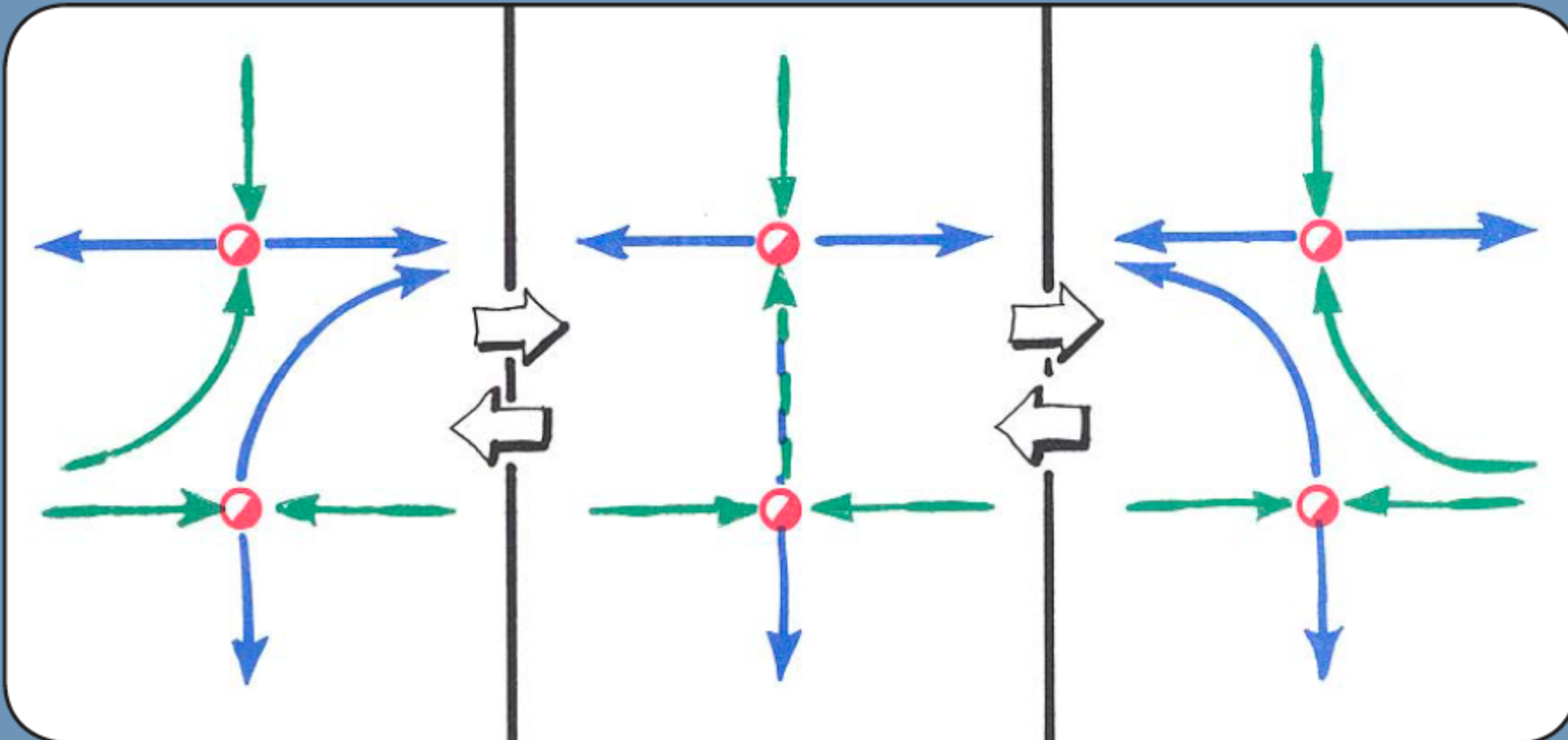
Meeting #5Th, May 4

- 3D Flows, continued
  - Dynamics: The Geometry of Behavior
    - Part 3: Global Behavior
      - Chs. 13, 14, and 15

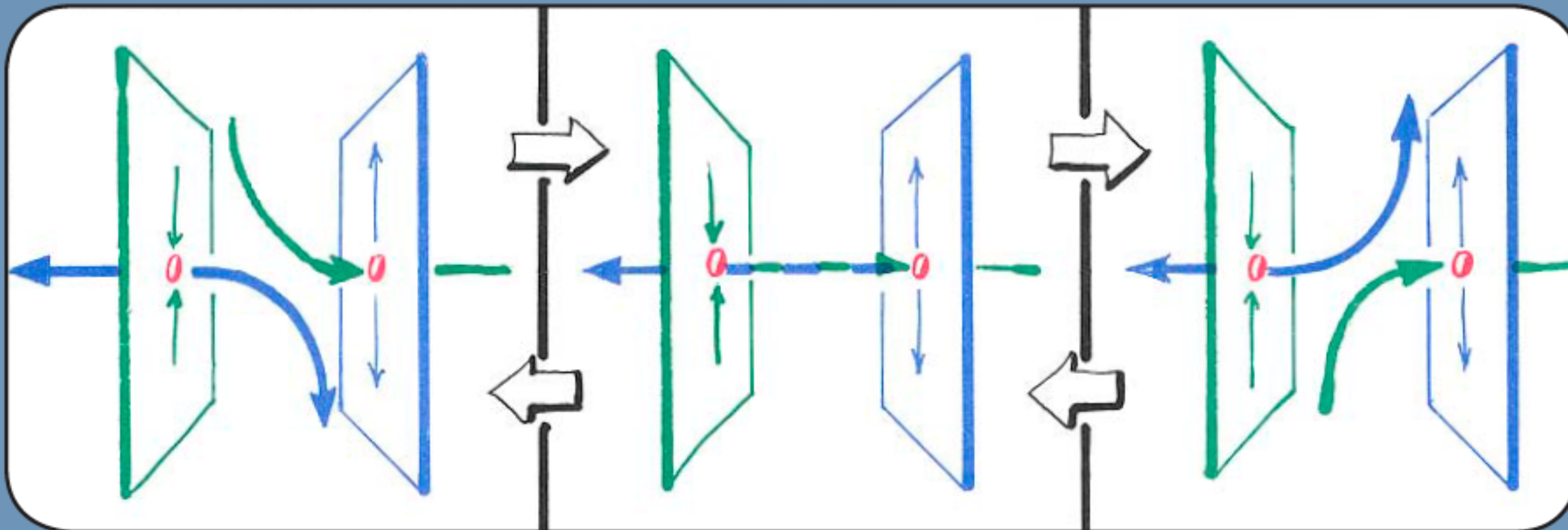
Ch. 13: Heteroclinic Tangles

- Sec. 13.1. Point to point
- Sec. 13.2. The Lorenz attractor
- Sec. 13.3. Point to cycle
- Sec. 13.4. Cycle to cycle
- Sec. 13.5. Birkhoff's signature

Sec. 13.1
Point to Point

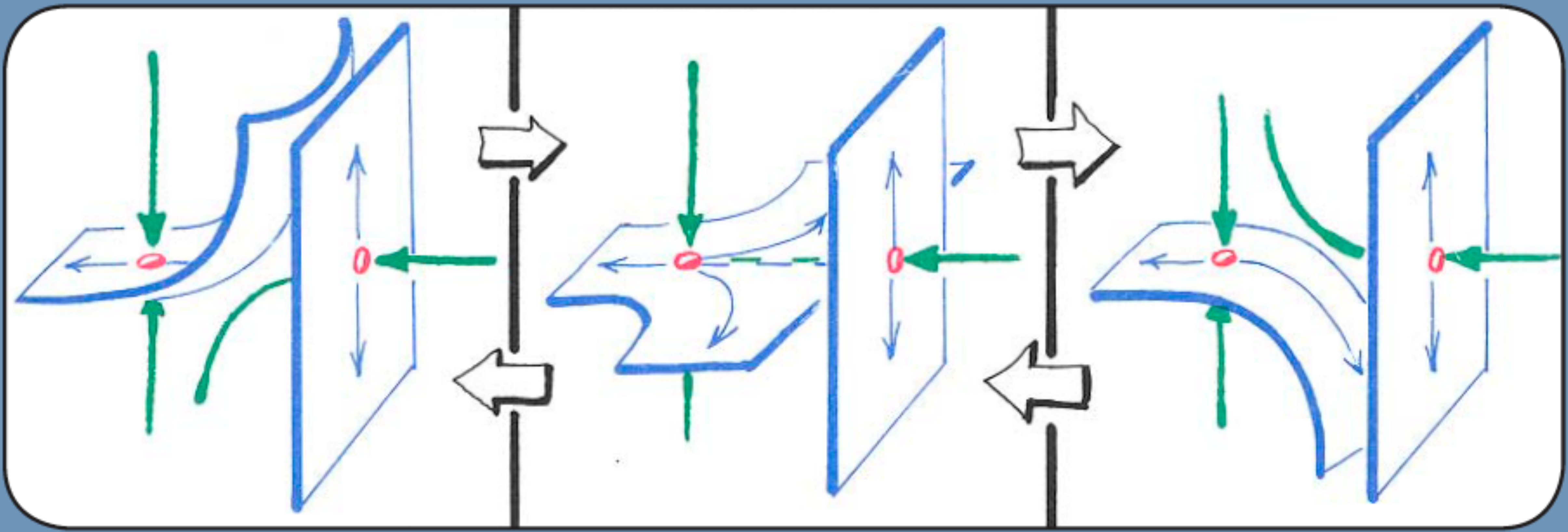


13.1.1.
These three phase portraits each have two hyperbolic limit points of saddle type. The end ones have no saddle connection, while the one in the center has a single heteroclinic trajectory. The sequence has occurred previously in Part One, under the name *saddle switching*. It represents the actual coincidence of the outset from the left saddle and the inset to the one on the right. The transverse intersection of two curves in the plane must be in isolated points. Therefore, this intersection is not transverse. It is a *nongeneric* saddle connection. There are not transverse saddle connections in the two-dimensional case.



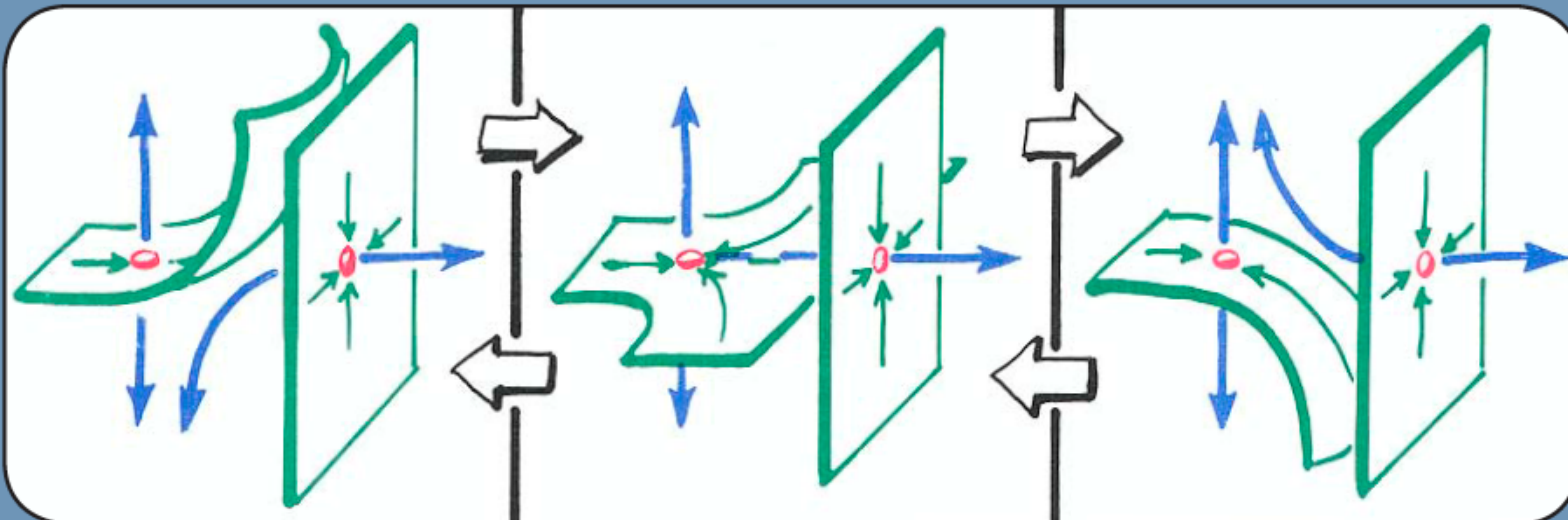
13.1.2.

A saddle point of index 1 cannot have a transverse connection to a saddle point of index 2, in three dimensions. Three closely related portraits are shown here, in analogy to saddle switching in the two-dimensional case. The one in the center has a nontransverse heteroclinic trajectory connecting the two saddle points.

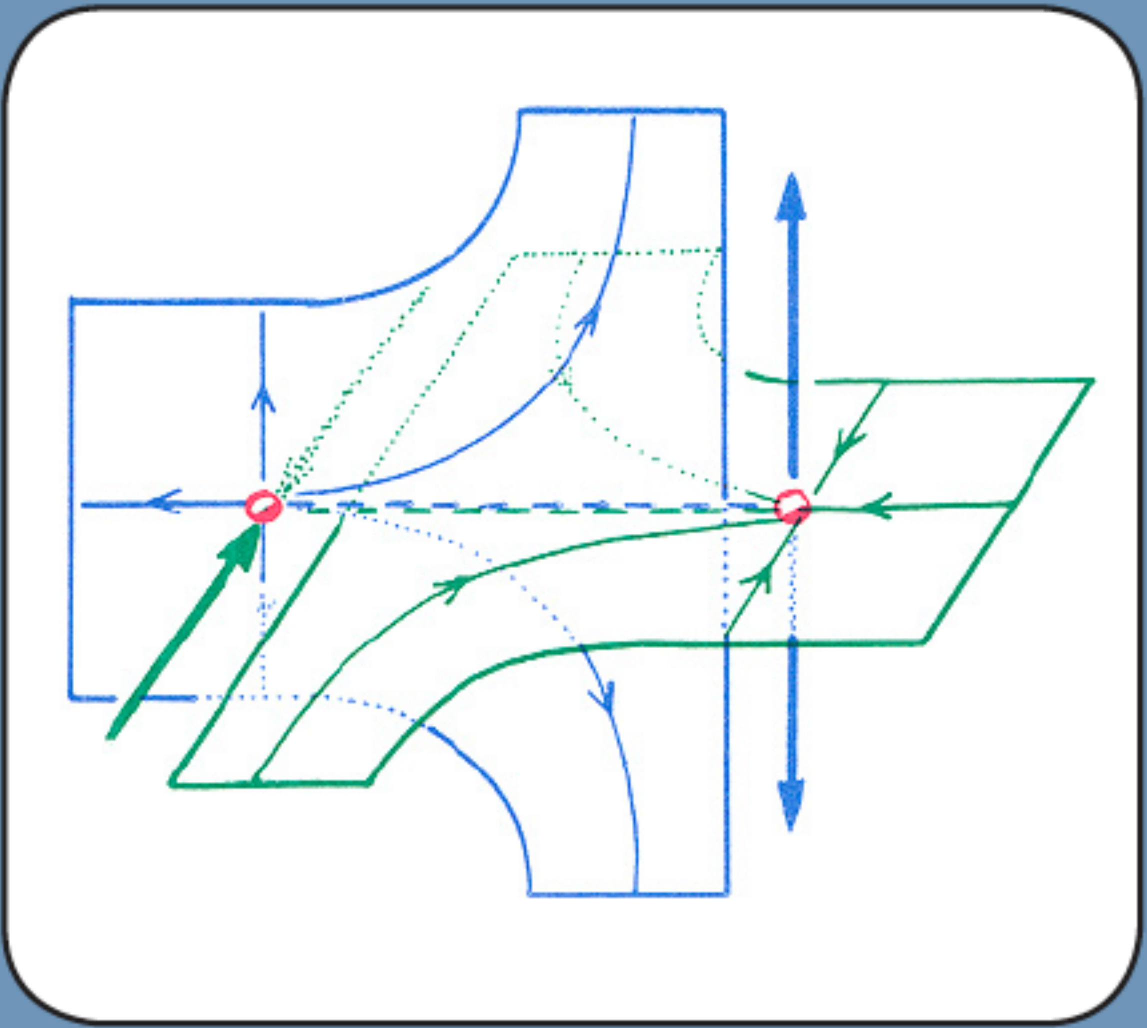


13.1.3.

The next donor, a saddle point of index 2, cannot have a transverse connection to a saddle point of index 2 (same receptor as above), in three dimensions. Here again, three similar portraits are shown. The one in the center is an example of a nontransverse heteroclinic trajectory.

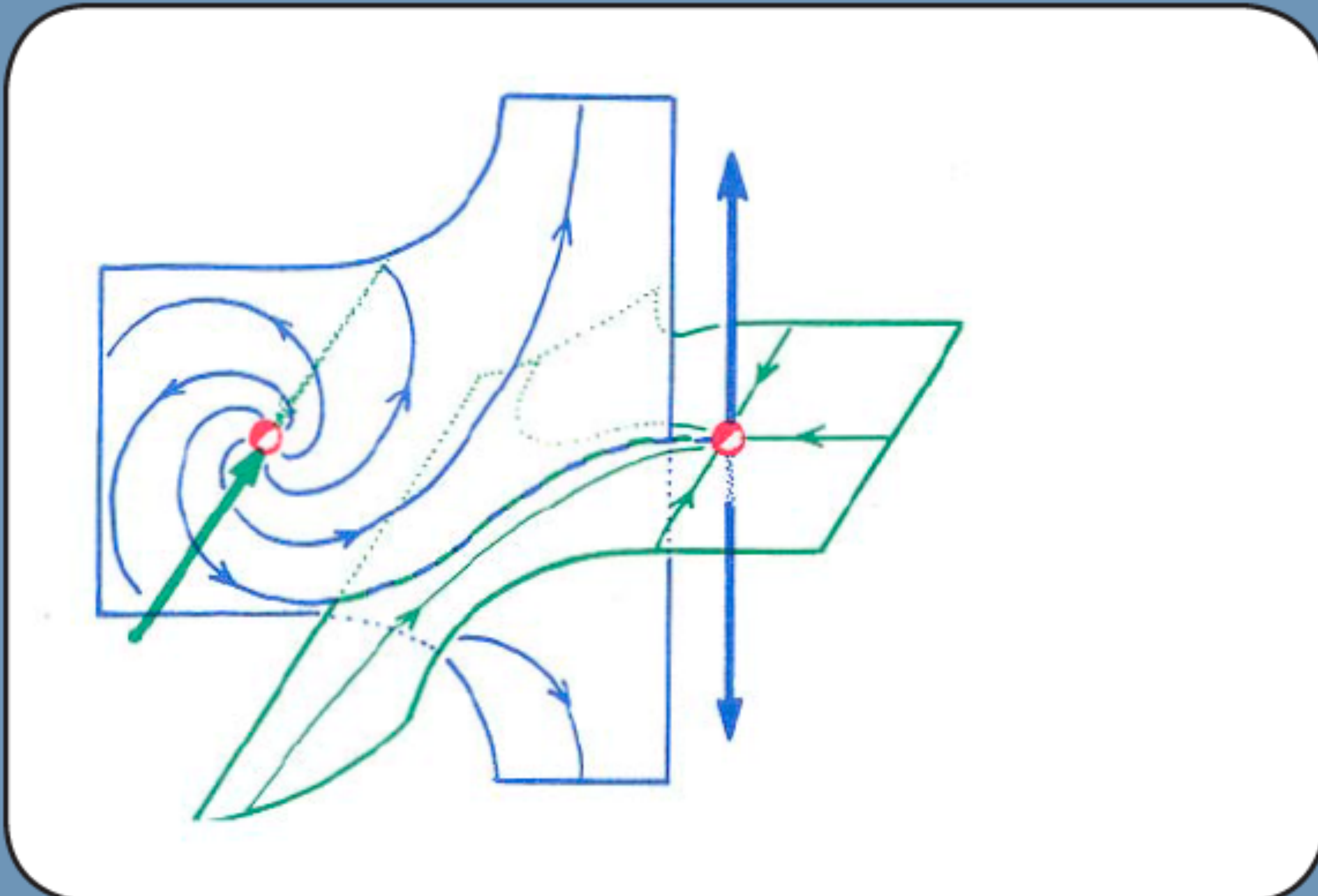


13.1.4.
Transverse connection from a saddle point of index 1 to a saddle point of index 1 (like the case of index 1 to index 2, and index 2 to index 2, described above) cannot occur in three dimensions.

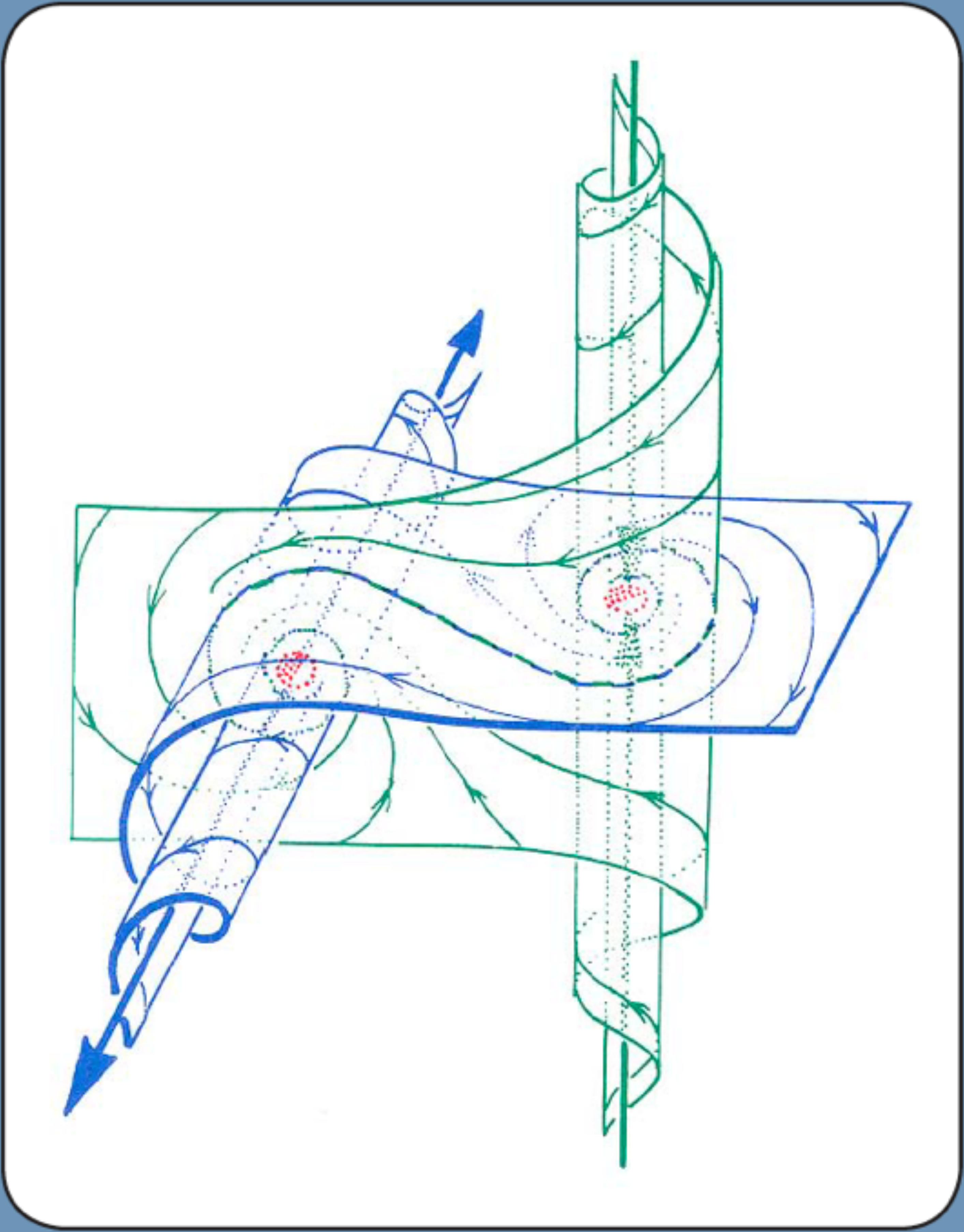


13.1.5.

