Math 145
Chaos Theory

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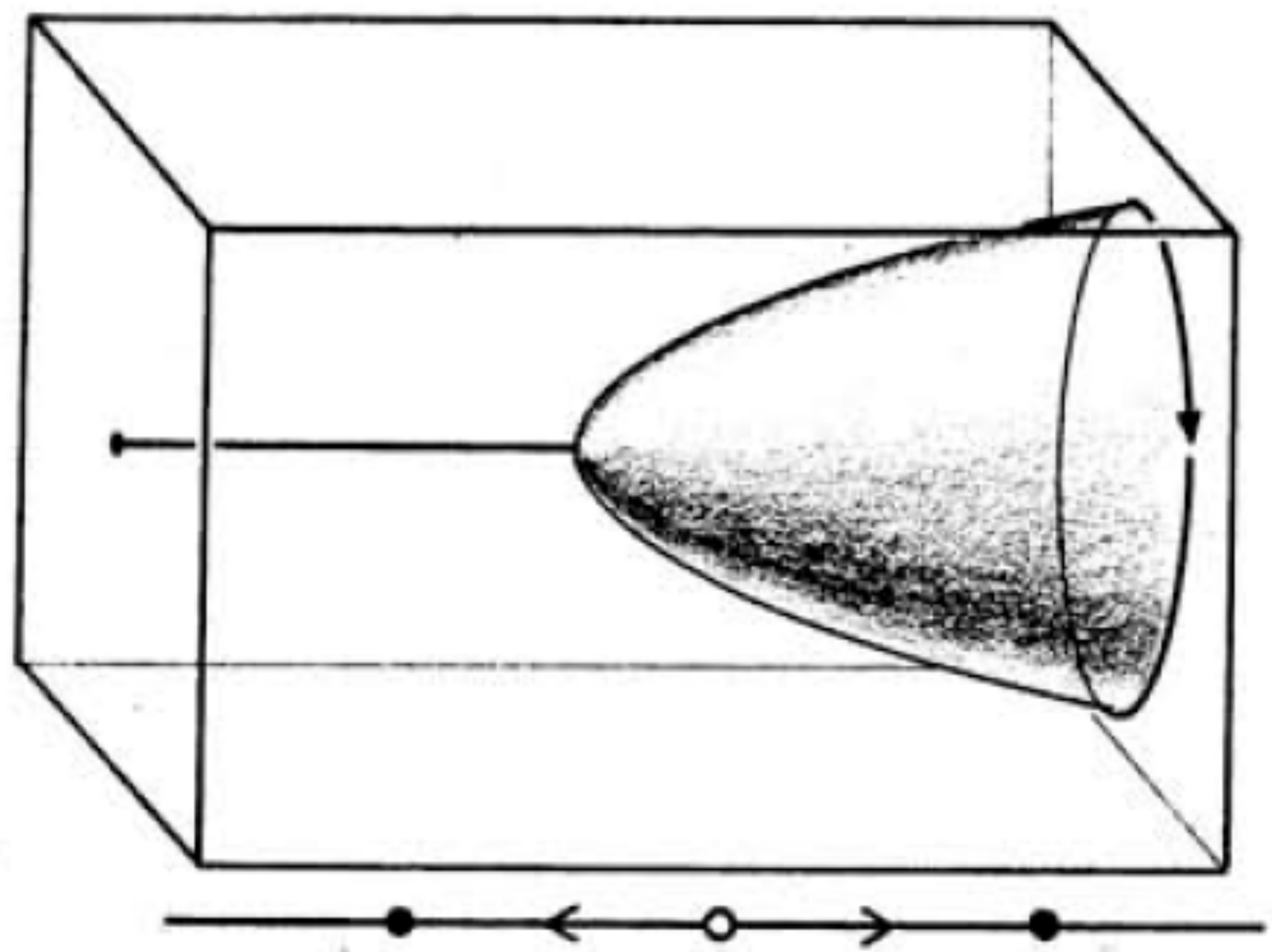
- Complex Dynamical Systems
  - Self-regulation
  - Guidance
  - Serial coupling
  - Examples

Self-regulation

FIGURE 66. SELF-REGULATION OF ONE CONTROL. Given that we have a dynamical system depending on a control parameter, who turns the control knob? If it turns itself, through action of a dynamical subsystem on the control space itself, the model is called a self-regulating system. Here a typical dynamical system with one control, producing a Hopf bifurcation, is shown with a generic self-regulation vectorfield on the one-dimensional control space. Because this AB portrait on the line has only two point attractors, their basins separated by a point repellor, the control knob will seek one of these two attractive settings.