In this fourth case, a heteroclinic trajectory leads from a saddle point of index 2 to one of index 1. The outset of the donor and the inset of the receptor are both two-dimensional. Thus, a transverse intersection of them in a one-dimensional curve (necessarily a trajectory of the dynamical system) is possible. A nontransverse intersection along a heteroclinic trajectory is also possible - for example, the two surfaces could be tangent to each other, along their intersection. Here, the transverse case is illustrated. This is the only generic (transverse) connection between saddle points in three dimensions.

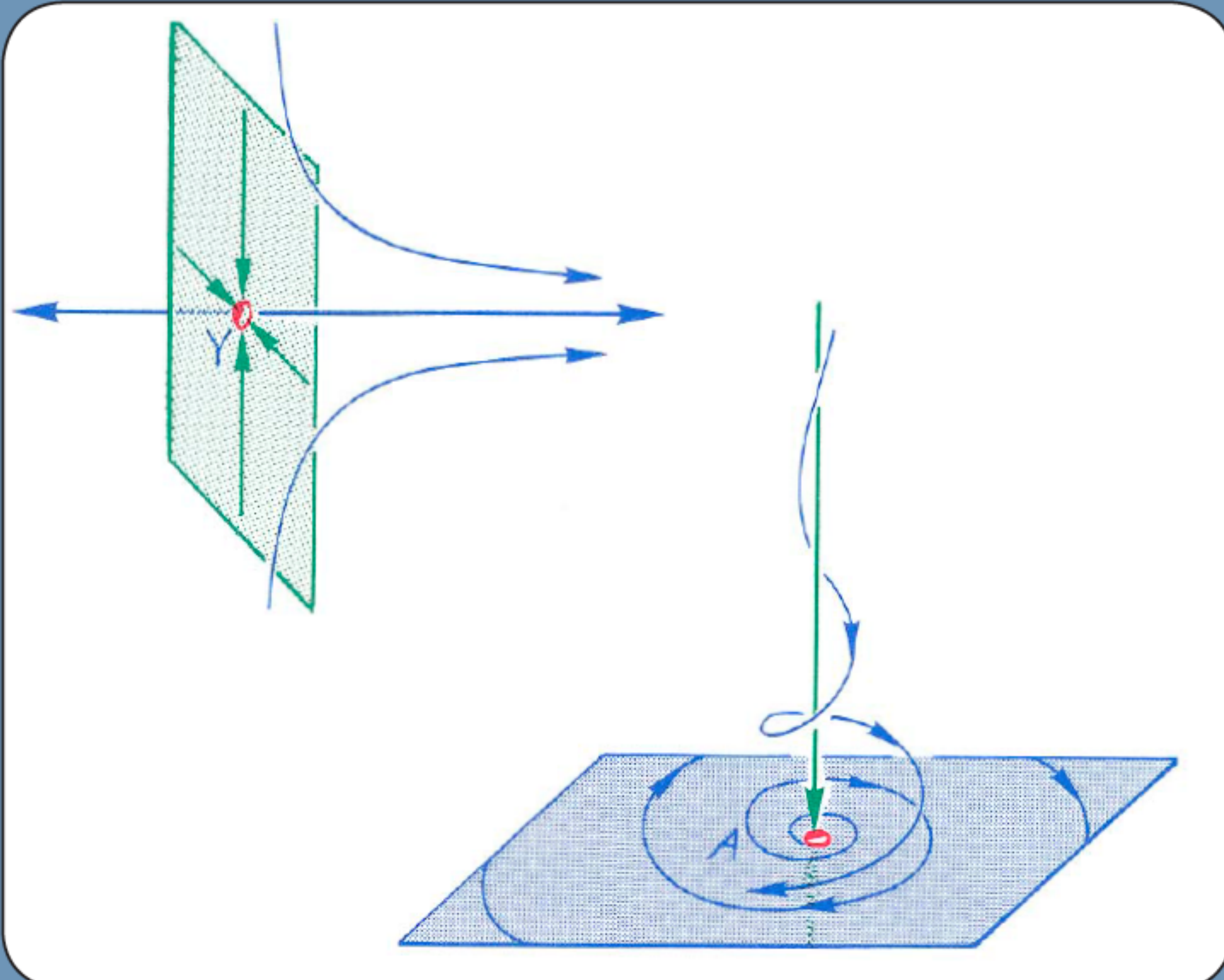


13.1.6.
The preceding illustration shows the transversely connected saddle points, assuming both are the radial (nonspiral) type. Here, the donor has been replaced by a spiral type. This is topologically equivalent to the preceding portrait.

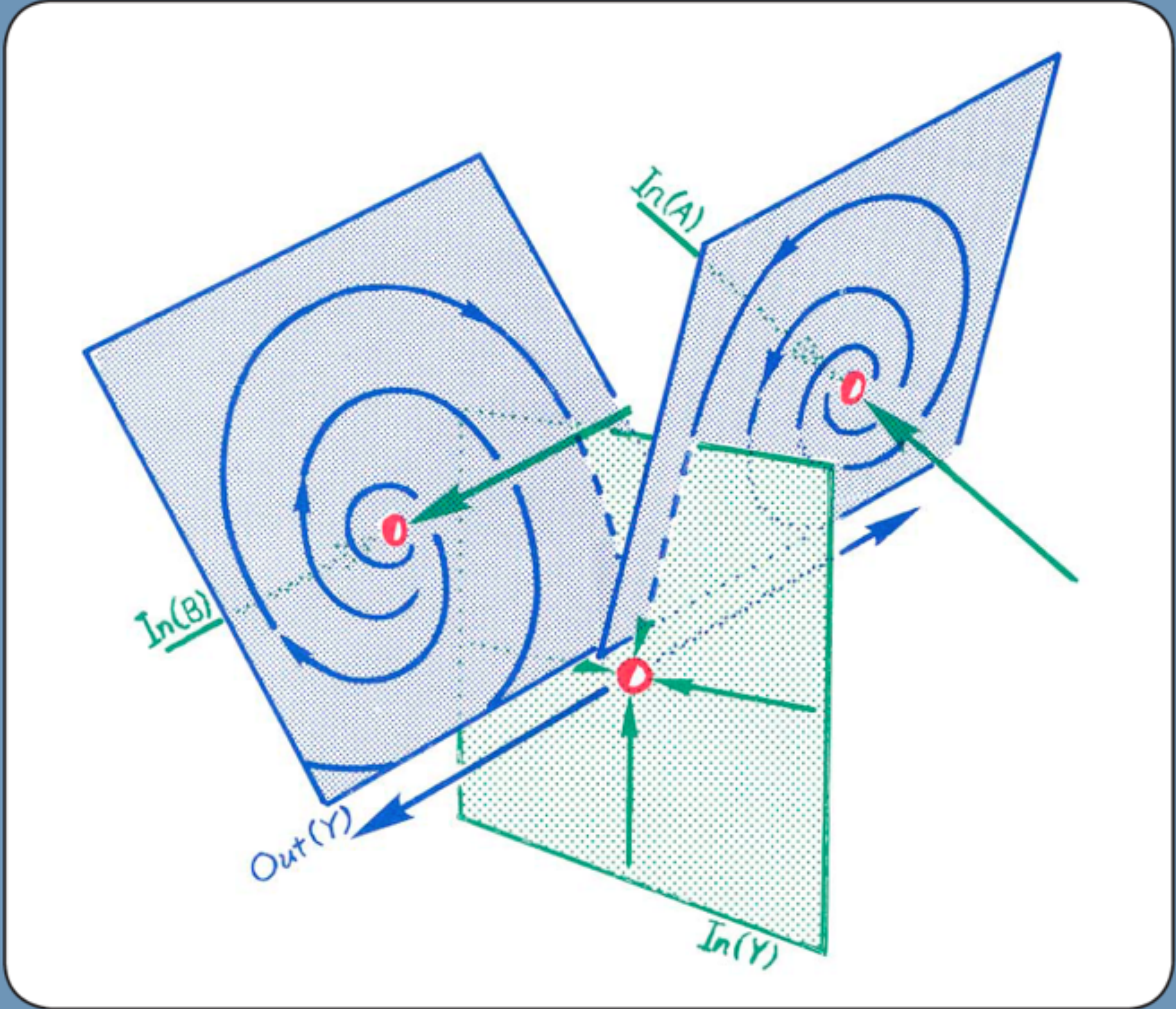


13.1.7.
In this example, both the donor and the receptor are of the spiral type. Again, this is topologically equivalent to the preceding portraits.

Sec. 13.2
The Lorenz Attractor

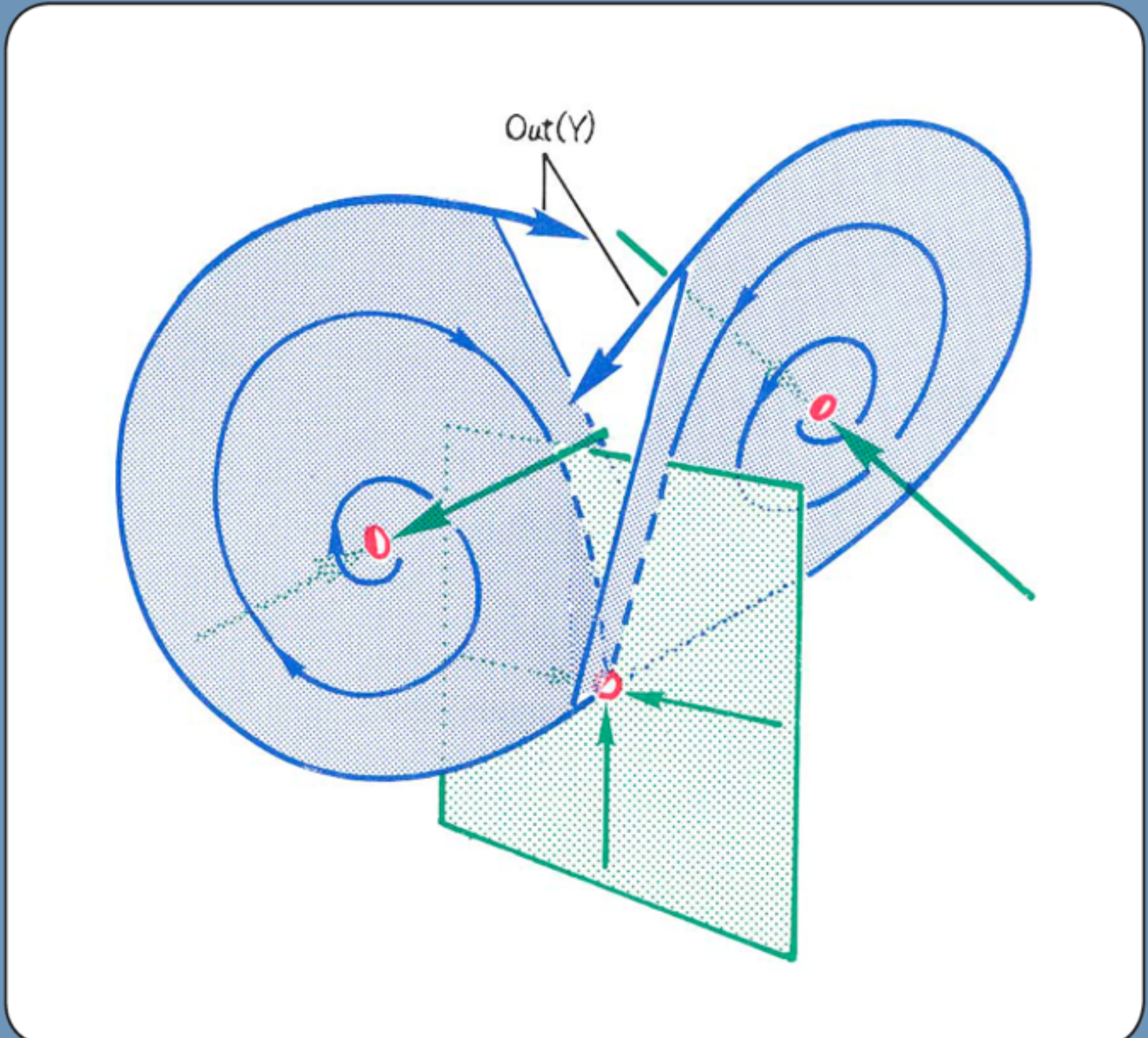


13.2.1.
Here are two saddle points, A and Y. They are hyperbolic, in three dimensions. One, A, has index 2, with spiral dynamics on its planar outset (shaded), Out(A). The other, Y, has index 1, with nodal dynamics on its planar inset (dotted), In(Y). The two outsets are attractive, as shown by the neighboring trajectories. As Out(A) and In(Y) are both two-dimensional, they could intersect transversely in three space. If they did, the transversal intersection would have to be a trajectory, called a *heteroclinic trajectory*.

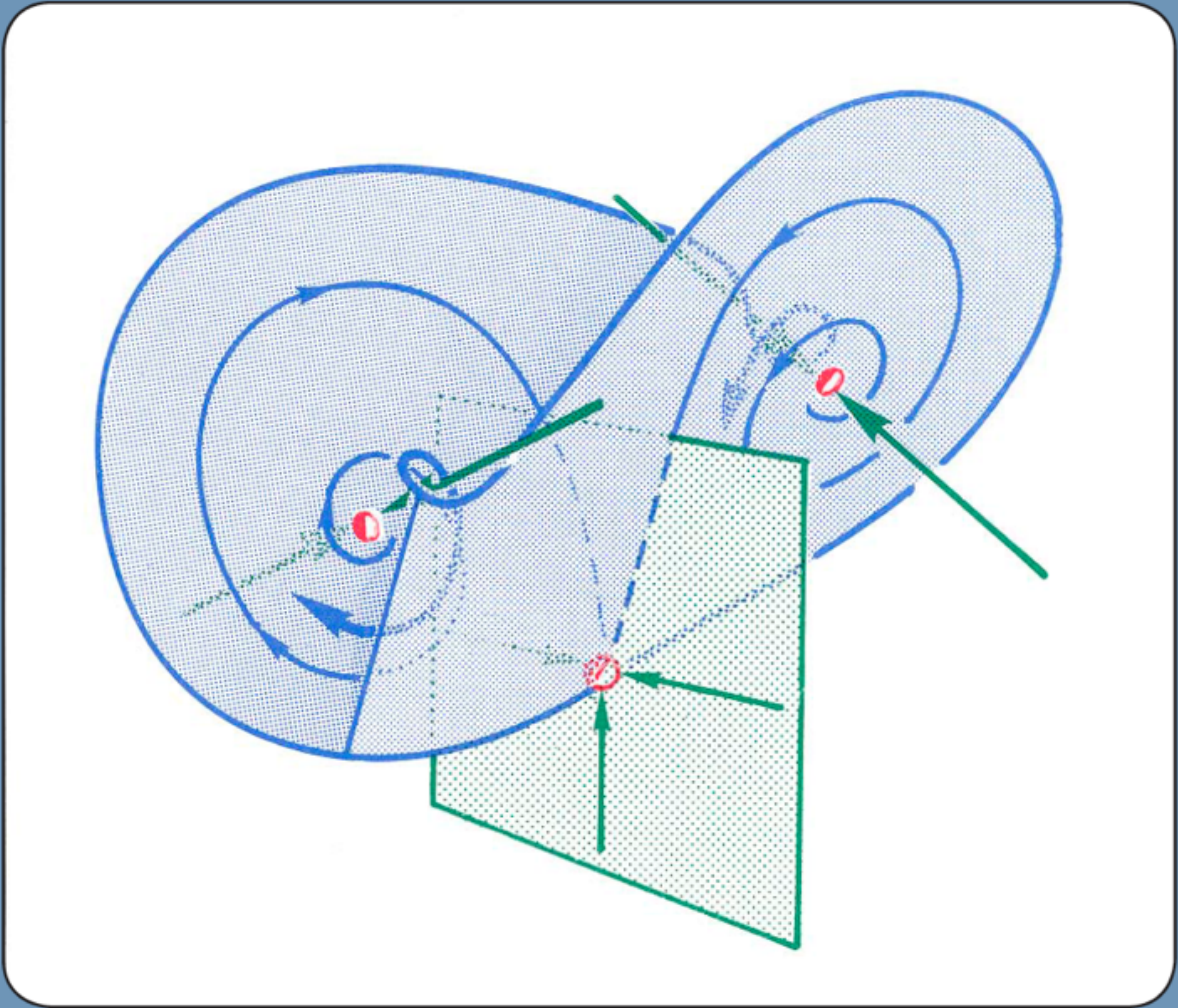


13.2.2.

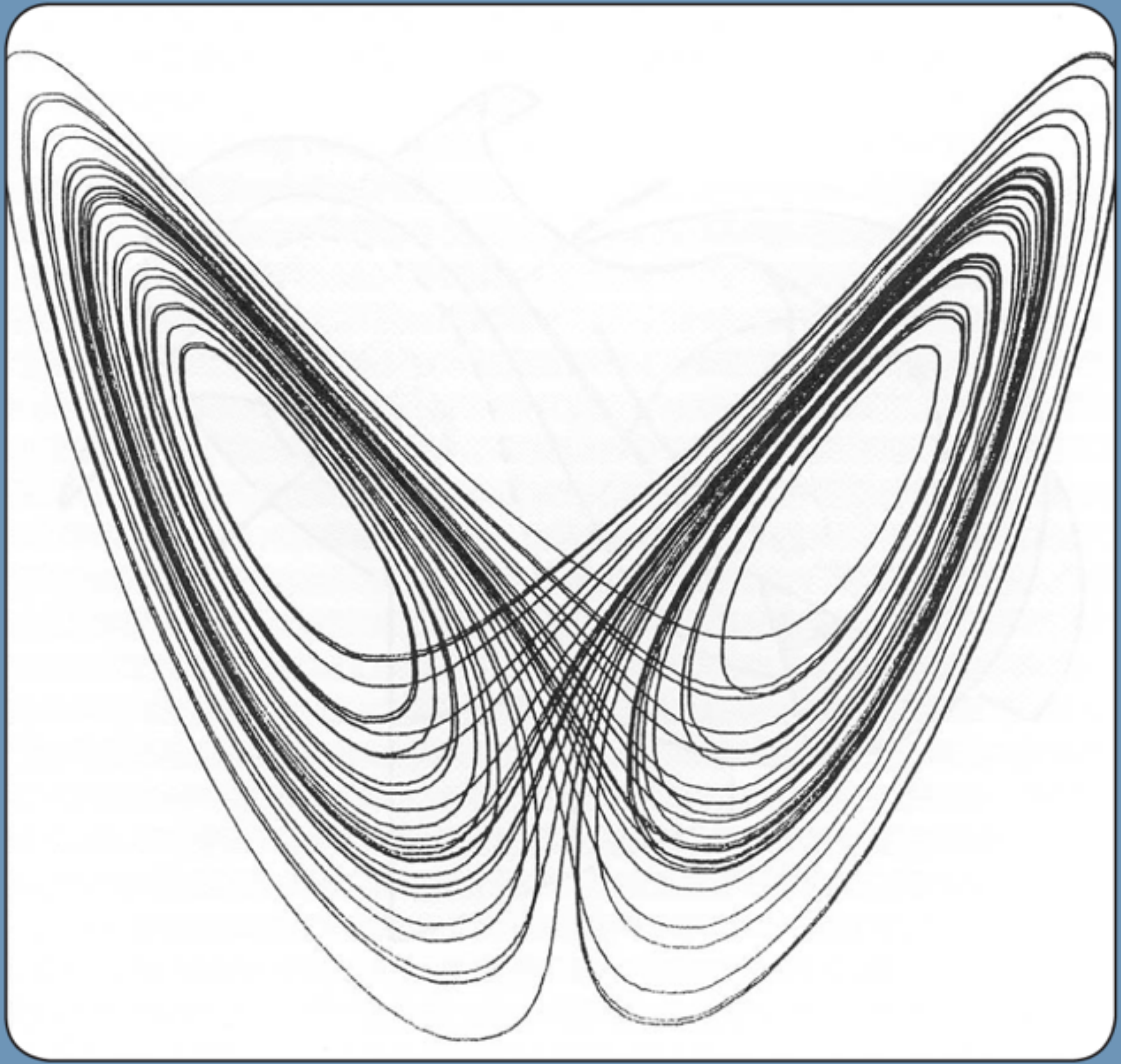
Adding another saddle point, B, essentially identical to A, we make a yoke like this. Both A and B are heteroclinic to Y. They are *transversely heteroclinic*, as the two planar outsets (shaded) intersect the planar inset (dotted) transversely. There are *two heteroclinic trajectories* in this yoke. Note that the arriving outsets are incident upon the departing outset, at Y. We call this a *neat yoke*. Next, we will see where these outsets end up.



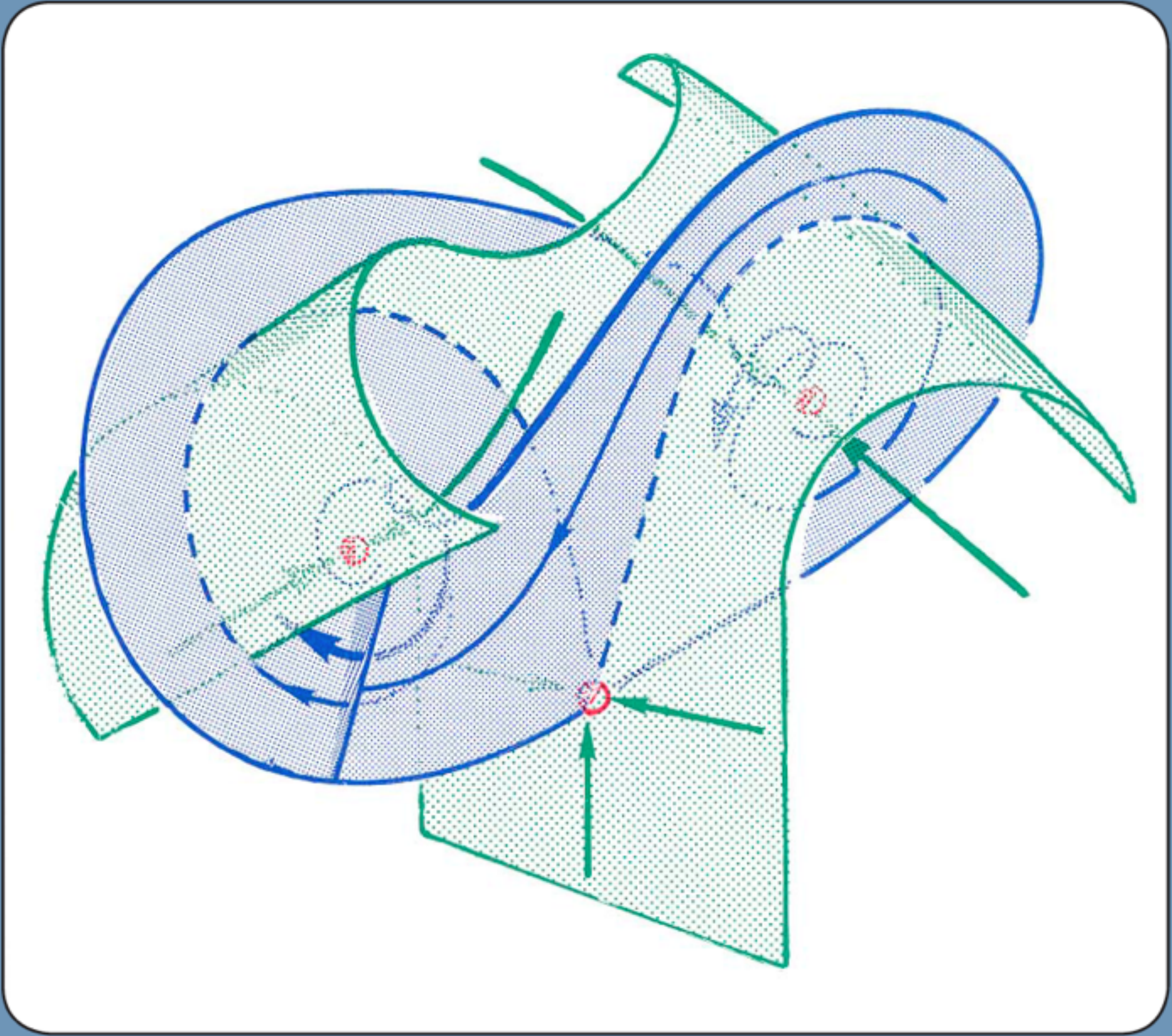
13.2.3.
As the arriving outsets, Out (A) and Out (B), both have spiral dynamics, the departing outset that bounds them, Out (Y), swirls around and reinserts, as shown here. It cannot go off to infinity, as the Lorenz system has a repellor at infinity.



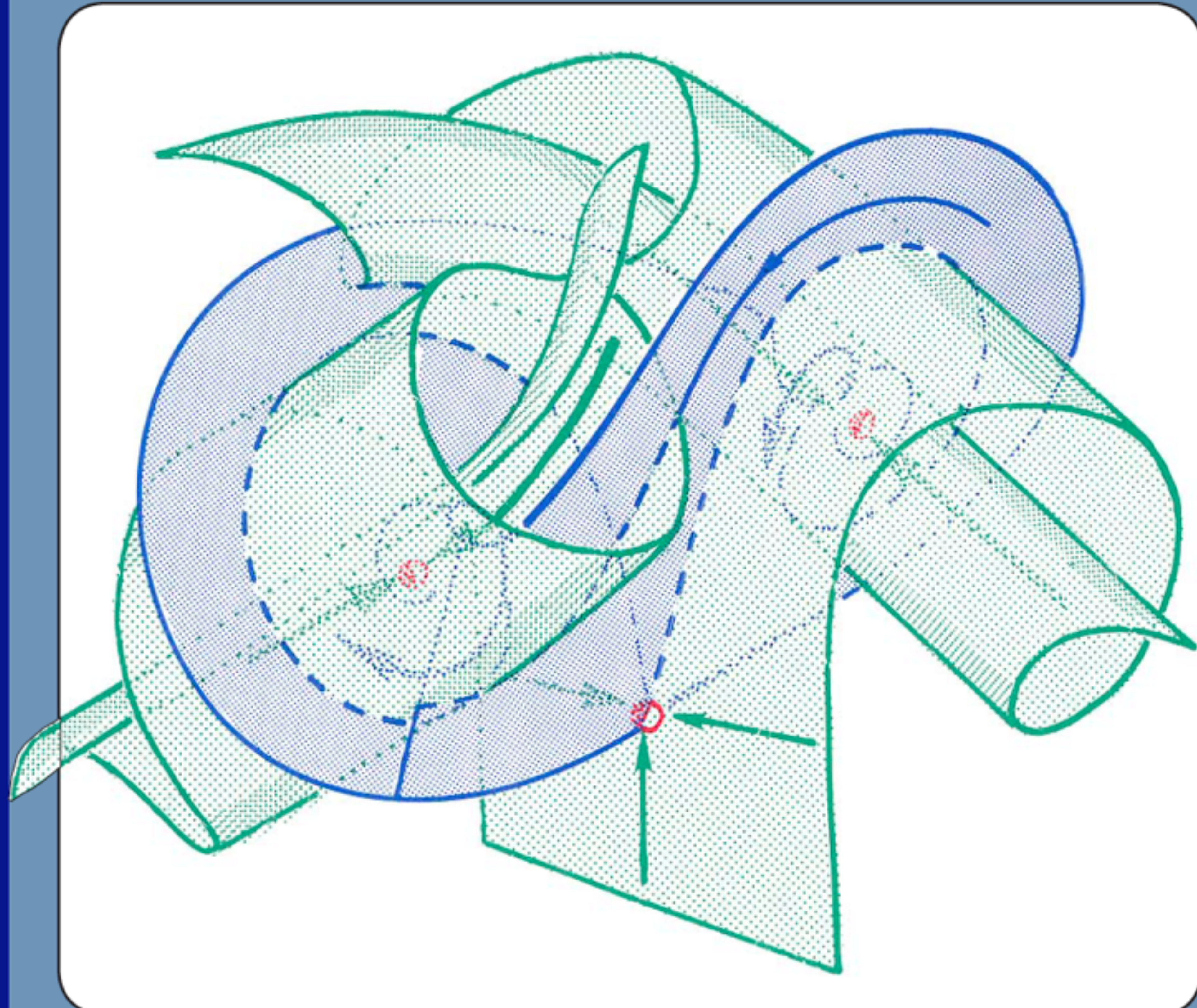
13.2.4.
The result of reinserting is this: as each branch of Out (Y) swirls around one of the shaded outsets, it approaches near the other shaded outset. It gets attracted, as outsets are attractive. Thus, the omega limit set of Out(Y) is within the closure of the union of the three yoked outsets.



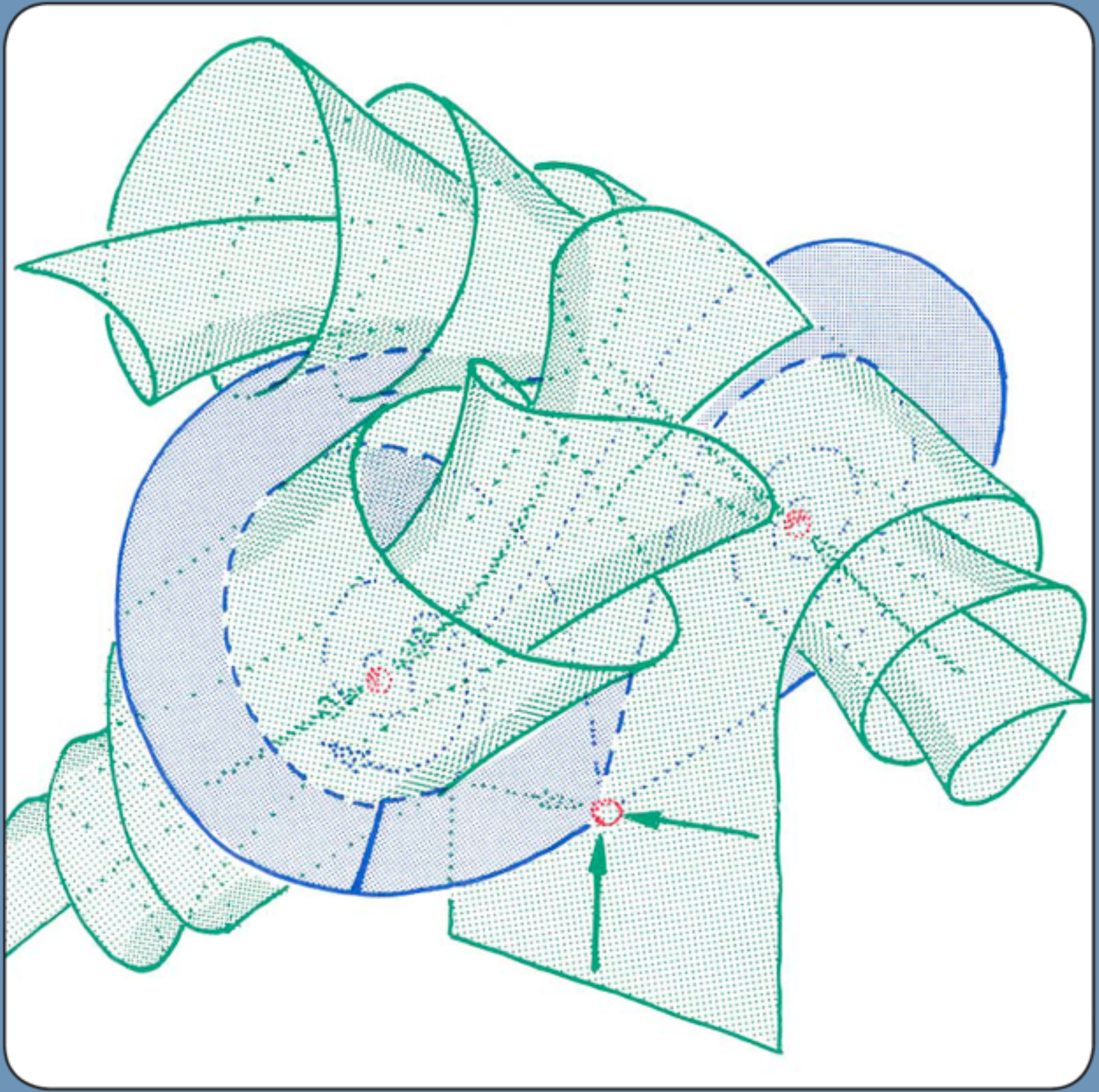
13.2.5.
And here, for comparison, is a computer drawing by Robert Shaw of the Lorenz attractor. Inspection of the equations reveals the three distinguished saddle points, right where we want them. But the planar inset of the saddle point in the lower center is qualitatively invisible. It is a kind of a separatrix. Now we will add it to the picture, with its full extension.



13.2.6.
Referring to Figure 13.2.4, we run the flow backwards in time, to extend the planar (dotted) inset outward from Y. It follows the heteroclinic trajectories (dashed) back to the yoked saddles, A and B, scrolling as it goes.



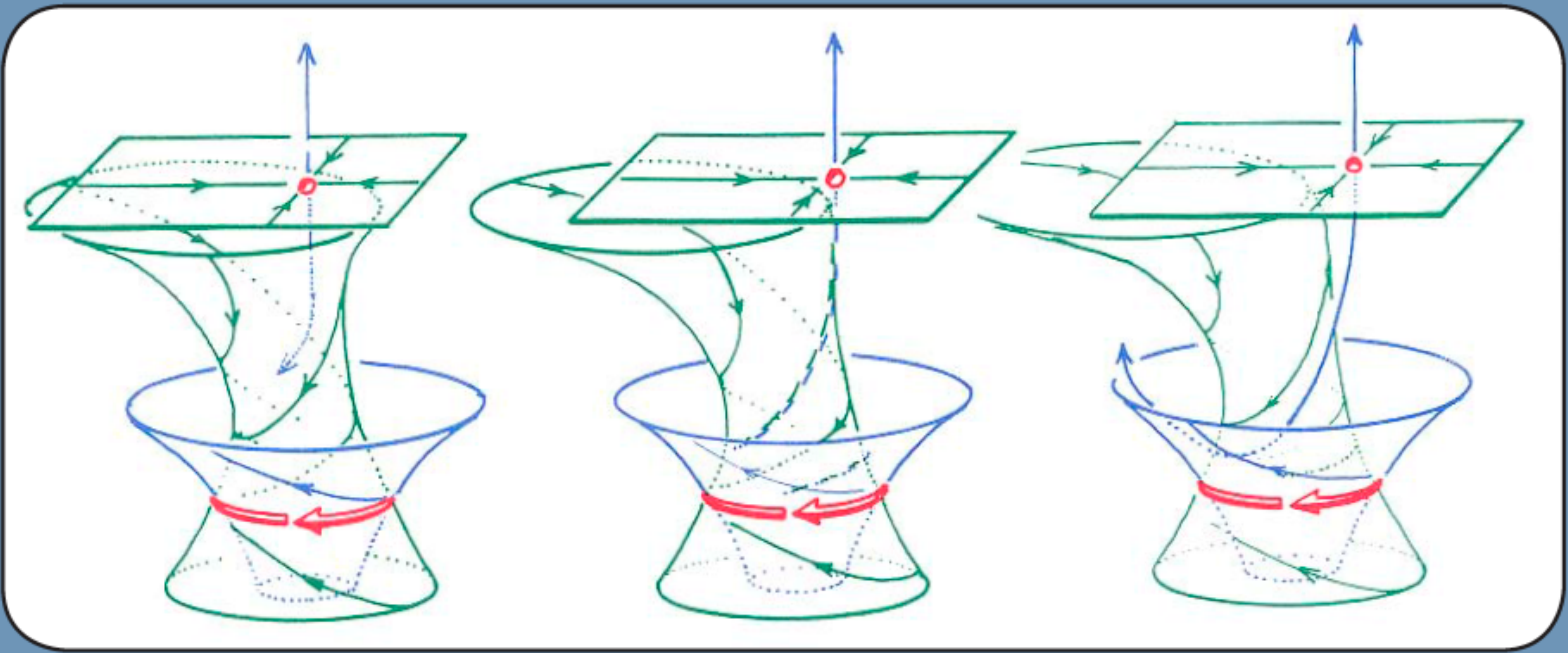
13.2.7.
Extending the dotted inset farther backwards in time, it scrolls up tightly around the *one-dimensional insets* of A and B, In (A) and In (B).



13.2.8.

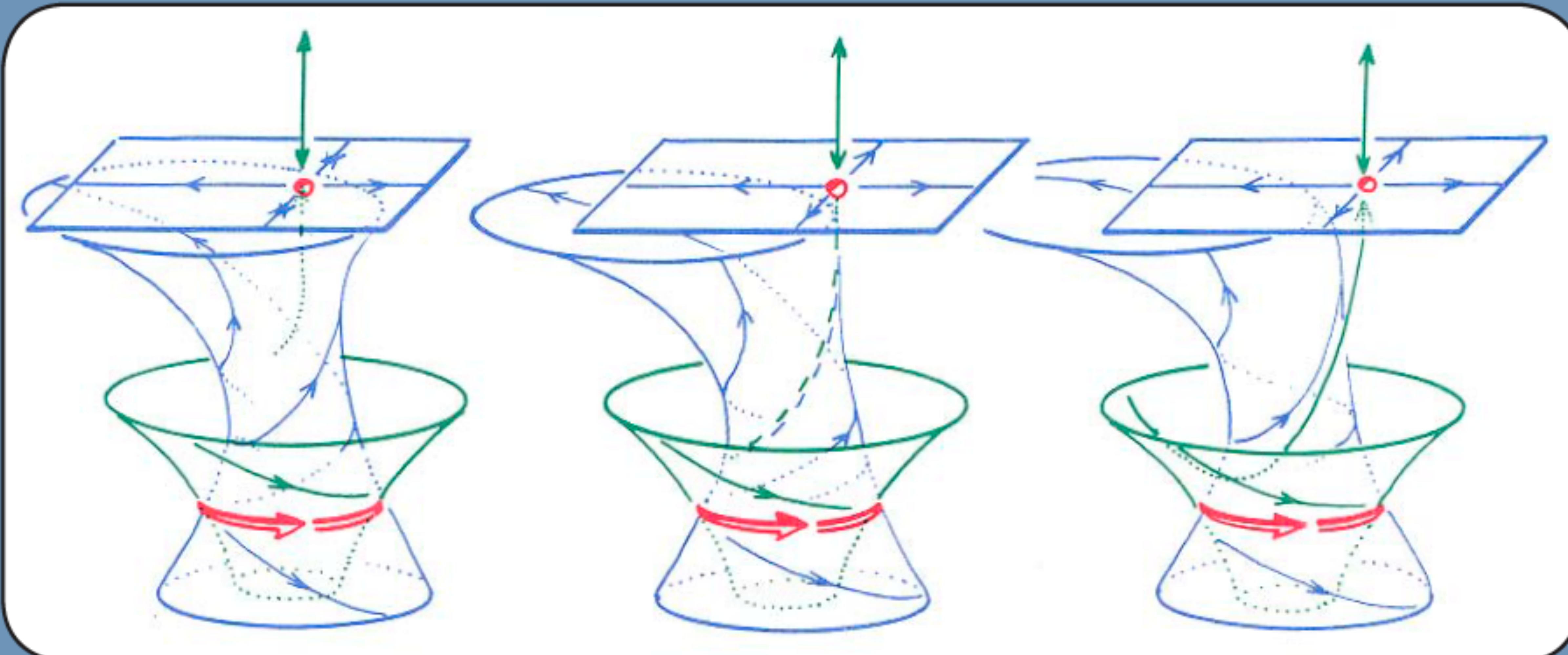
Extending the dotted inset farther backwards still, the four ends of the scrolls are pulled out along the curves, In (A) and In (B), toward their source at infinity.

Sec. 13.3
Point to Cycle

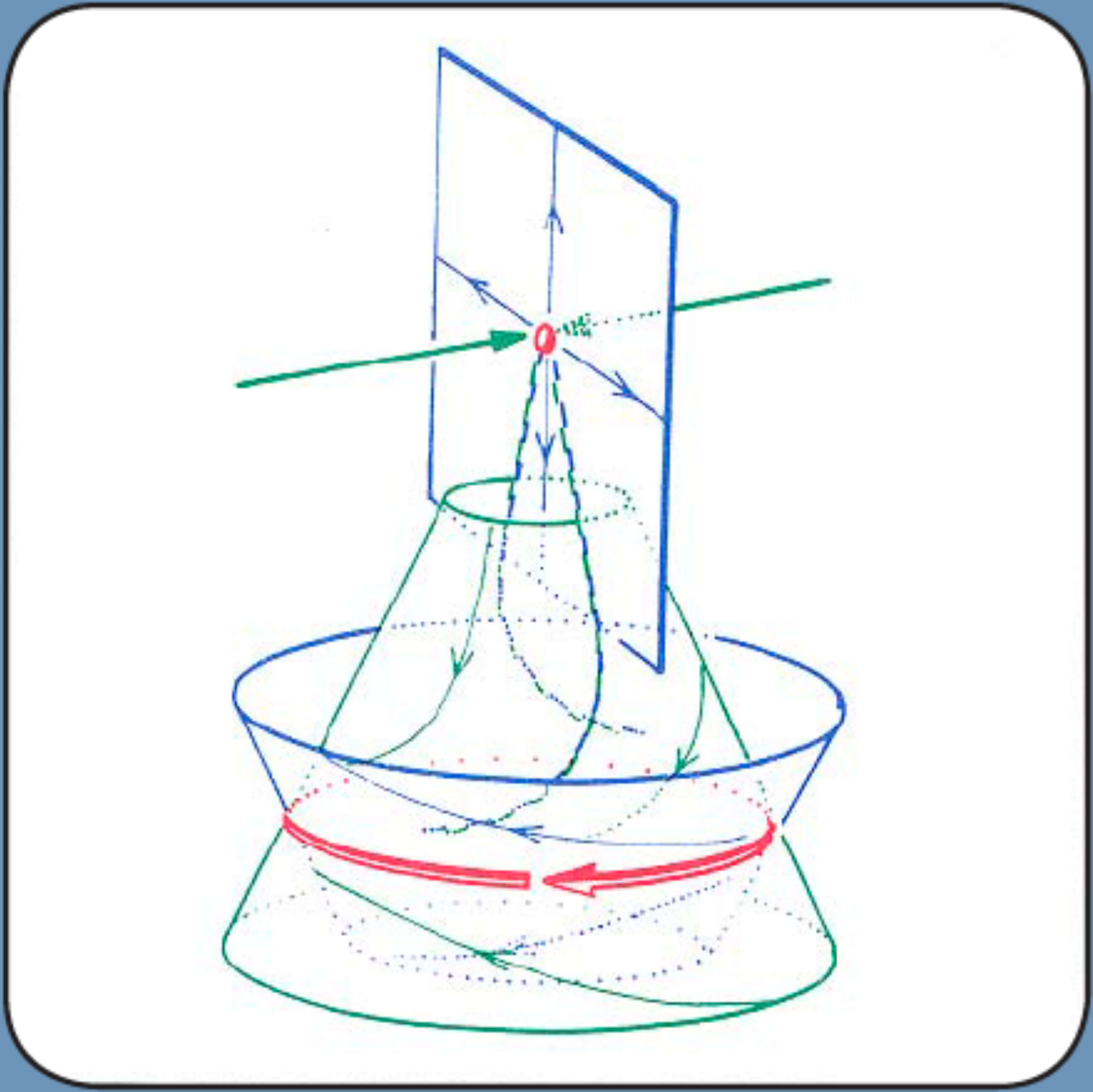


13.3.1.