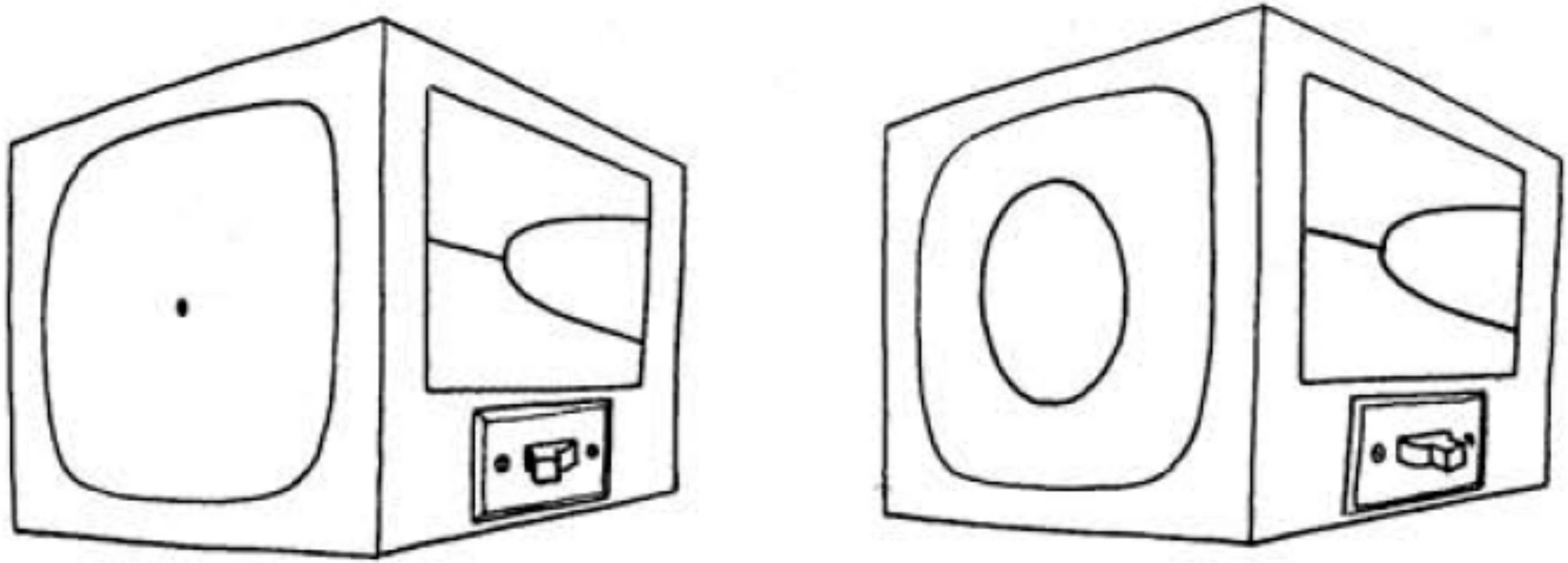


FIGURE 67. A MACHINE REPRESENTATION of the self-regulation system described above is this "electric stopwatch."