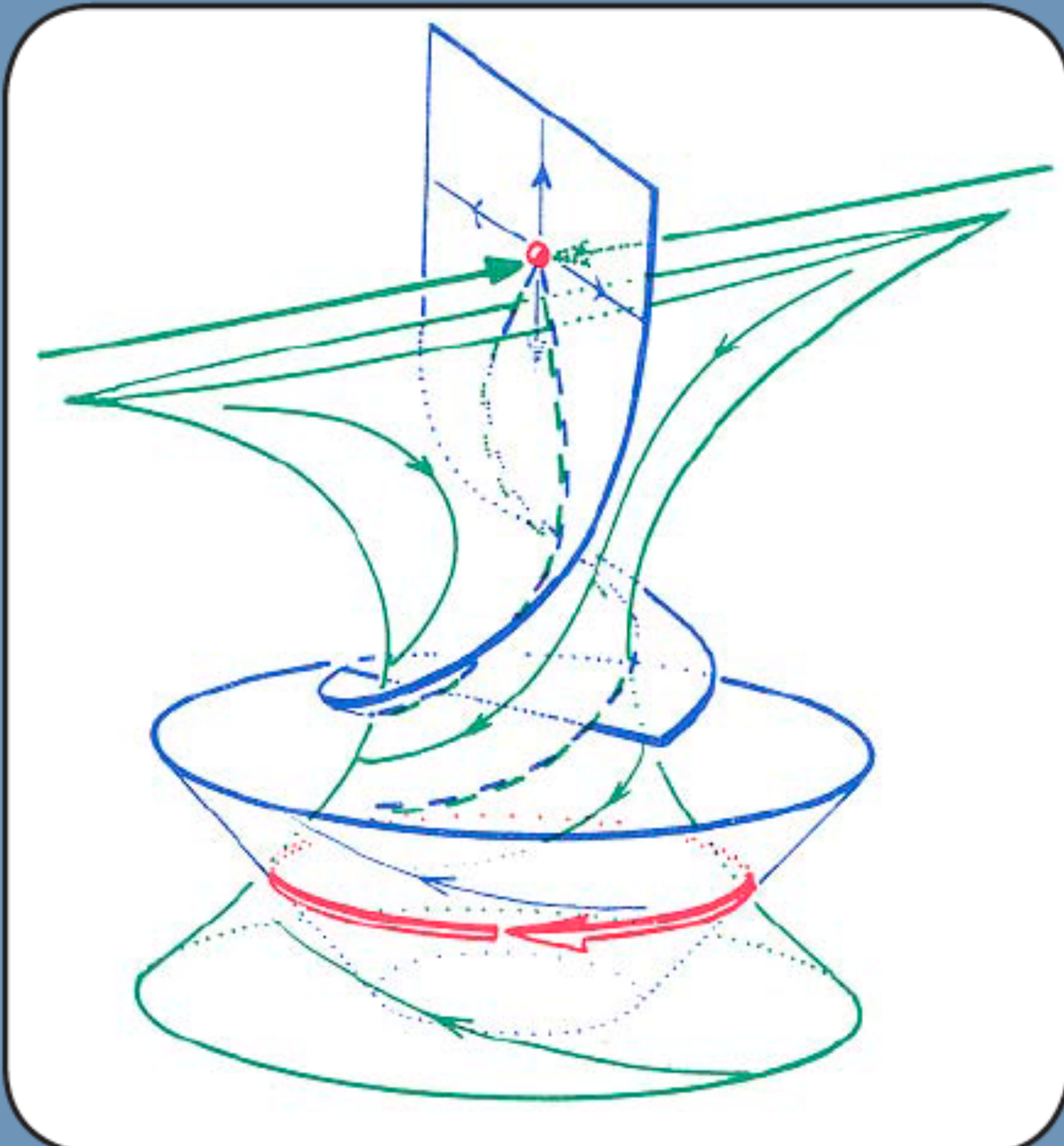
A heteroclinic trajectory from a saddle point of index 1 to a saddle cycle can never be transverse in three dimensions. Here is a nongeneric portrait, in the center, flanked by two nearby generic ones.



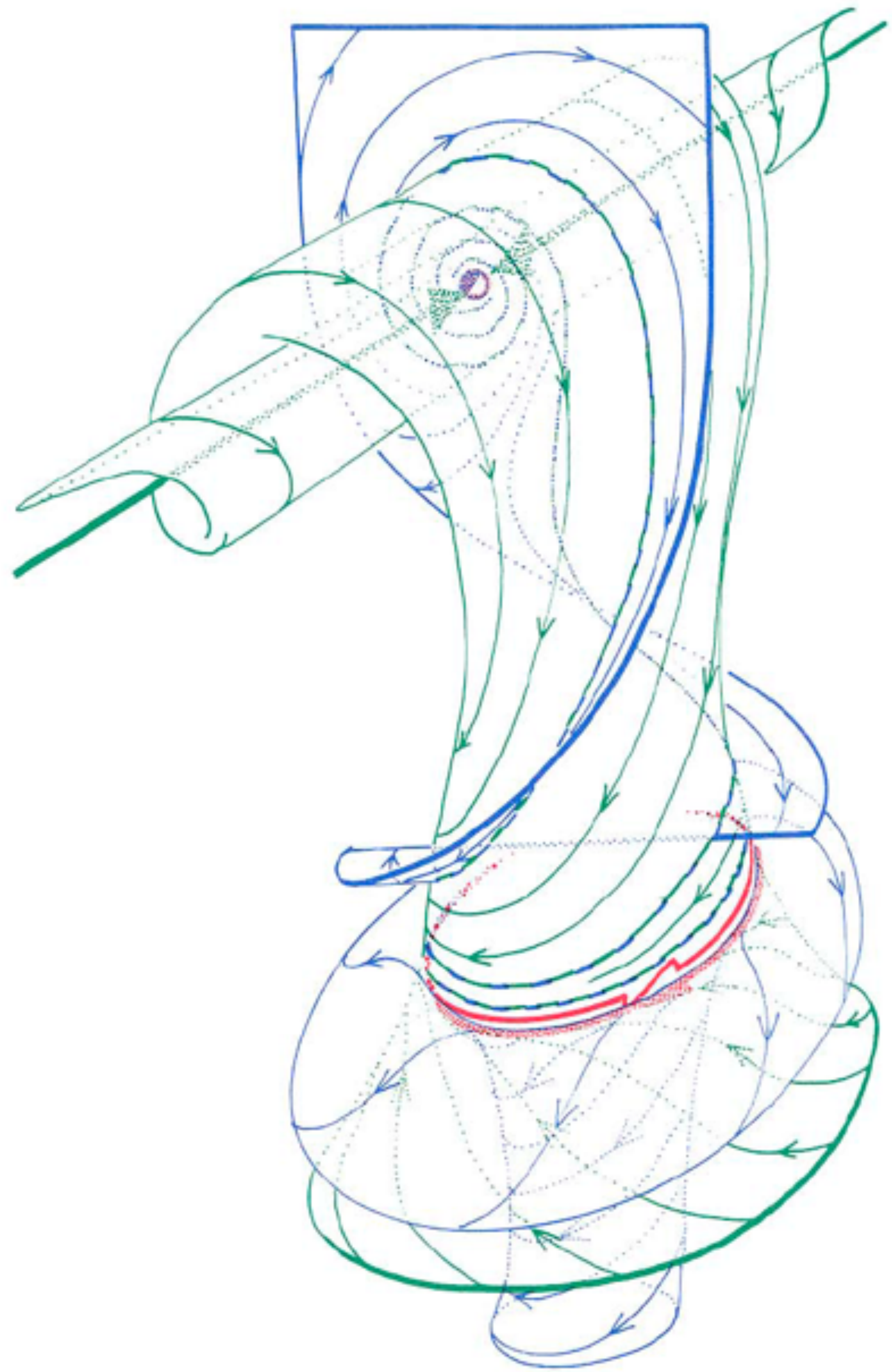
13.3.2.
Similarly, a heteroclinic trajectory from a saddle cycle to a saddle point of index 2 is nongeneric.



13.3.5. Nevertheless, heteroclinic connection from a saddle point of index 2 to a saddle cycle can occur generically in three dimensions. Here is the first step in the visualization of this configuration.



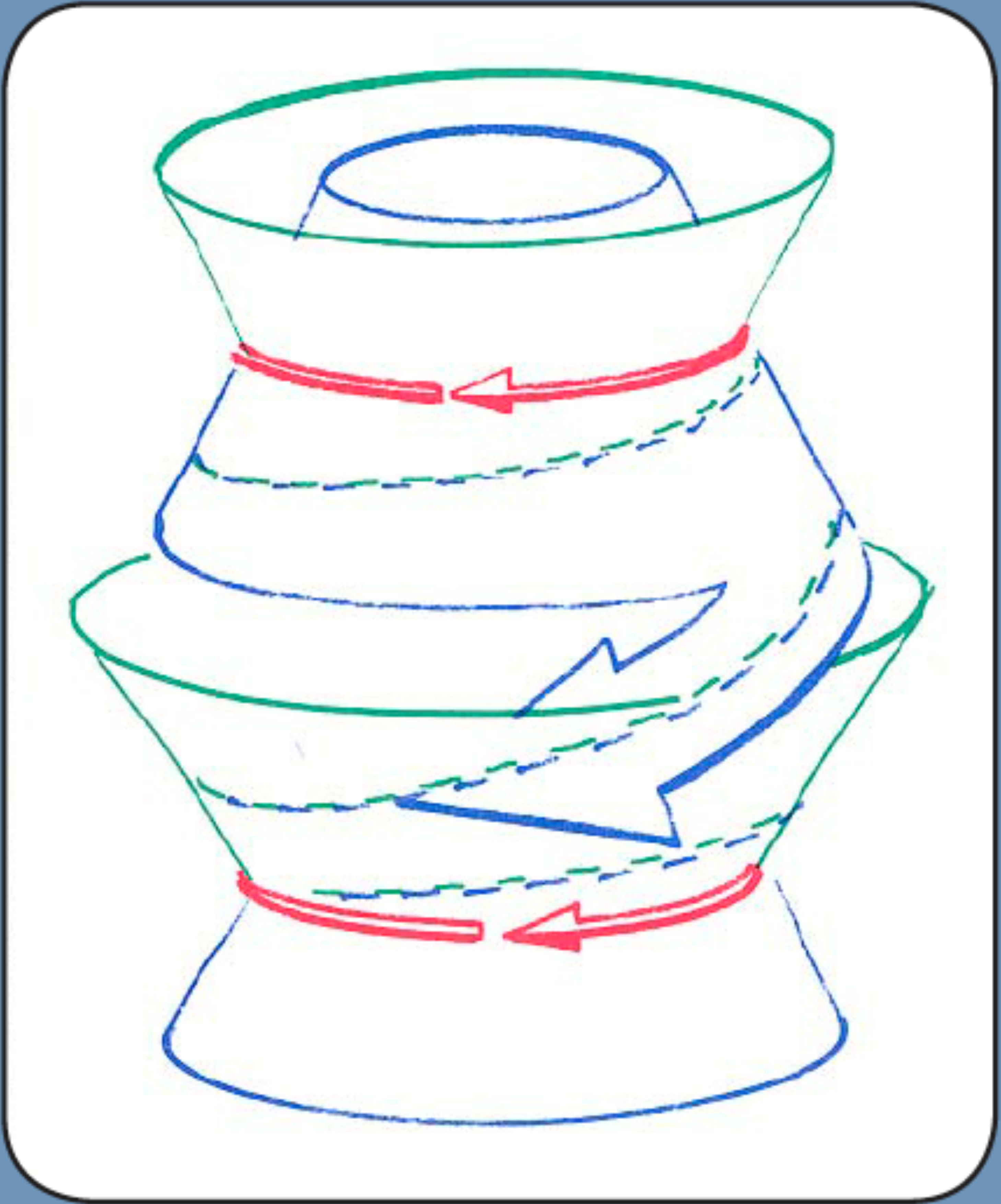
13.3.6.
To generate more of the picture, the inset of the limit cycle (upper cone above) must be extended further into the past, to see how the trajectories spiraling into the limit cycle must have come from near the inset trajectories of the limit point.



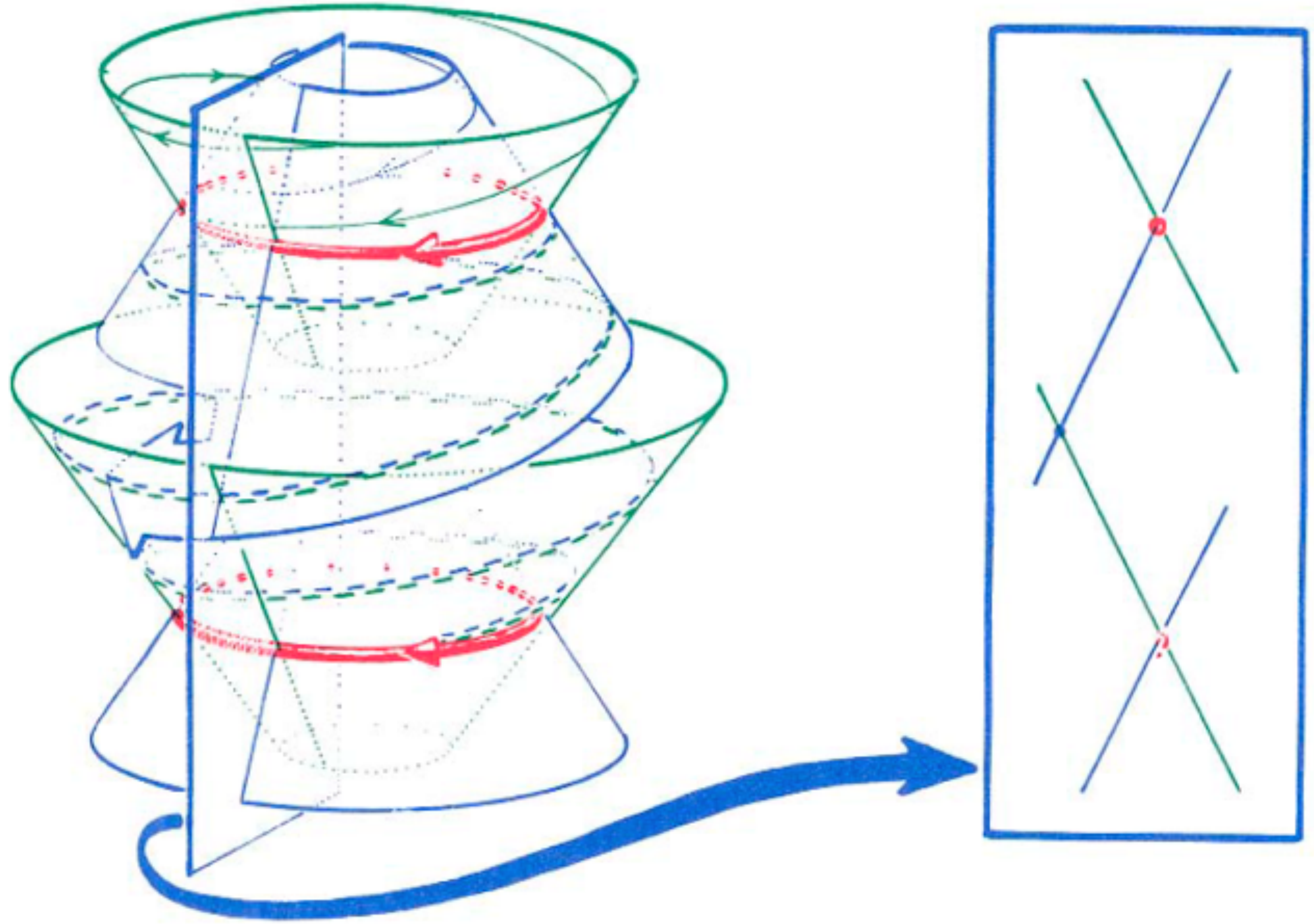
13.3.7.

Before, the saddle point of radial type was shown. Here, it has been replaced by a spiraling one. These two distinctive types of heteroclinic behavior are topologically equivalent, however.

Sec. 13.4
Cycle to Cycle

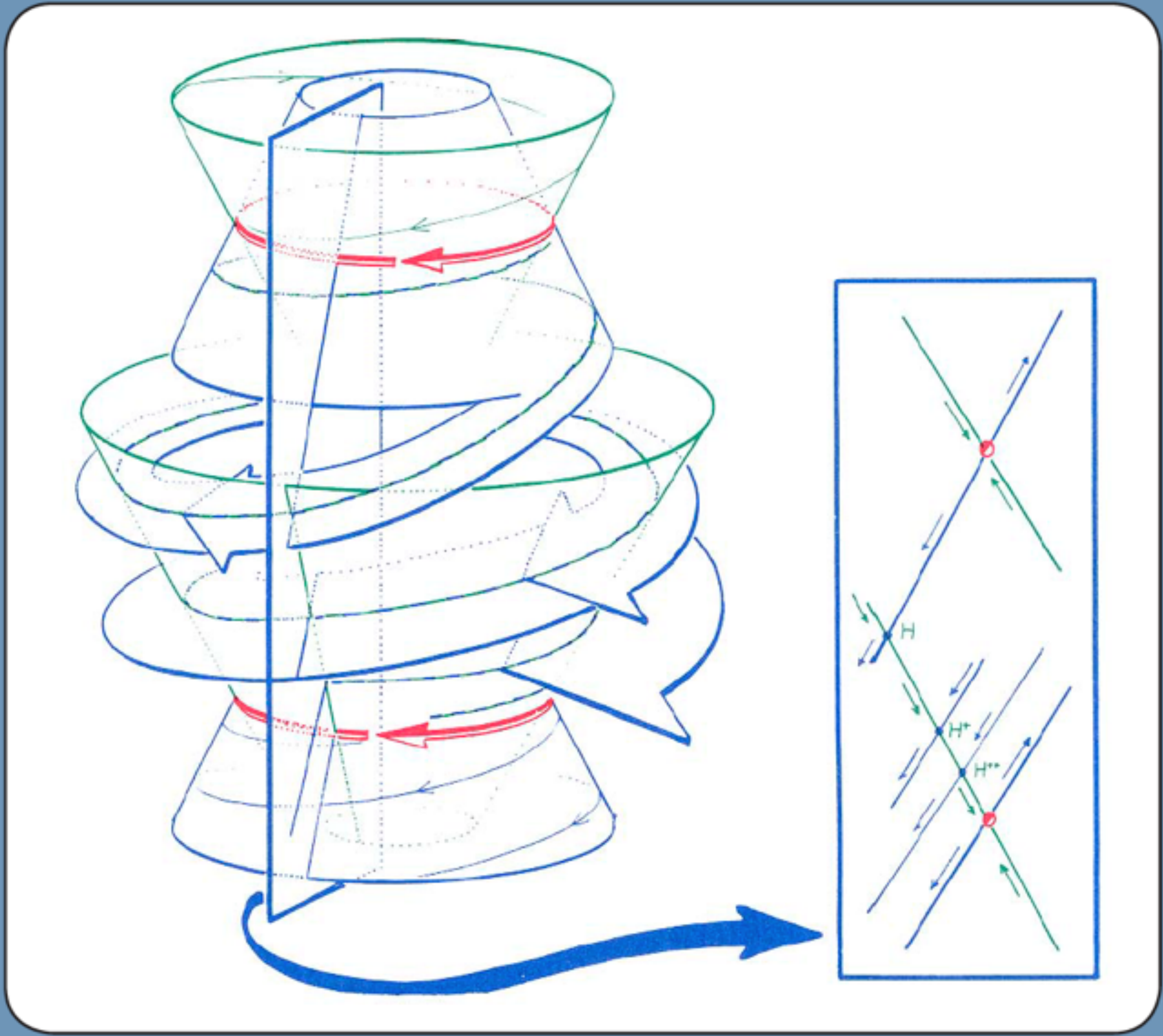


13.4.1.
The outset of a saddle cycle (two-dimensional) can intersect the inset of another saddle cycle (also two-dimensional) transversely, in a (one-dimensional) curve of intersection, necessarily a spiraling trajectory. This fourth type of generic heteroclinic behavior is decidedly complicated.



13.4.2.

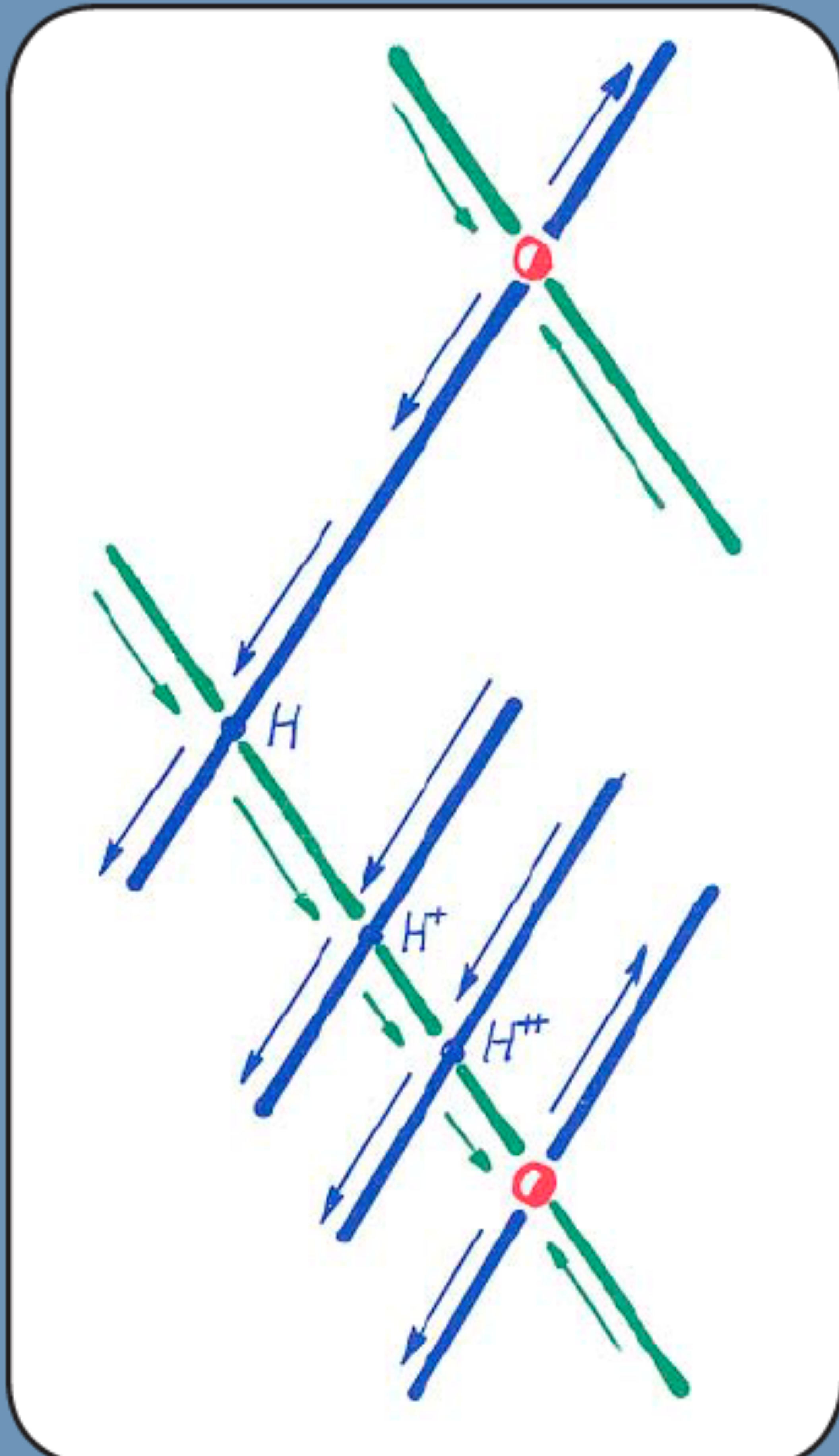
To dissect the complicated structure of such a connection between limit cycles, Poincaré introduced the *transverse section*, and the *first return map*. Within the cross-section (the *Poincaré section*) the two limit cycles are represented by points, and their insets and outsets by curves. The intersection of the outset of the donor cycle (above) and the inset of the receptor cycle (below) is a heteroclinic trajectory, represented in the Poincaré section by the point designated *H*.



13.4.3.

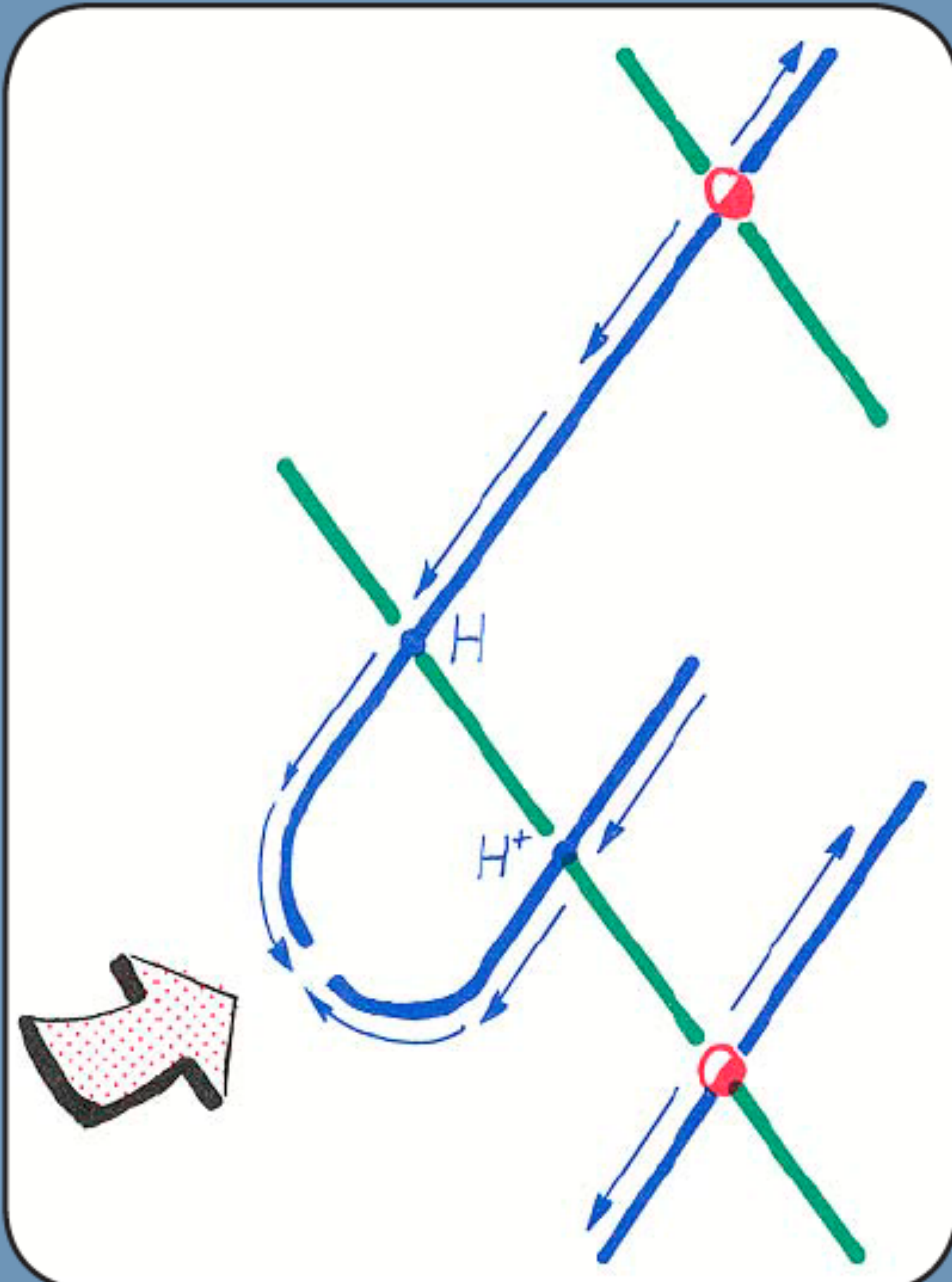
This picture, understood by Poincaré and fully analyzed by Birkhoff and Smith,[3] involves a doubly infinite sequence of intersections of the curves representing the inset and outset. For the marked point, *H*, representing the heteroclinic trajectory, is mapped by the Poincaré first return map into another point, H+, which is also in both curves. This point, H+, is actually on the same heteroclinic trajectory as H, at a later time. Further, the image of H+ is another point, H++, through which both curves must cross.

Sec. 13.5
Birkhoff's Signature

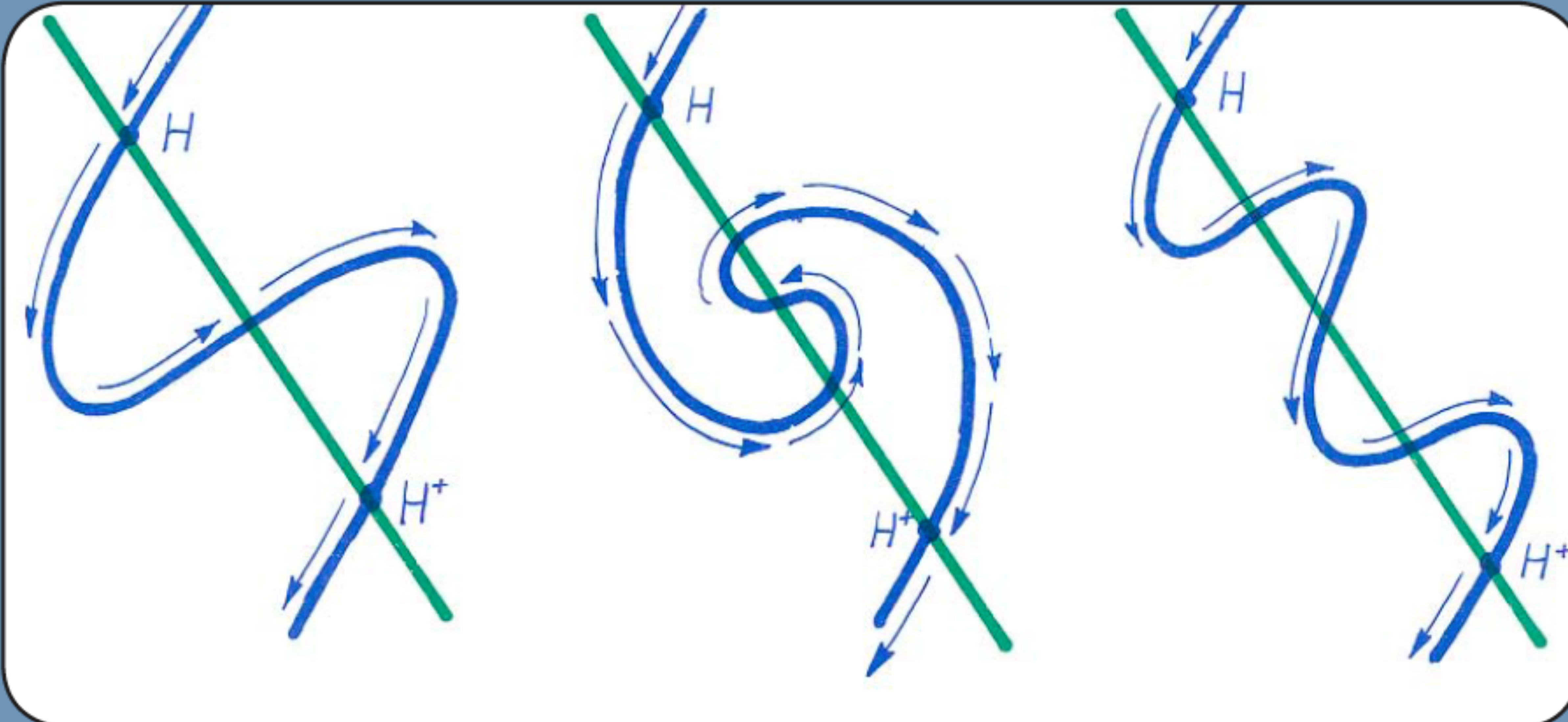


13.5.1.

Here is a close-up view of two successive intersections, *H* and *H+*, belonging to a single heteroclinic trajectory. They are shown here on a piece of the inset curve of the saddle point on the right, representing the receptor saddle cycle. Through *H+* passes a short piece of the outset curve of the saddle point on the left, representing the donor saddle cycle. How can we fill in the entire donor outset curve, connecting these short segments?

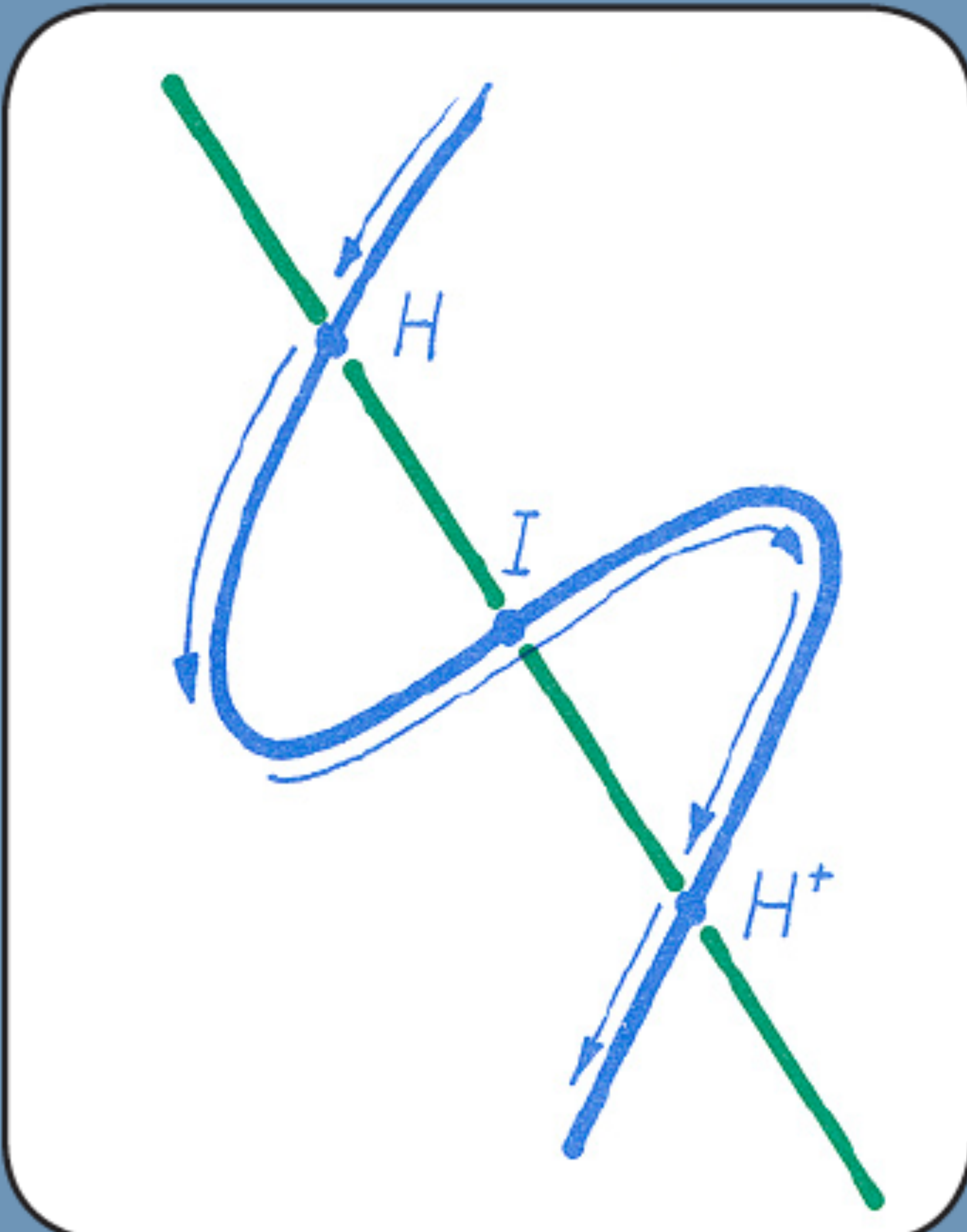


13.5.2.
The simplest solution might be just to connect up the loose ends, as shown here. Unfortunately, this does not work. The out-directions must connect properly, without conflict.

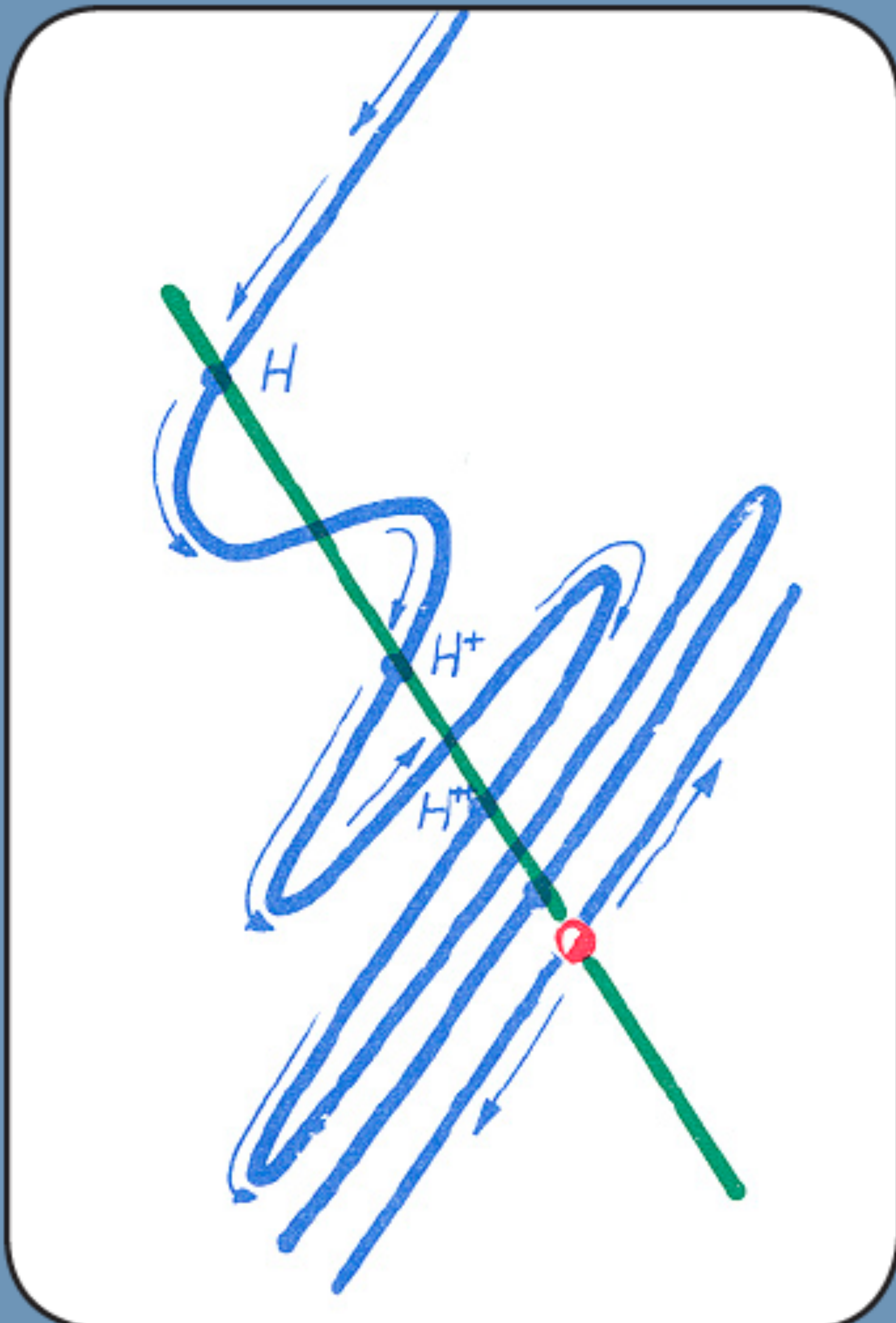


13.5.3.

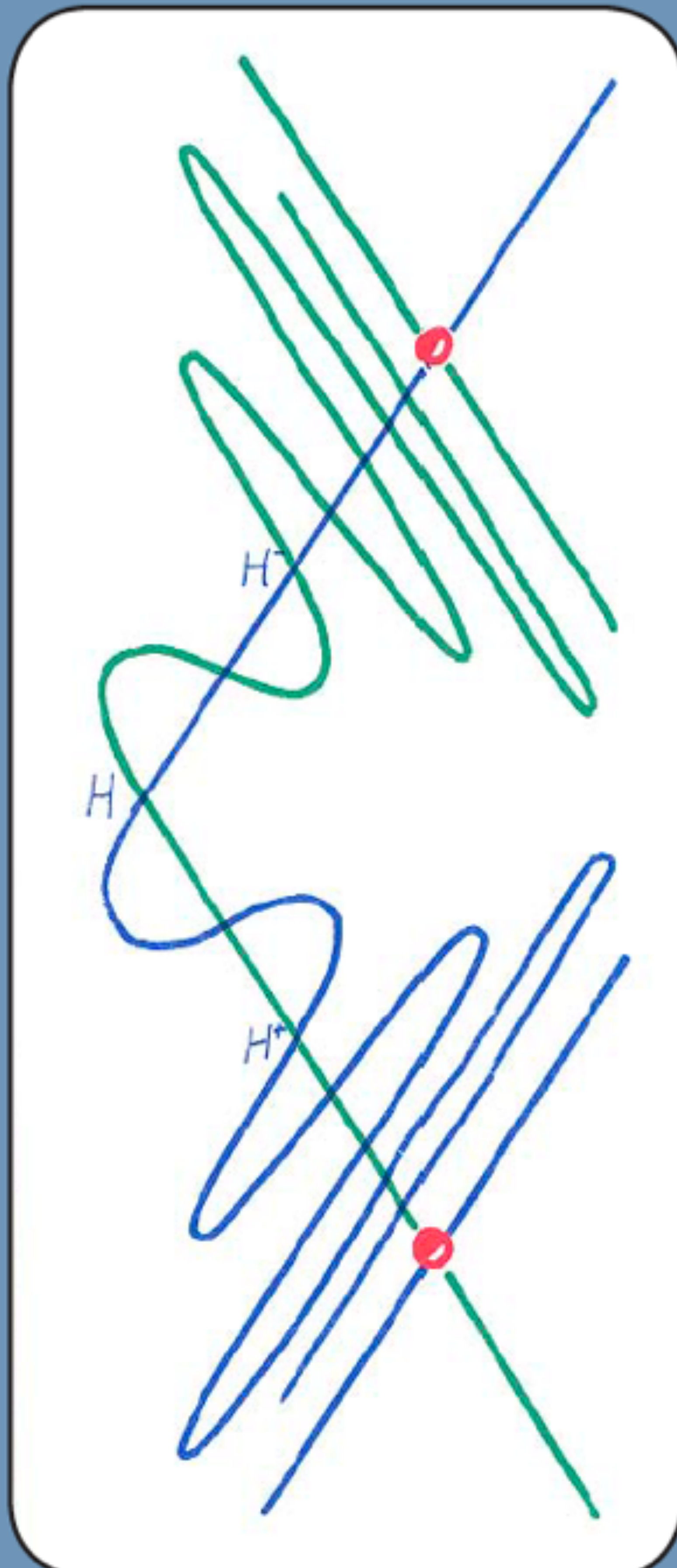
This drawing shows three possible connections for the outset curves, joining the short segments without conflict of the out-directions. The complete outset segment, joining two successive points corresponding to the same heteroclinic trajectory, *H* and *H+*, cuts through the inset segment joining the same two points in an odd number of points, all heteroclinic, but belonging to different heteroclinic trajectories. The two complete segments, joining *H* and *H+*, comprise the figure Birkhoff called the *signature* of the saddle connection.



13.5.4.
This shows the simplest possible Birkhoff signature. The odd number of interpolated heteroclinic points is only 1. This point, 1, represents another heteroclinic trajectory, sharing the same donor and receptor, and possessing its own signature (not shown).

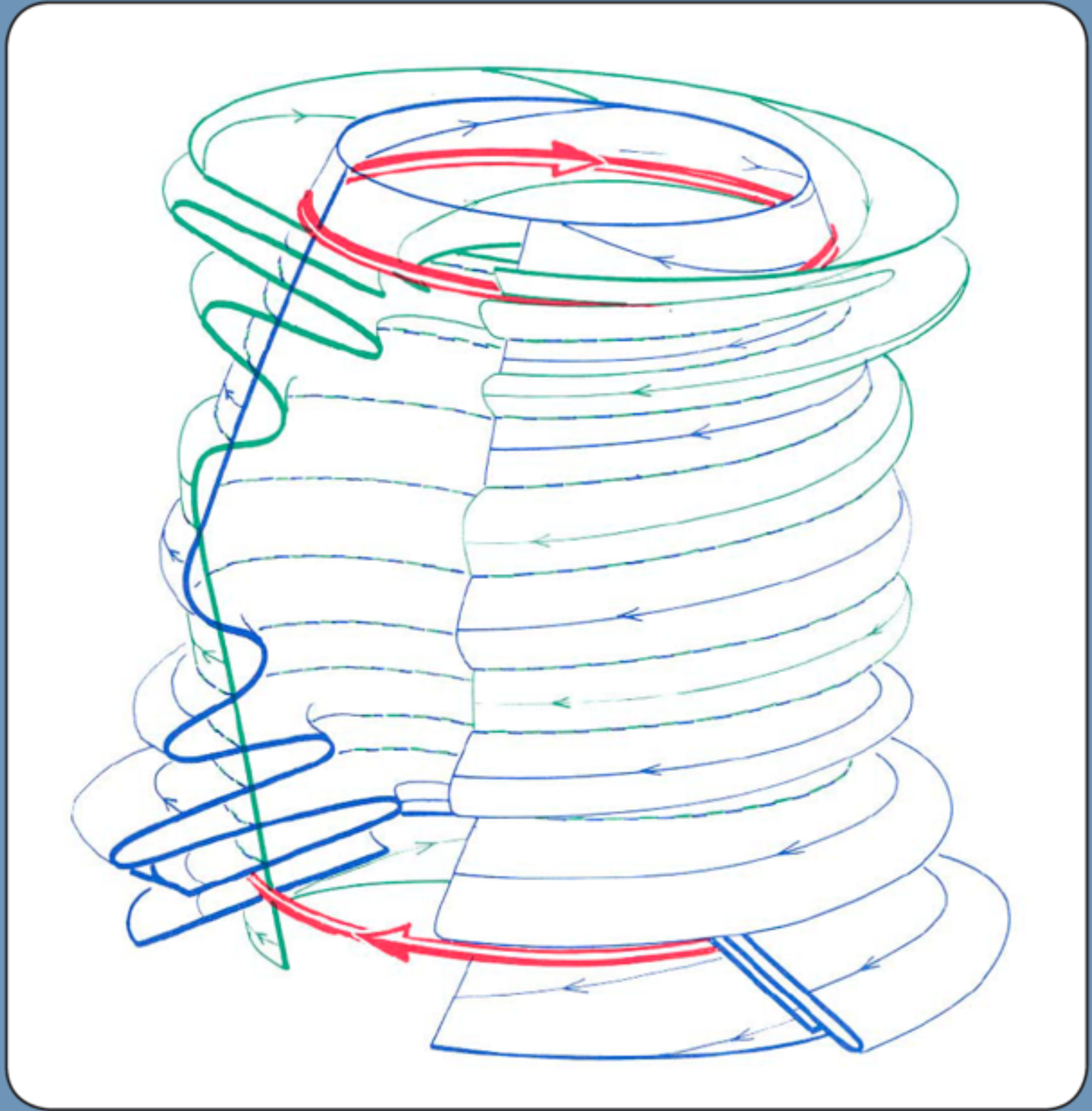


13.5.5.
Reinserting this Birkhoff signature into the starting picture of this section, together with two of its forward images under the first return map, we have a roughly complete idea of the donor outset. There are many possibilities for the future of the outset, but here we have used only the simplest signature, as shown in the preceding panel. In this case, there is an infinite sequence of points of intersection, *H*, *H+*, *H++*, … , all belonging to a single heteroclinic trajectory.



13.5.6.