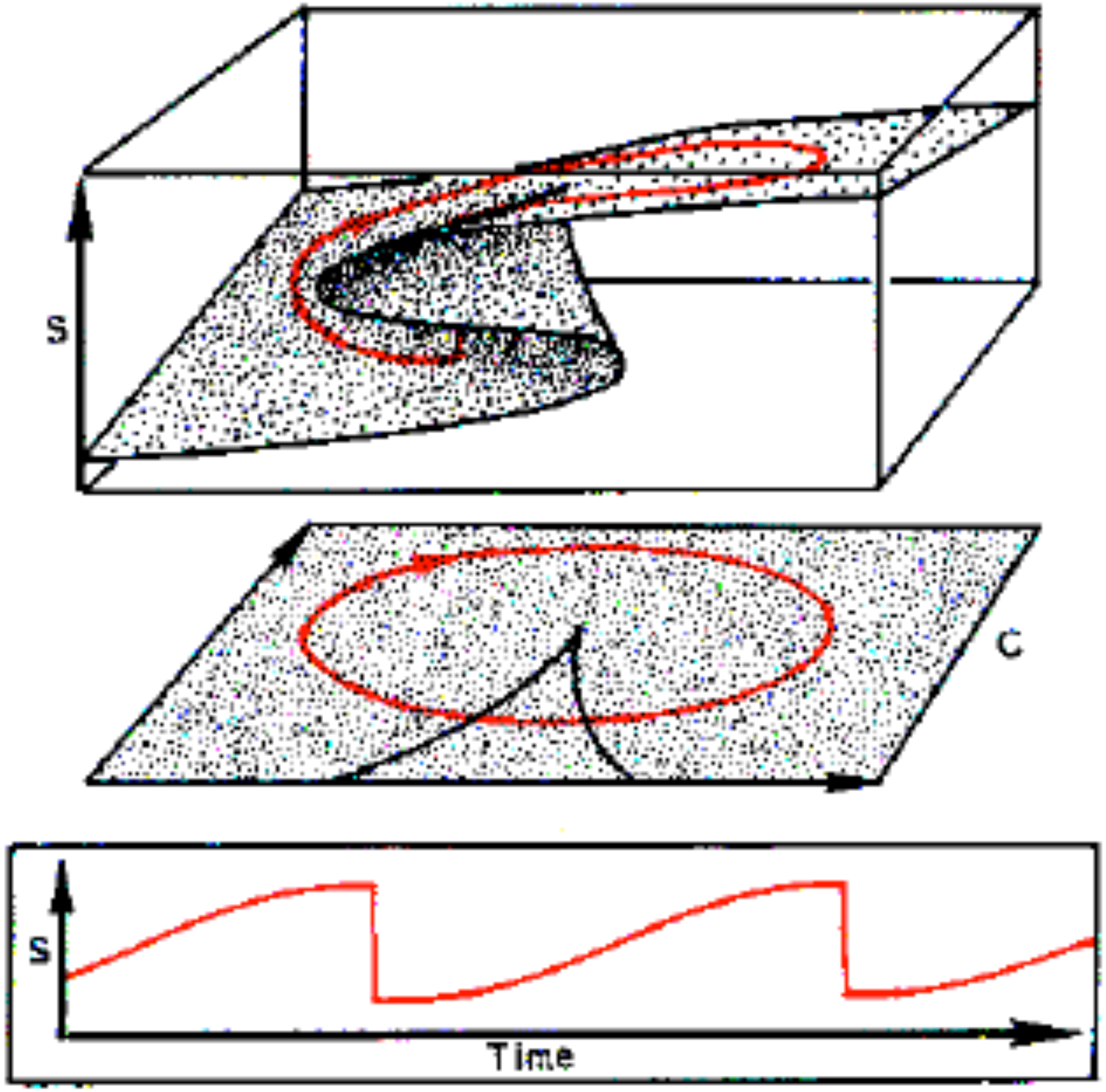


FIGURE 68. SELF-REGULATION OF TWO CONTROLS. Here a simple dynamical system with two-dimensional control space, the cusp of Figures 62 and 63, is endowed with a self-regulation vectorfield. This particular system has a single periodic attractor in the control space, *C*, with a virtual separatrix (a point repellor) within it. We always assume in these contexts that *the separatrices of the self-regulation dynamic are in "general position" with respect to the bifurcation set of the controlled dynamic*, in the sense that their intersection is as small as possible. The behavior of this system, introduced by Zeeman (1977) in a study of the electrical activity of neurons, is shown in the sawtooth-like time series recording the motion in the one-dimensional state space, *S*.