Extending the receptor's inset backwards in time, we obtain the predecessor of *H*, *H-*, its predecessor, *H–*, and so on. This completes a doubly infinite sequence, *corresponding to one full heteroclinic* trajectory. Likewise, the interspersed heteroclinic trajectory contributes a complementary doubly infinite sequence *as shown here, in the Poincaré section*.

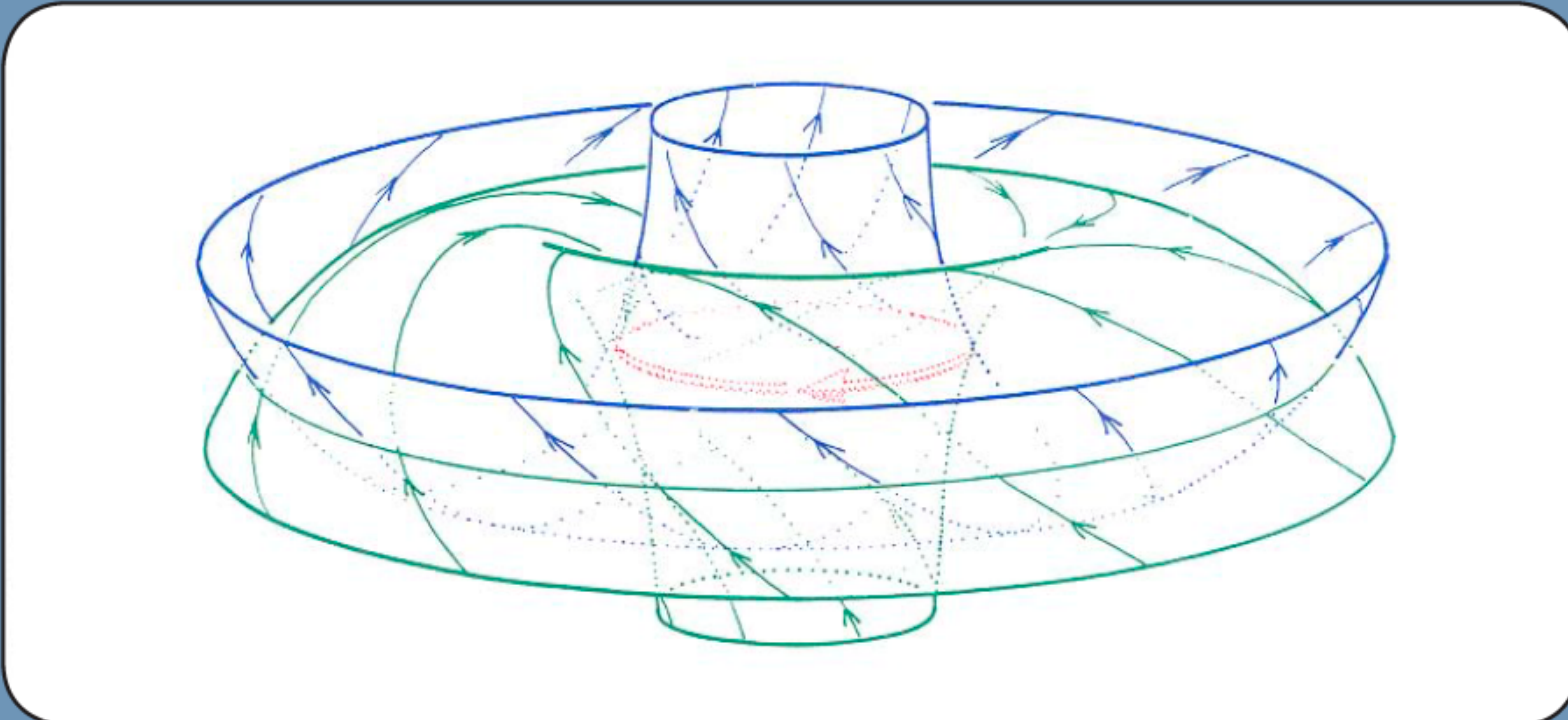


13.5.7.
The doubly infinite sequences each correspond to a heteroclinic trajectory of intersection of the donor's outset and the receptor's inset, in the original three-dimensional context. Here, the generic connection of saddle cycles in three dimensions is shown, with all its complex structure. A section has been removed here, for improved visibility.

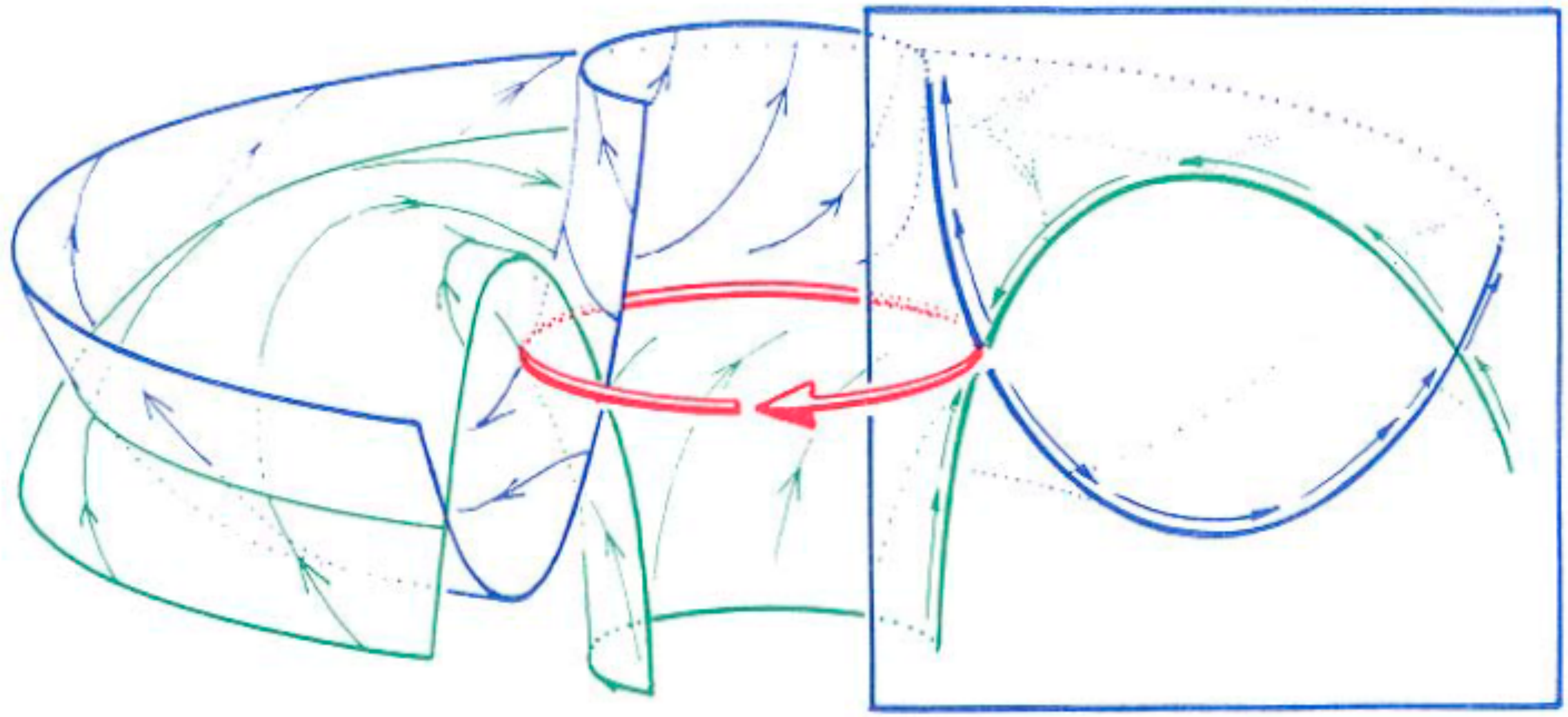
Ch. 14: Homoclinic Tangles

- Sec. 14.1. Homoclinic cycles
- Sec. 14.3. Horseshoes

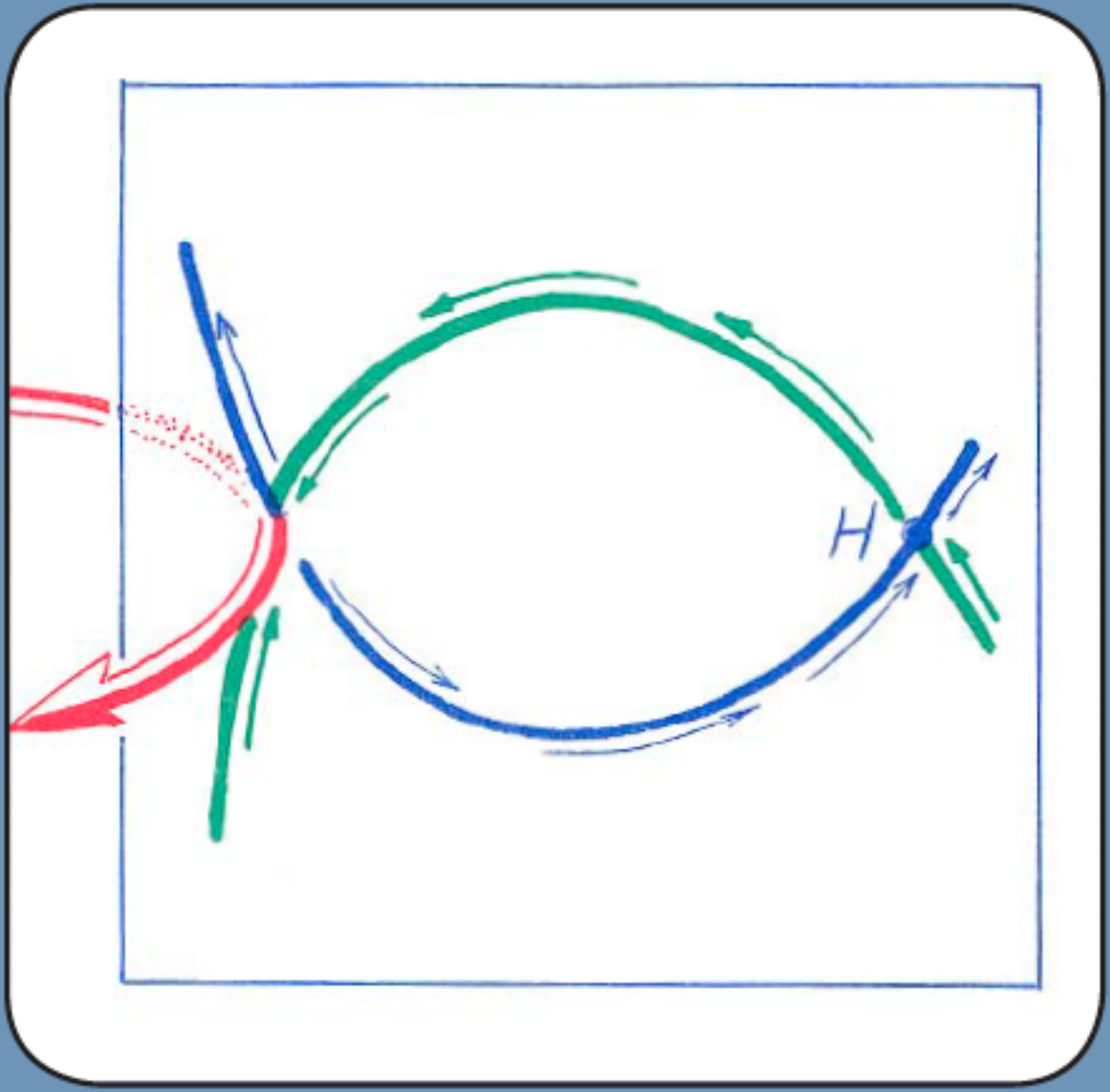
Sec. 14.1
Homoclinic Cycles



14.1.1.
Here the outset of the limit cycle, at the top, is pulled down like a sleeve turned inside out. The inset, below, is likewise pulled up. Then, they are pushed through each other, to produce the beginning of an extensive intersection.

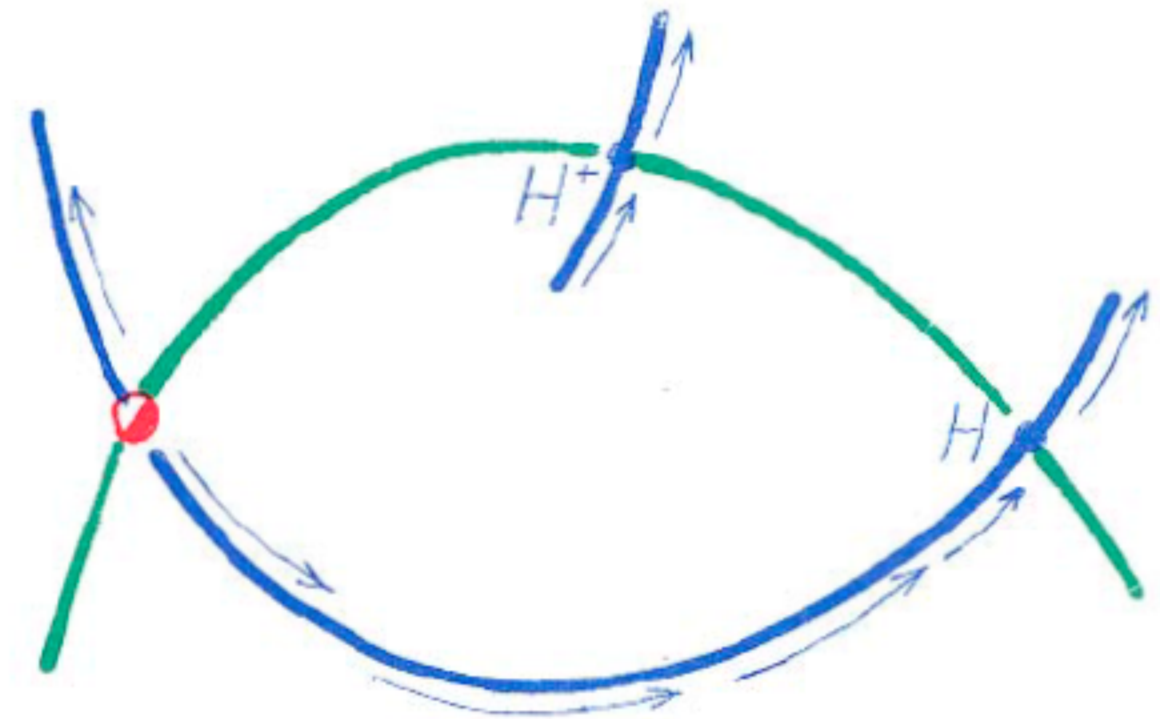


14.1.2.
To visualize the intersection, we cut through it with a Poincaré section. The procedure is the same as the heteroclinic case, described in the preceding chapter (see 13.4.2.).



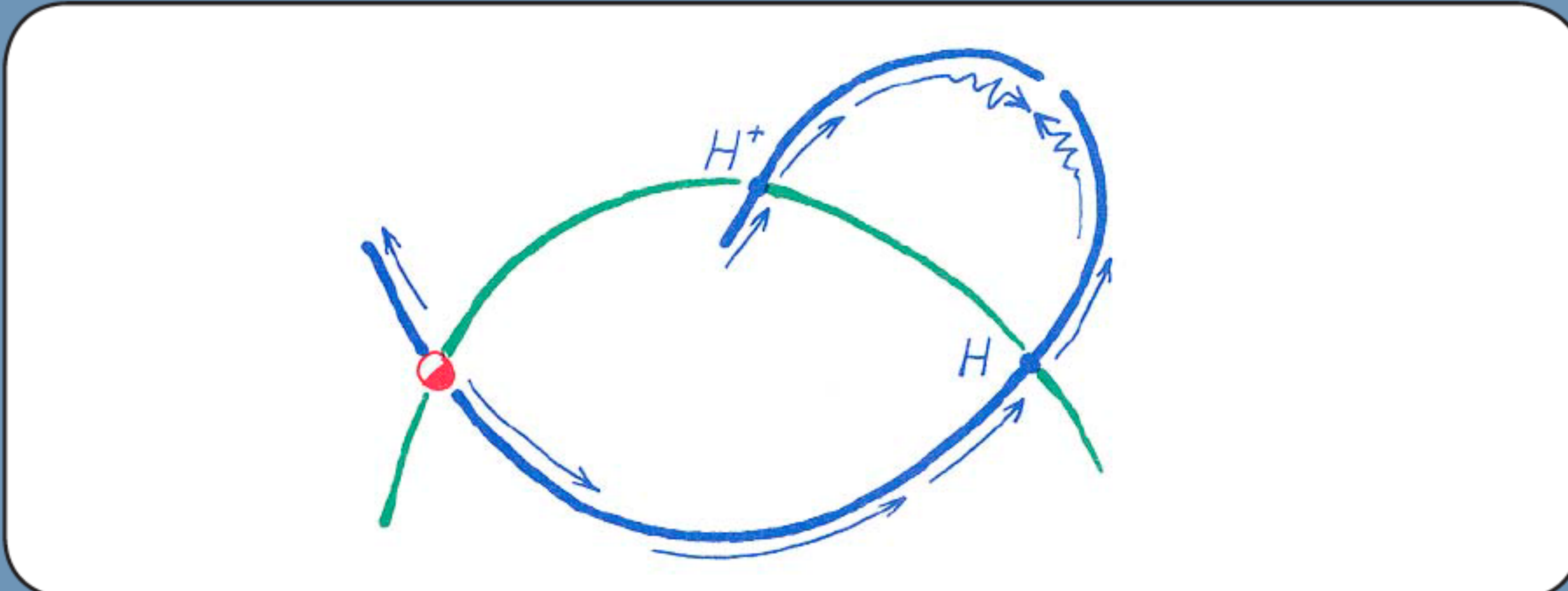
14.1.3.

As in the preceding chapter (see 13.5.1.), the outset surface of the receptor limit cycle (in this case, they are the same cycle) intersect the Poincaré section in two curves, the outset and inset curves. These curves intersect once at the point cut by the limit cycle (shown as a curved arrow here), and again at a point cut by the homoclinic trajectory, such as the homoclinic point *H*, shown here.

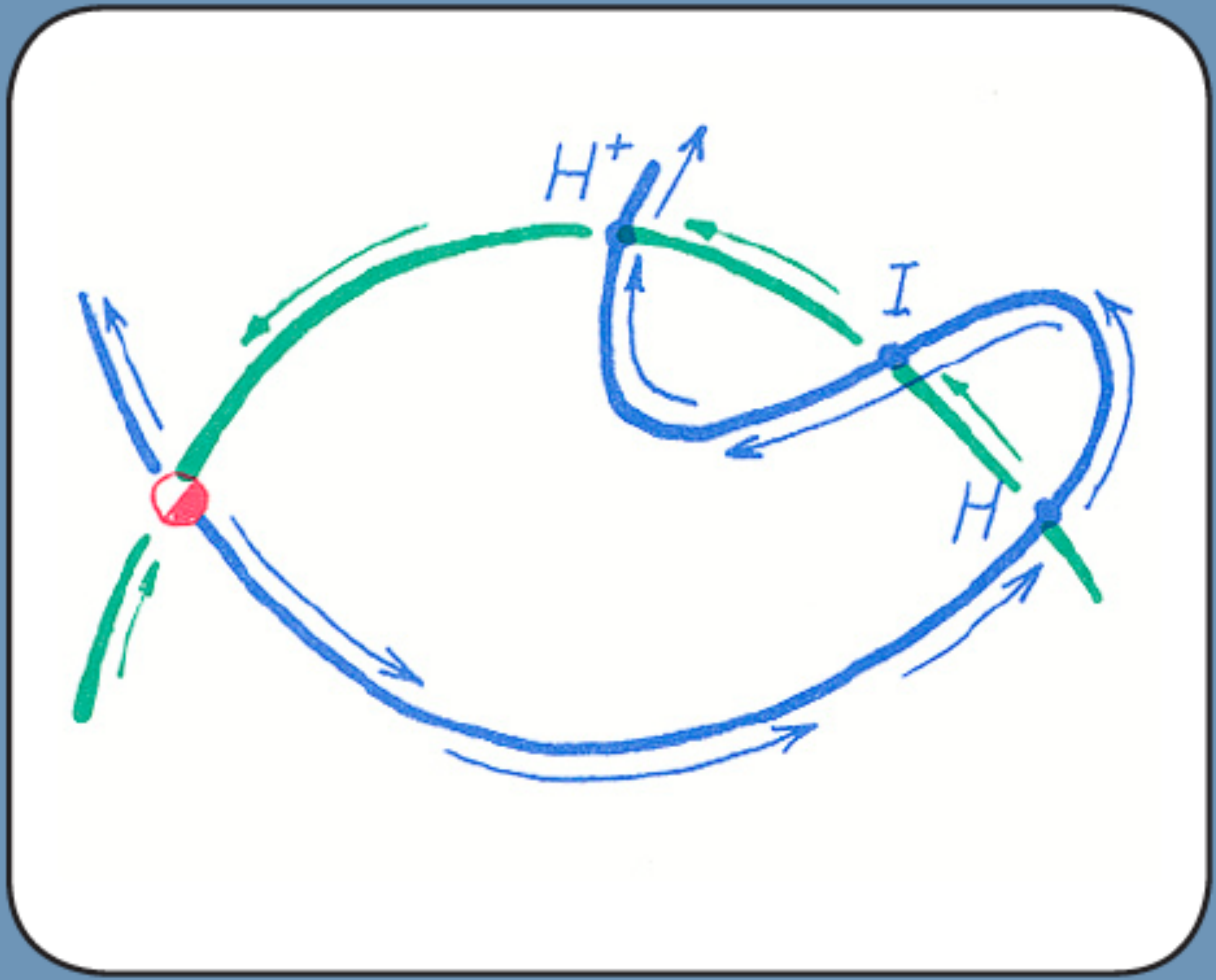


14.1.4.