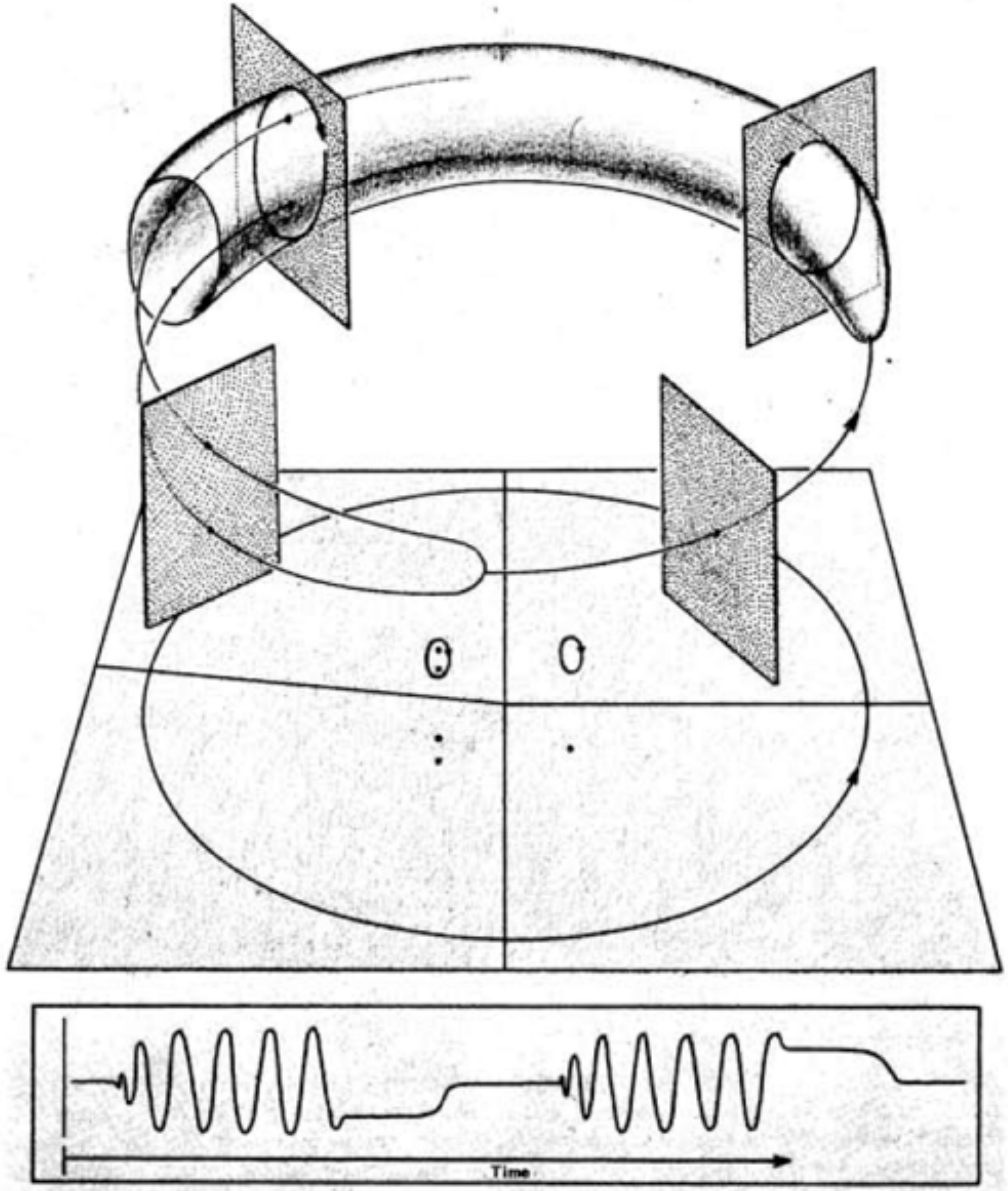


FIGURE 69. SELF-REGULATION WITH PERIODIC BEHAVIOR. The preceding example had a periodic attractor for its controls, but only a point attractor for its state dynamic. We show here a richer example, with the same control space and self-regulation dynamic, but with a state space of two dimensions. The bifurcation diagram is that of Andronov and Takens, shown in tableau form in Figures 64 and 65. Here we show the actual bifurcation diagram, obtained when the controls are restricted to the values along the periodic attractor on the control space. The time series corresponding to one of the two state variables also is shown. It resembles that of a stringed instrument rhythmically plucked. The direction in which it is plucked varies from cycle to cycle.

Guidance Systems

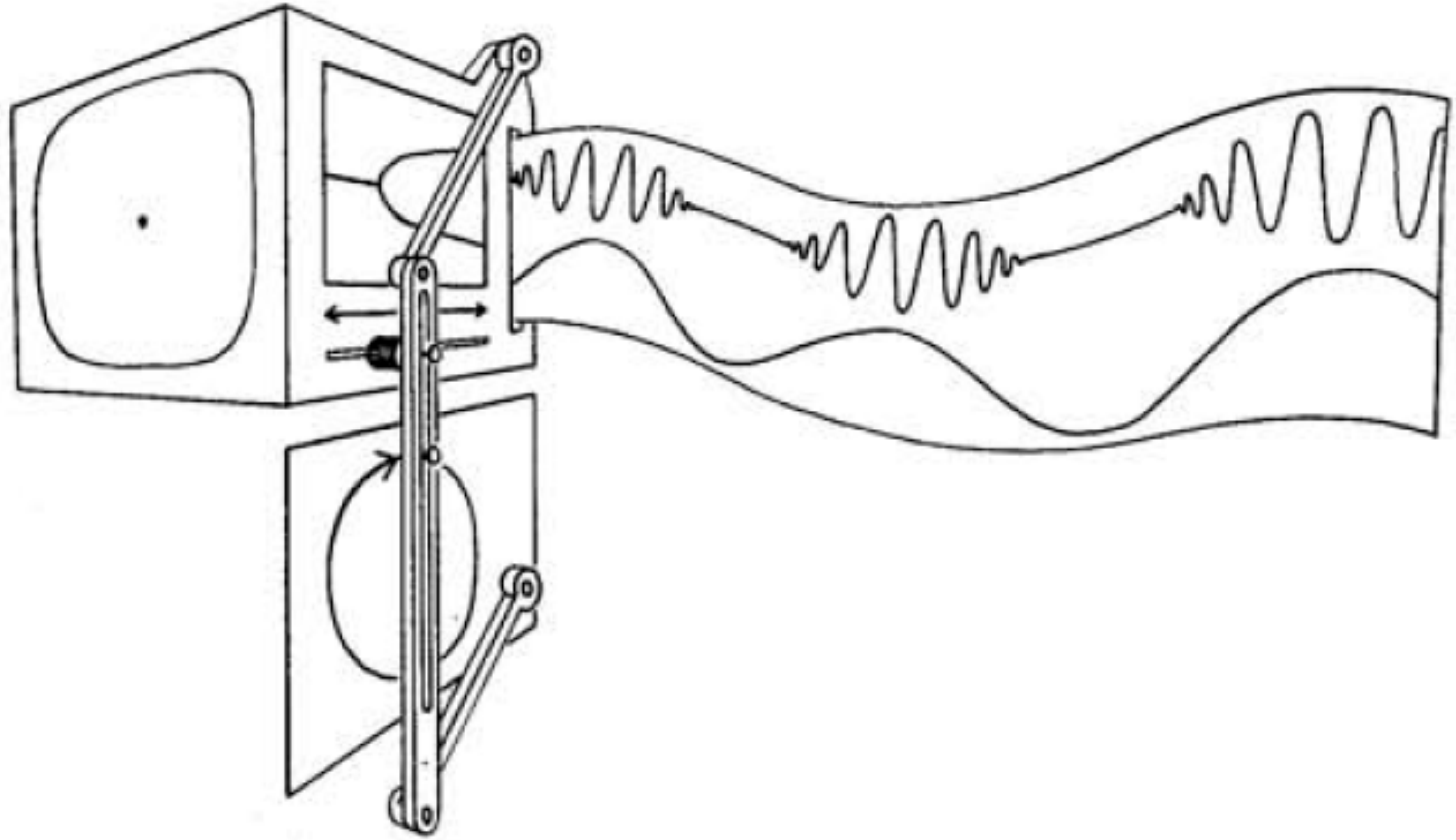


FIGURE 70. GUIDANCE SYSTEMS FOR ONE CONTROL. The self-regulation system is too limited for many applications, as this example shows. In this figure a dynamical system with one knob is controlled by a regulating vectorfield on an auxiliary space of two dimensions. A smoothly periodic variation of the control knob therefore is possible. In this example the controlled dynamic is again a Hopf bifurcation diagram, and the output time series of one of the two state variables is an intermittent oscillation. A similar system, described by Crutchfield, emits intermittent noise.