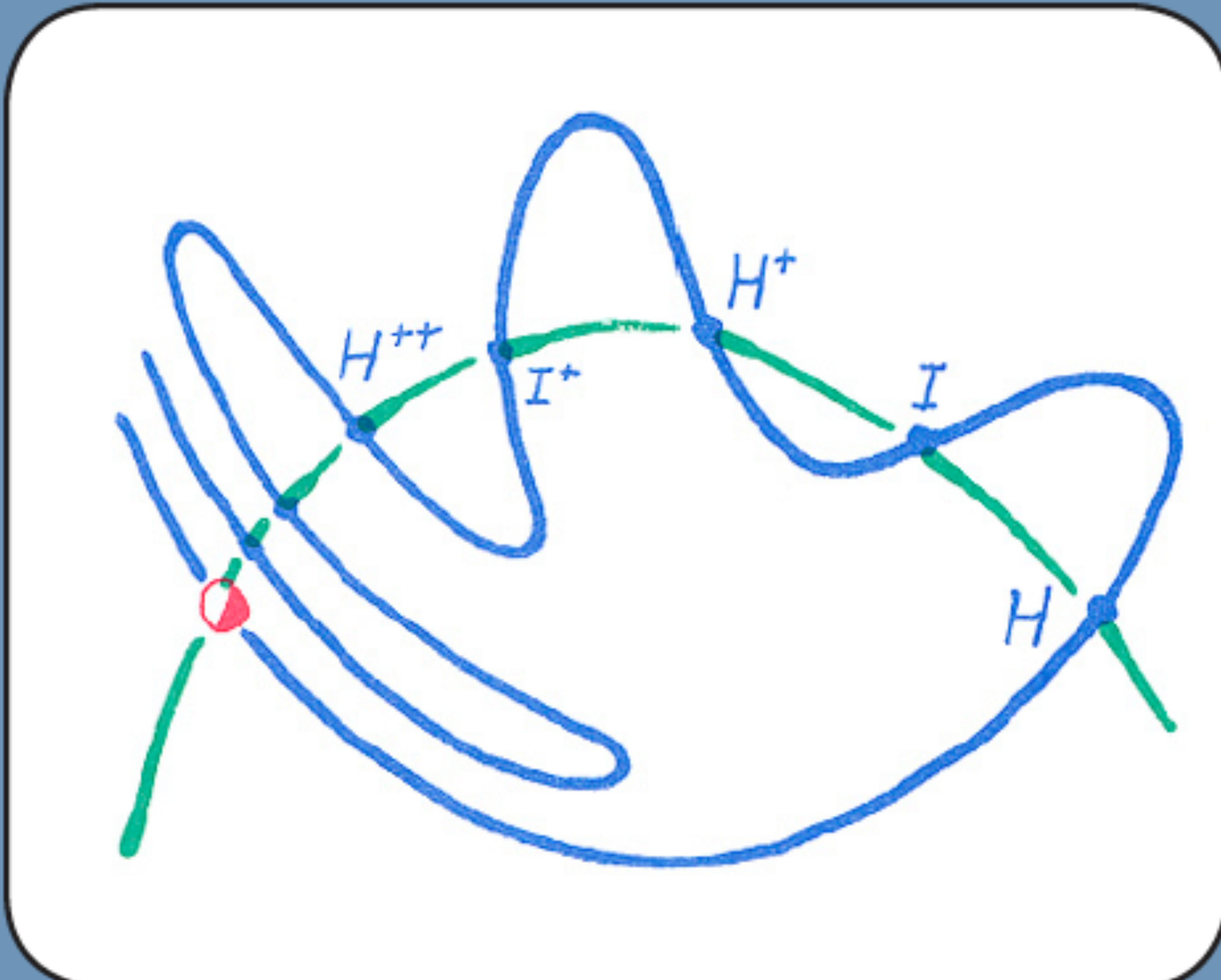
As in the heteroclinic case (again, see 13.5.1.), this point is mapped to another, *H+*, closer to the limit point. This image point is on the inset curve, as this curve is mapped into itself by the first return map. Further, this curve consists of all the incoming points. However, the image point must also be on the outset curve, which is also mapped into itself by the first return map, and which consists of all outgoing points. The homoclinic points, *H* and *H+*, are both outgoing and incoming, by assumption. Thus through the image point, *H+*, there must also pass a piece of the outset curve, shown here with its out-direction indicated by an arrow.



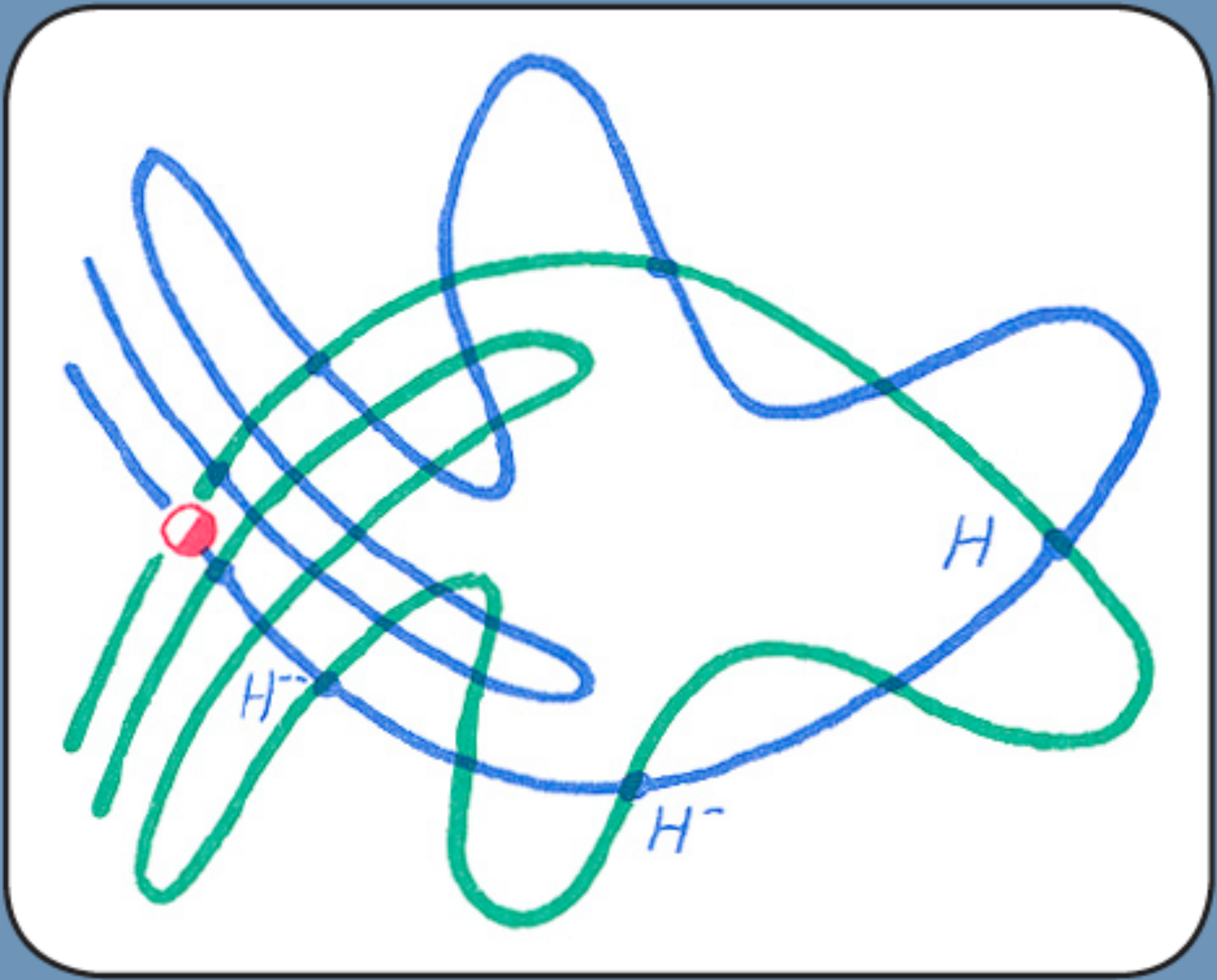
14.1.5.
As in the heteroclinic case (see 13.5.2.), direct connection leads to a conflict of out-directions. Thus ...



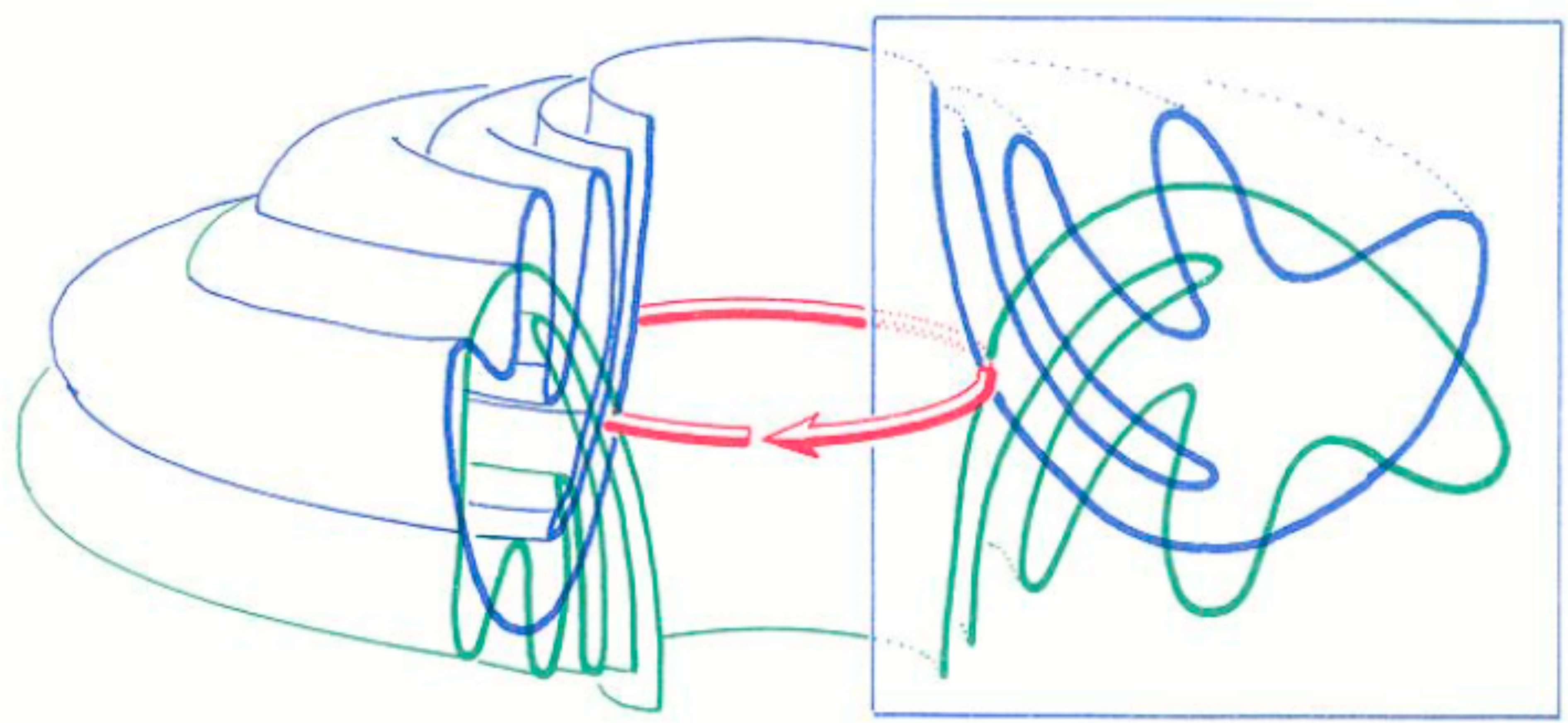
14.1.6.
… as in the heteroclinic case (see 13.5.3.), the outset segment from *H* to *H+* must cross the inset segment (between the same two points) an odd number of times. This is the simplest legal construction, illustrating the *Birkhoff signature* in the homoclinic case.



14.1.7.
Reiterating the first return map again and again, the outset segments push up against the inset curve, near the limit point.



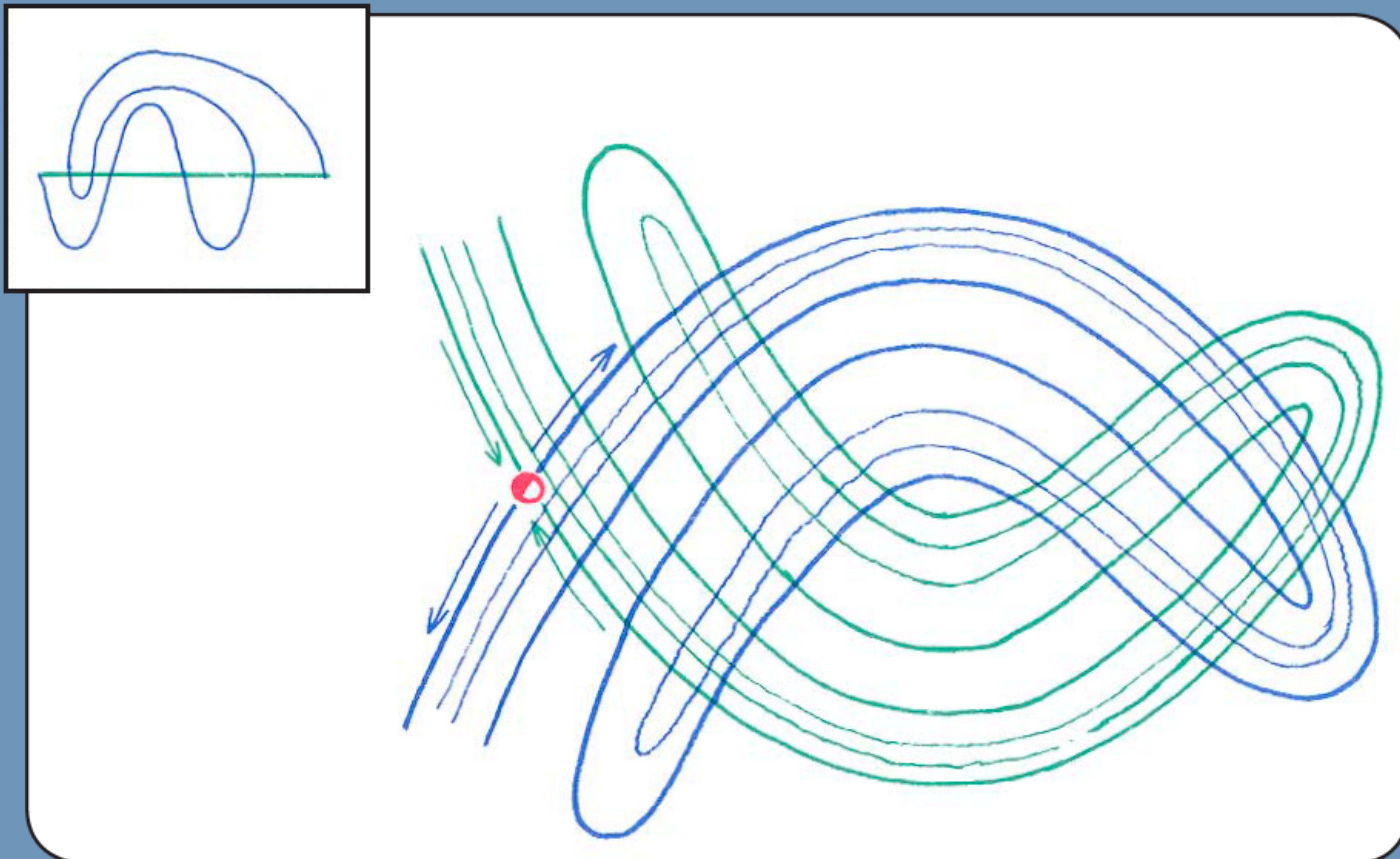
14.1.8.
Repeating the construction for negative times (iterating the *prior return map*), the inset segments pile up against the outset curve, again near the limit point. Thus, we obtain a full picture of the entire *homoclinic tangle*, as shown in this drawing of a tangle studied by Hayashi,[3] the greatest master of experimental tangle art.



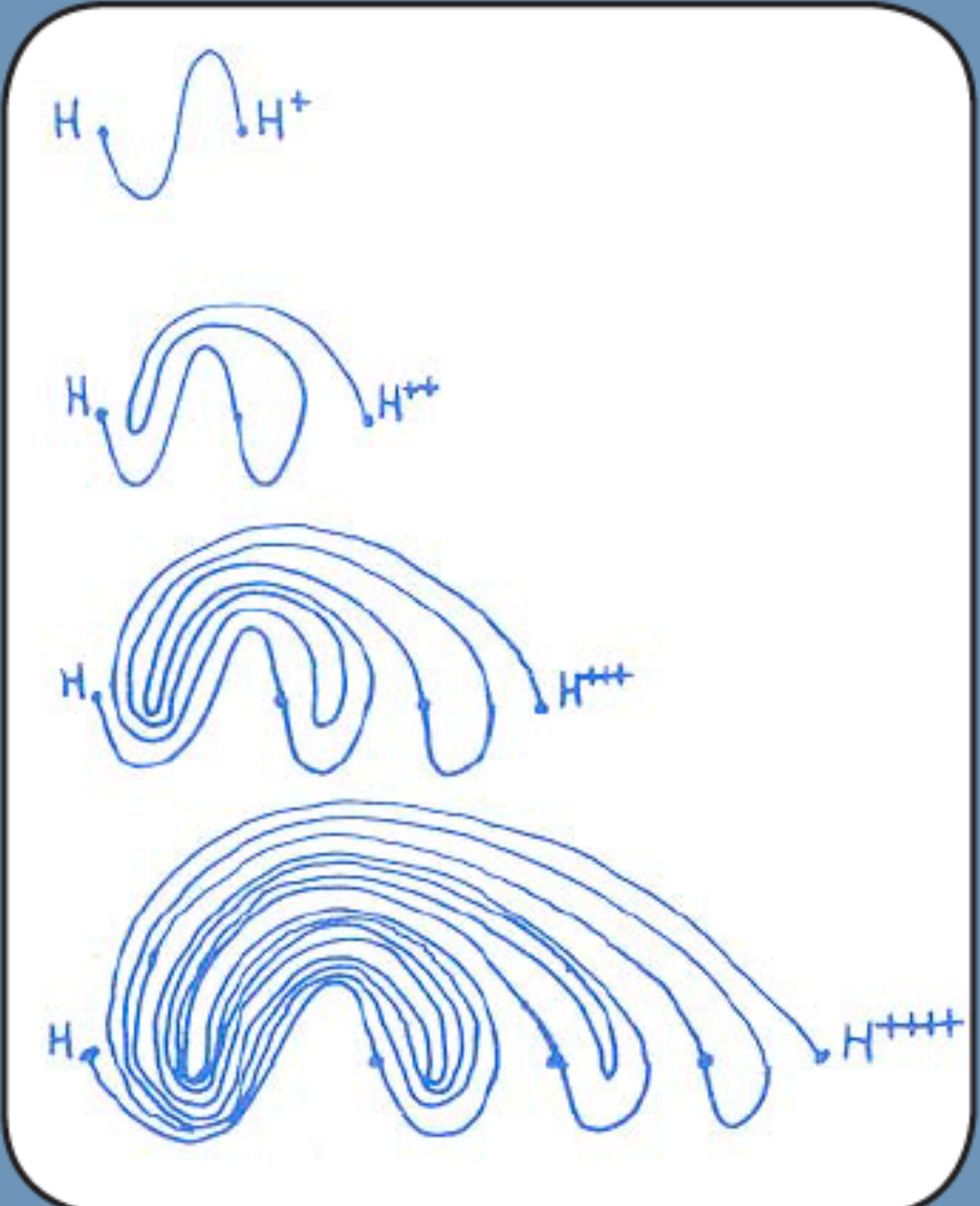
14.1.9.

Here is the tangle within the Poincaré section, replaced within the original 3D context (compare with 13.5.7.). The behavior of a nearby trajectory is a spiraling asymptotic approach, along the non-tangled half of the inset surface, followed by a period of chaotic motion, entrapped within the tangle, and finally a spiraling asymptotic escape, along the non-tangled half of the outset surface. Thus, the homoclinic tangle provides a model for *transient chaos*.

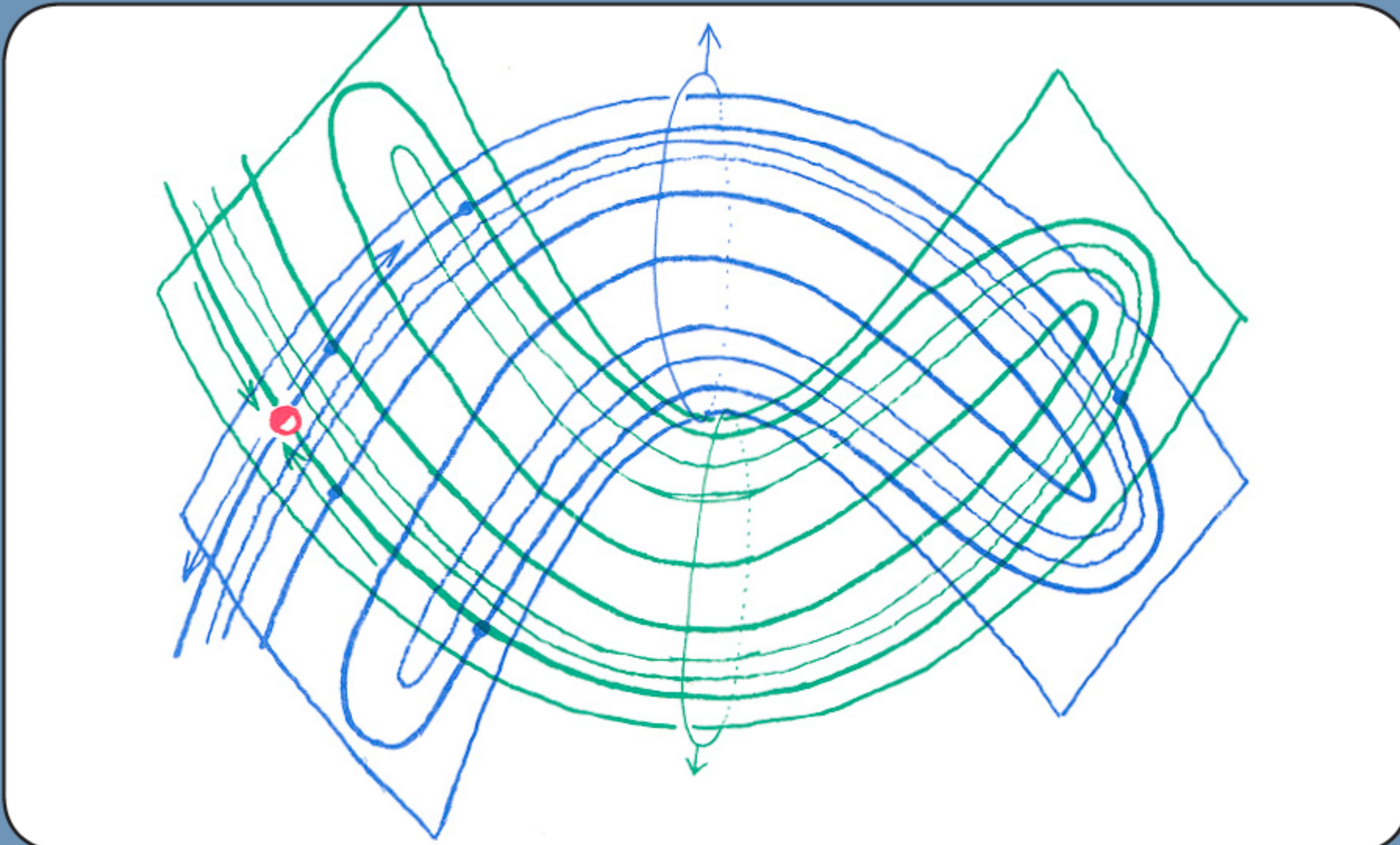
Sec. 14.3
Horseshoes



14.3.1.
Here is yet another homoclinic tangle, the famous *horseshoe of Smale*. Note that the first signature is the familiar simplest one. But in the second signature, shown in the inset, the hump has been twisted back, creating two new intersections. To further characterize this tangle, we must draw the *third signature.*

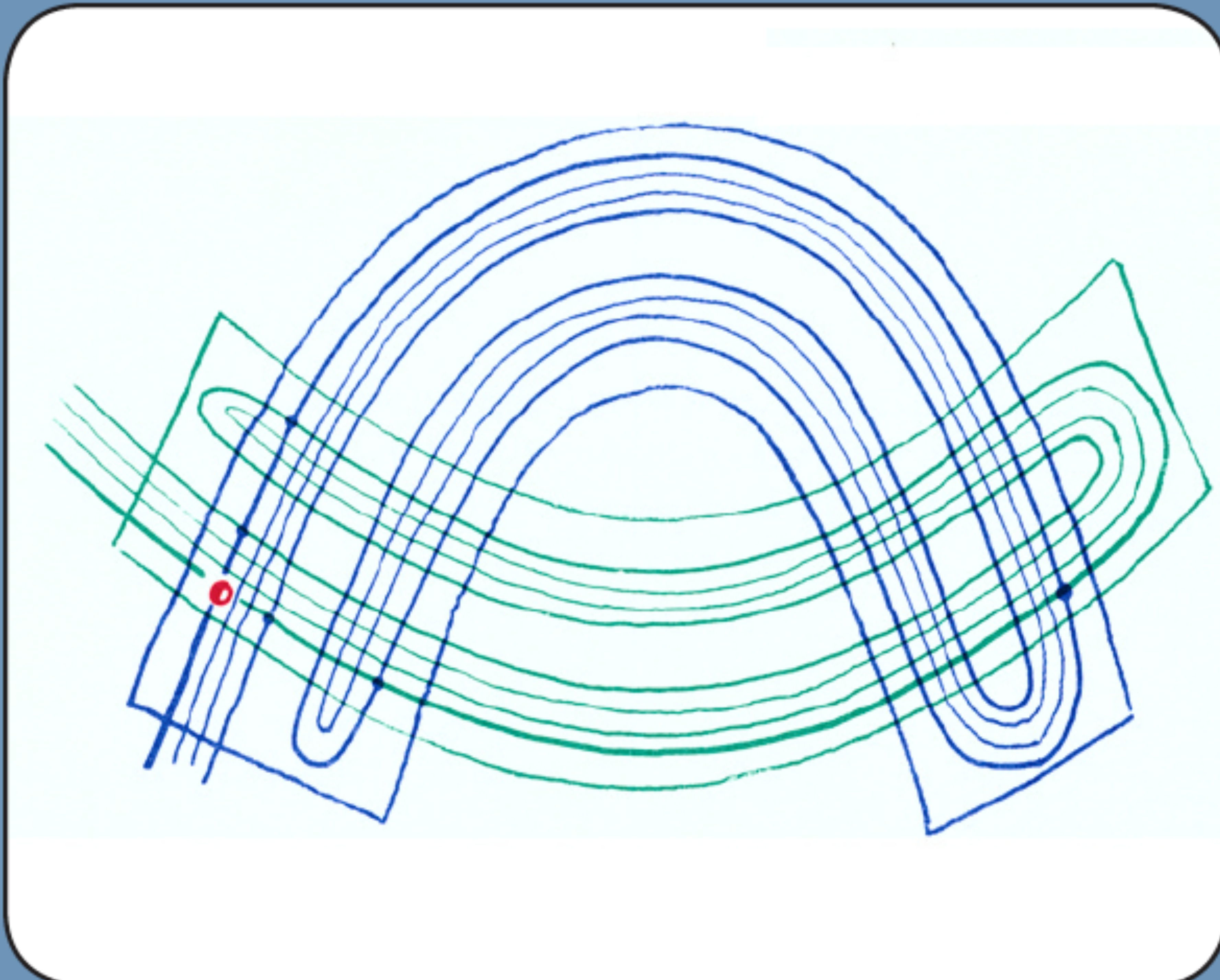


14.3.2.
Here are the first four signatures of our signature sequence, for Smale's horseshoe. The third signature is not identical to the third signature of the preceding example (try it and see).