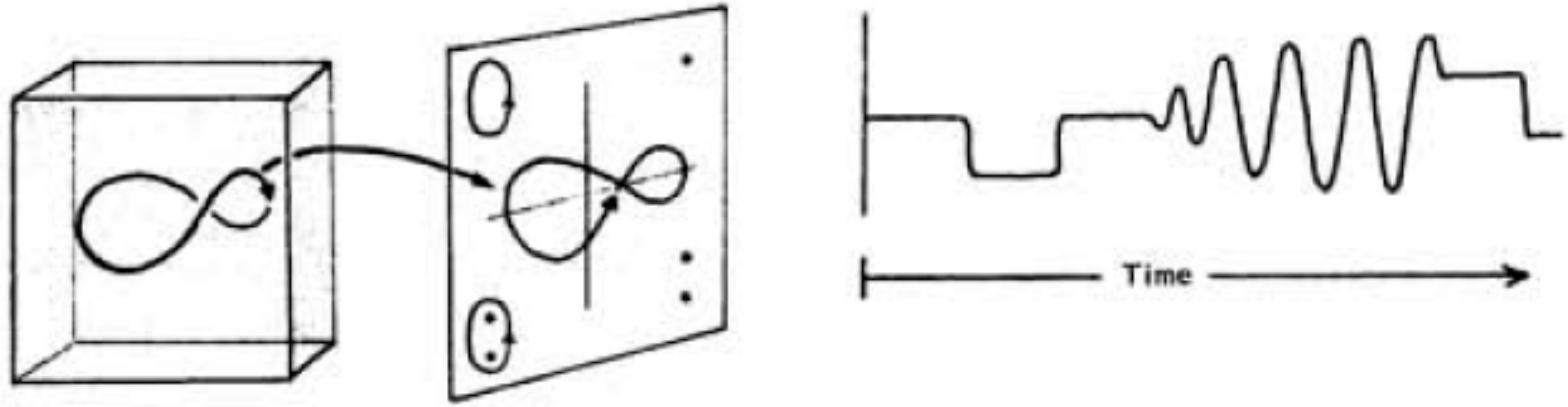


FIGURE 71. GUIDANCE SYSTEMS FOR MORE CONTROLS. By guidance system we mean a regulation vectorfield on an auxiliary space, B, and a *linkage map* from B to C, the control space of a dynamical system with controls, on a state space, S. Here is an example in which B, C, and S all have dimension two. The regulation dynamic on B has a periodic attractor, but the projection of this cycle from B into C by the linkage map is a loop that crosses itself. We shall call the projection of an attractor, in general, a *macron*. The controlled dynamic illustrated here is again the Andronov–Takens system of Figures 64, 65, and 69; the output is the time series shown. This is a periodic sequence of square waves and intermittent oscillation. Many other possibilities can be constructed with a guidance system as simple as this one.

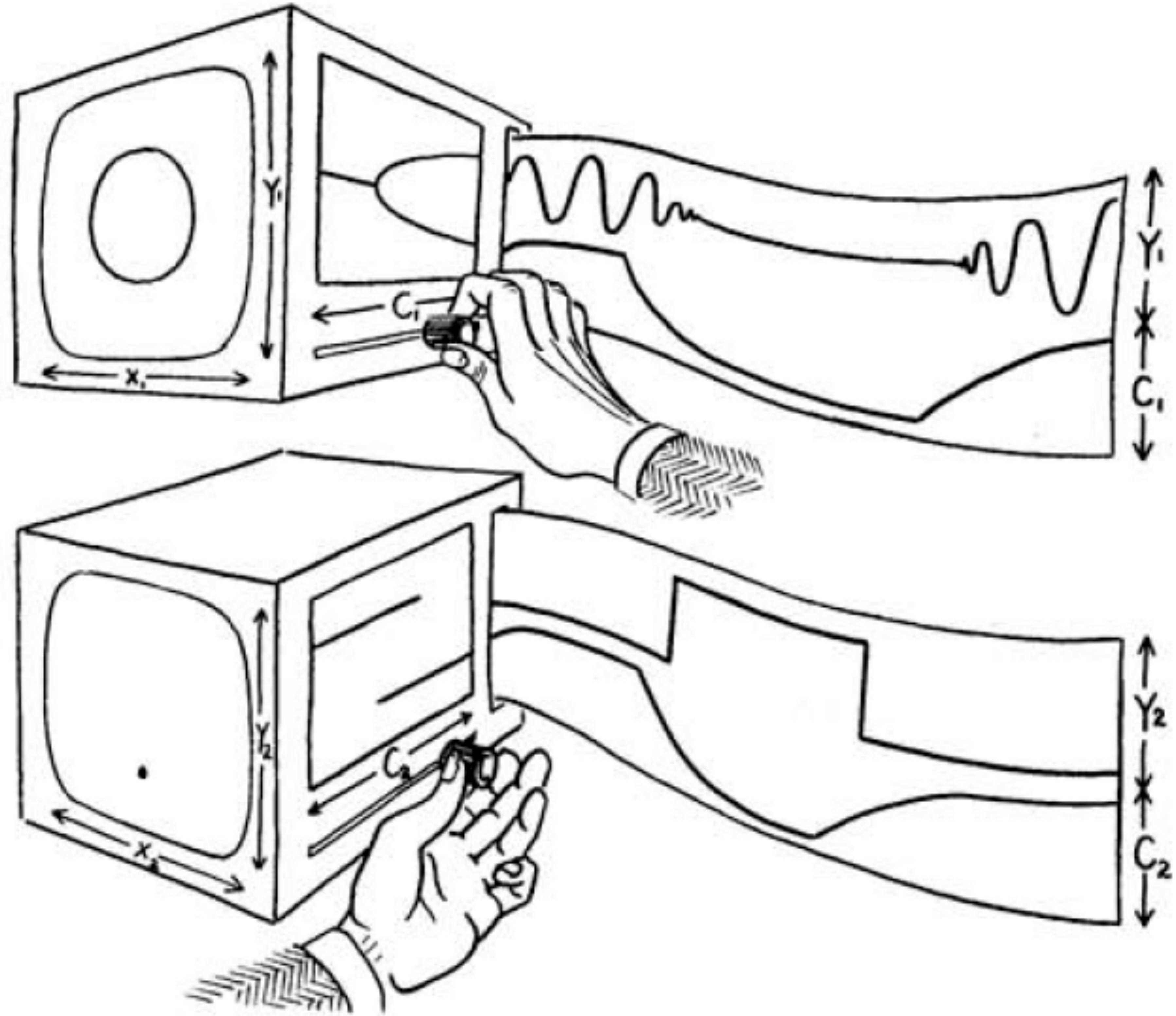


FIGURE 72. GUIDANCE WITH CONTROLS. In our sequence of portraits of gradually increasing complexity, we now consider two dynamical systems with controls. For example, let one be a subtle Hopf bifurcation (Figures 48–50) and the other a catastrophic kink exhibiting hysteresis (Figure 46, 47).

Serial Coupling

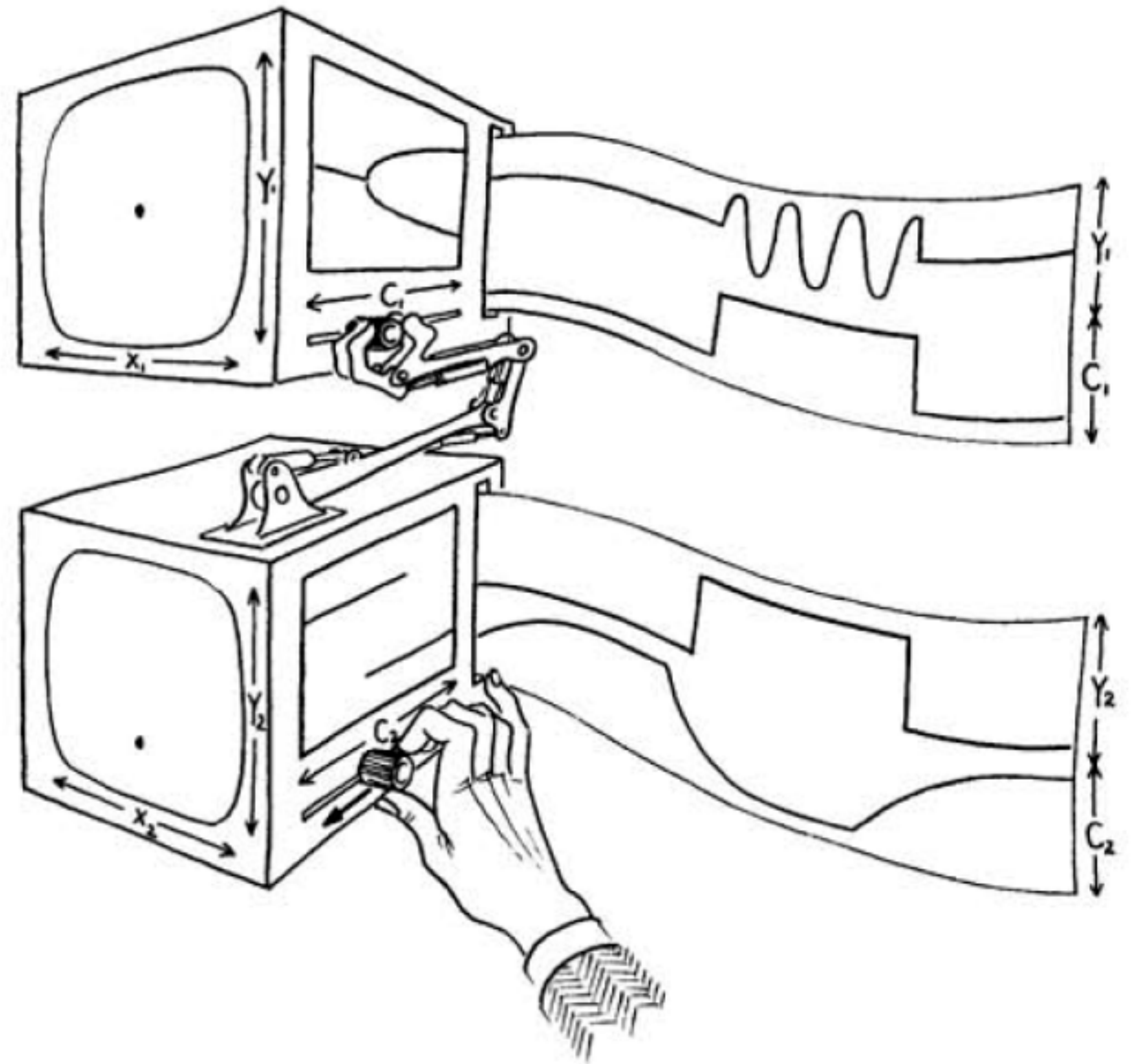


FIGURE 73. SERIAL COUPLING of these two systems means that a linking map is given from the state space of one, S^1, to the control space of the other, C^2. Thus, for a fixed value of the first control knob, the fixed dynamic on S^1 is a guidance system for the second system. But moving the first control knob changes the guidance dynamic. In the example illustrated here, moving the lower control knob full right, then full left, produces the time series shown for the guided system above. The lower control turns the oscillation abruptly on and off, with hysteresis.