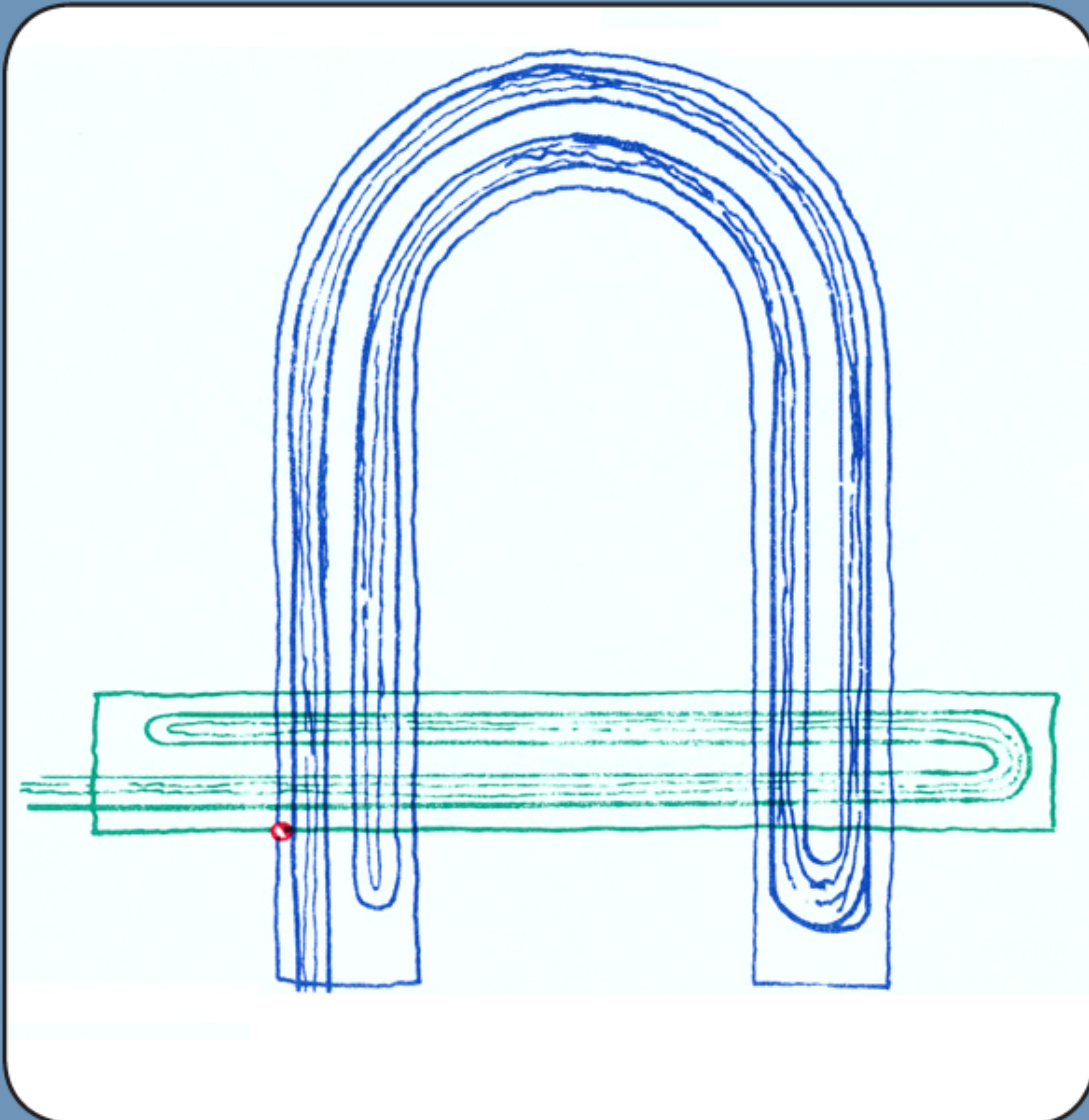


14.3.3.

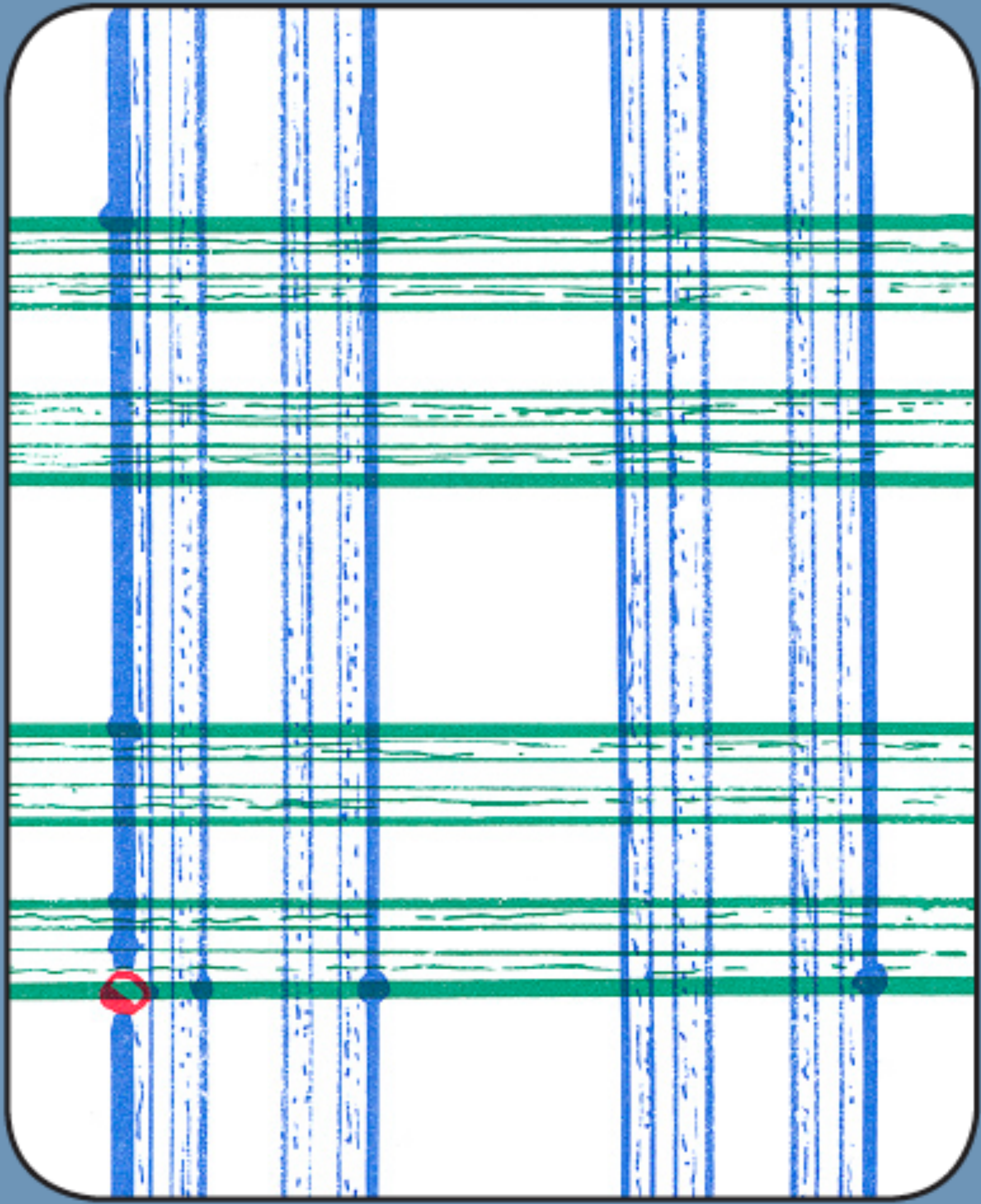
The horseshoe has been untangled by Smale[5] in a most ingenious way. Choosing a curved rectangular patch in the Poincaré section with some care, and applying the first return map yields another rectangular patch crossing the original patch at each end. Now, deform the whole picture by lassoing the two patches around the waist and pulling gently.



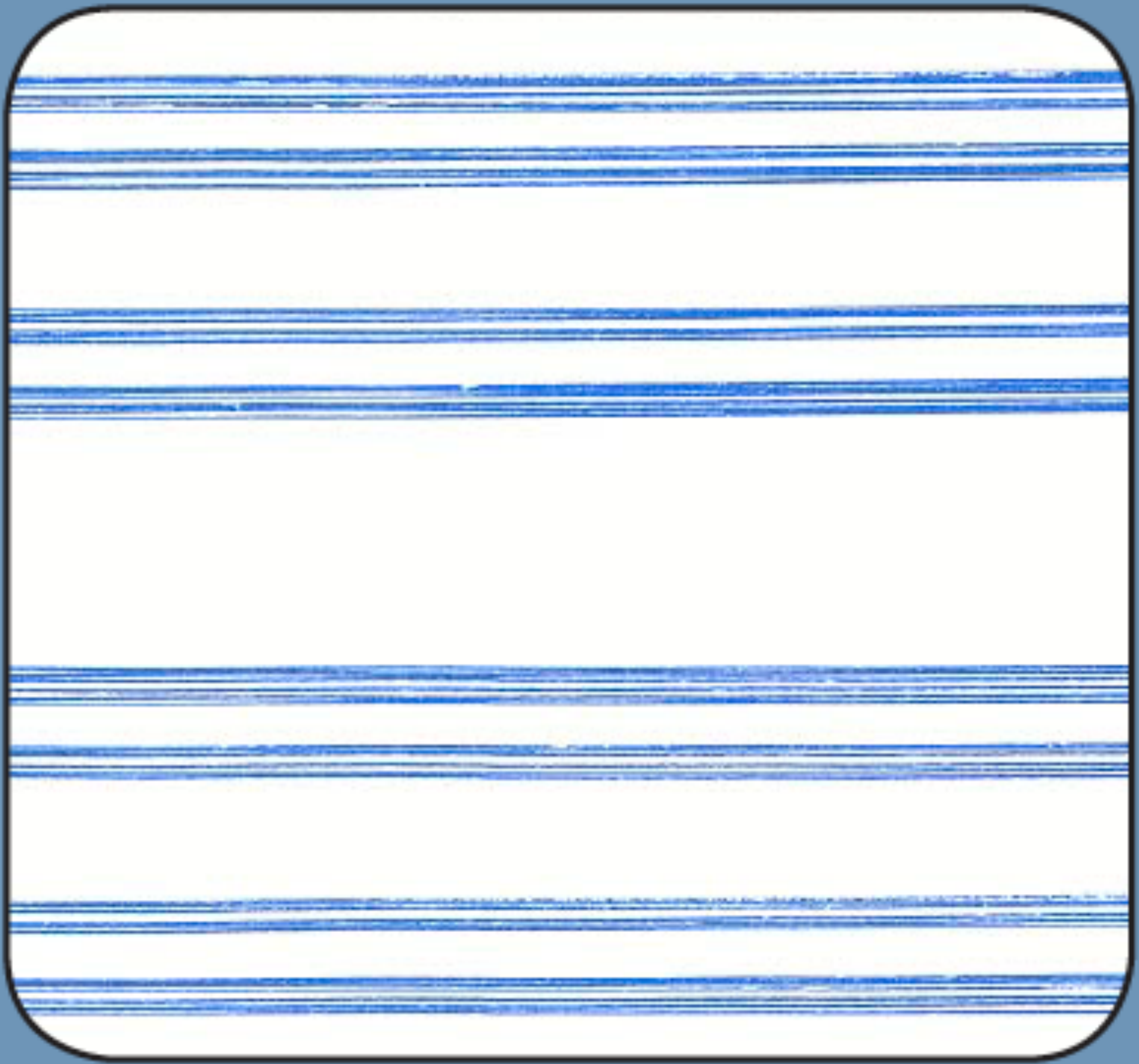
14.3.4.
Continue to pull the upper patch upwards by the waist, while pushing down on the ends. The idea is to straighten out the lower patch.



14.3.5.
There is the fully untangled tangle, the horseshoe of Smale. It is topologically equivalent to the messy original tangle, yet it admits a full analysis, as shown by Smale.



14.3.6.
The analysis is based upon a clever scheme for labeling all the points of intersection of the insets and outsets within the Poincaré section.

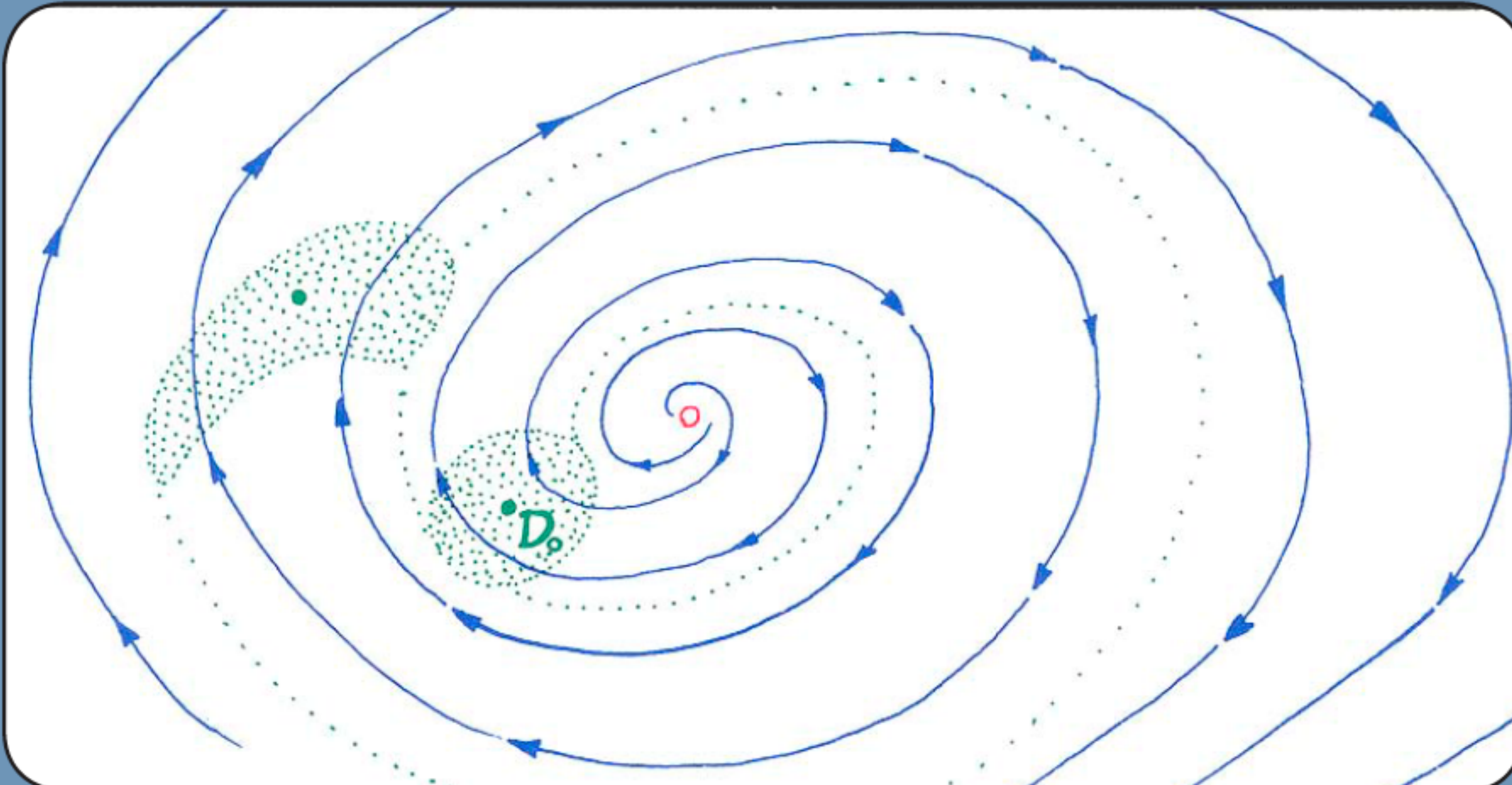


14.3.7.
Looking at a portion of the outset through a microscope, we see an infinite set of horizontal lines. Their intersection with a vertical line (such as the left edge of the box here) is much like Cantor's middle thirds set (see Figure 9.4.7.).

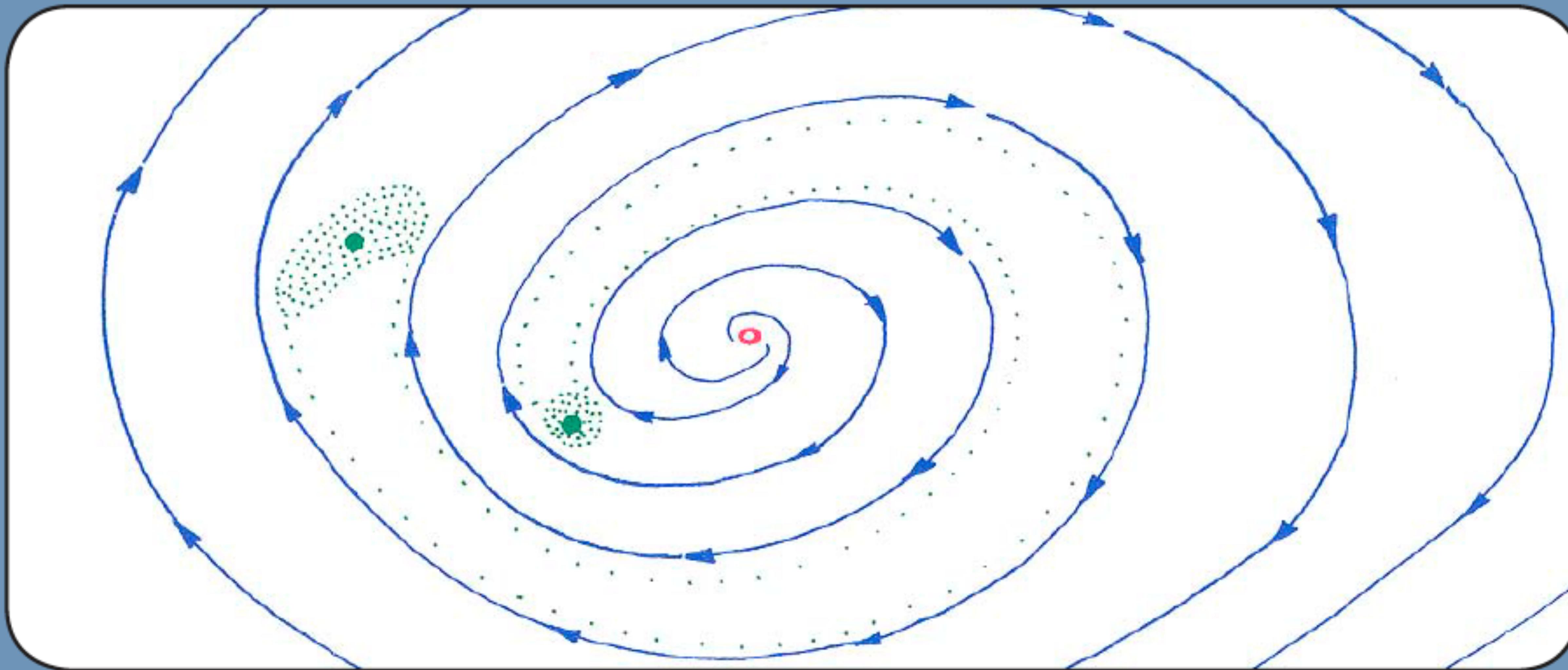
Ch. 15: Recurrence

- Sec. 15.3. Nonwandering points

Sec. 15.3
Nonwandering Points



15.3.1.
Suppose, having picked out a point in the state space and a little disk centered on it, that we follow the future meandering of the entire disk. If wide enough, it may meet up with itself along its meander.



15.3.2.

If so, start with a smaller disk, and repeat the contruction. If now the meandering disk leaves its original position, wanders away, and never returns to overlap its original position, then the original point at the center of the disk is called a *wandering point*.

Math 145 Spring 2017 Meeting #5Th

End