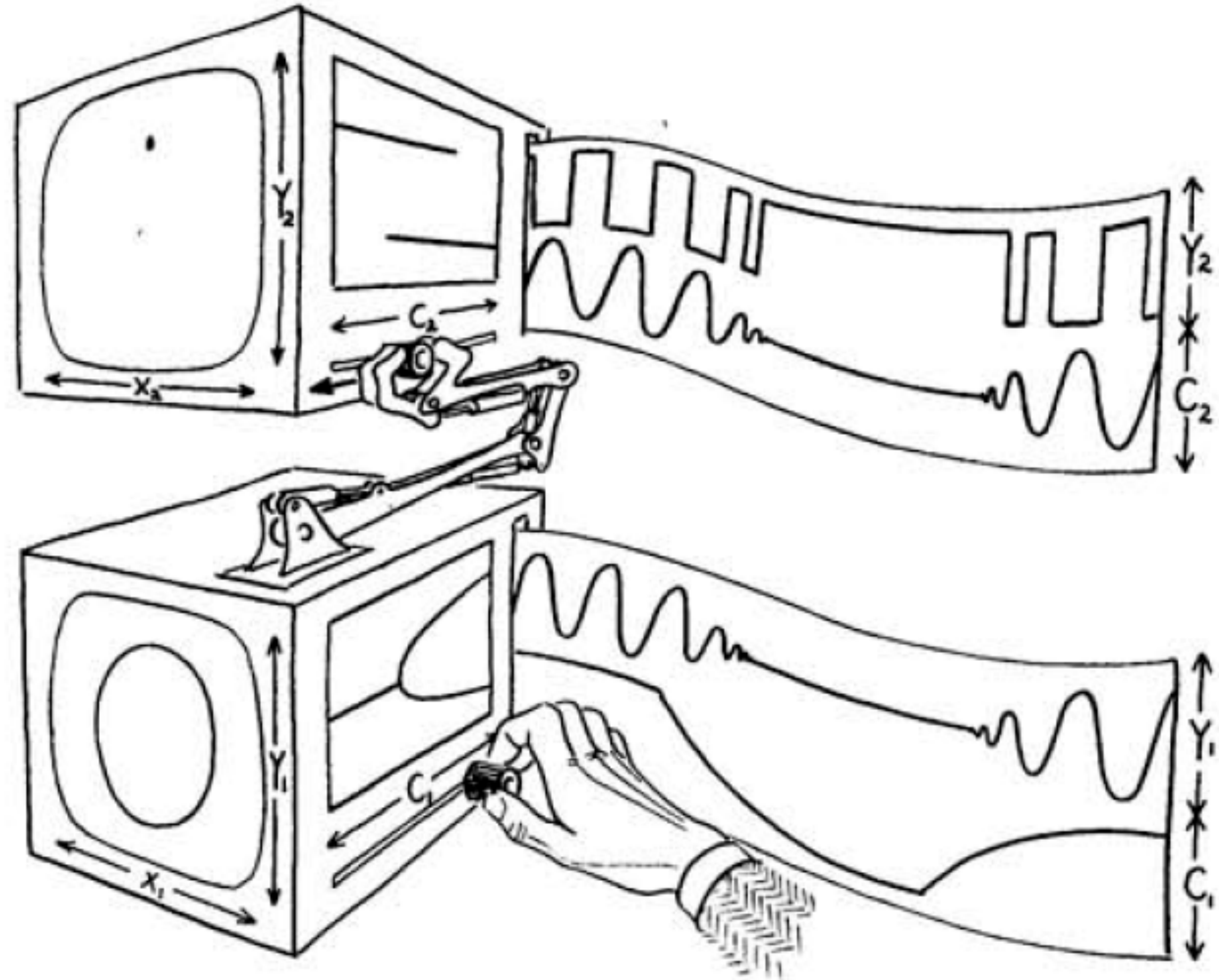


FIGURE 74. ANOTHER EXAMPLE is obtained by interchanging the roles of the two systems previously illustrated. Here the Hopf bifurcation controls the hysterical kink system. The lower control turns the square wave oscillator abruptly on and off, without hysteresis. (This type of system could be used, for example, to model the cAMP pulse relay activity of cellular slime mold—see Chapter 10.)

FIGURE 75. SYMBOLS for controlled dynamical systems may be made schematically by representing the control space as a horizontal line segment and the state space as a vertical line segment, regardless of the actual dimensions of these spaces. A single dynamical system with controls then is represented by a box with a line beneath it.

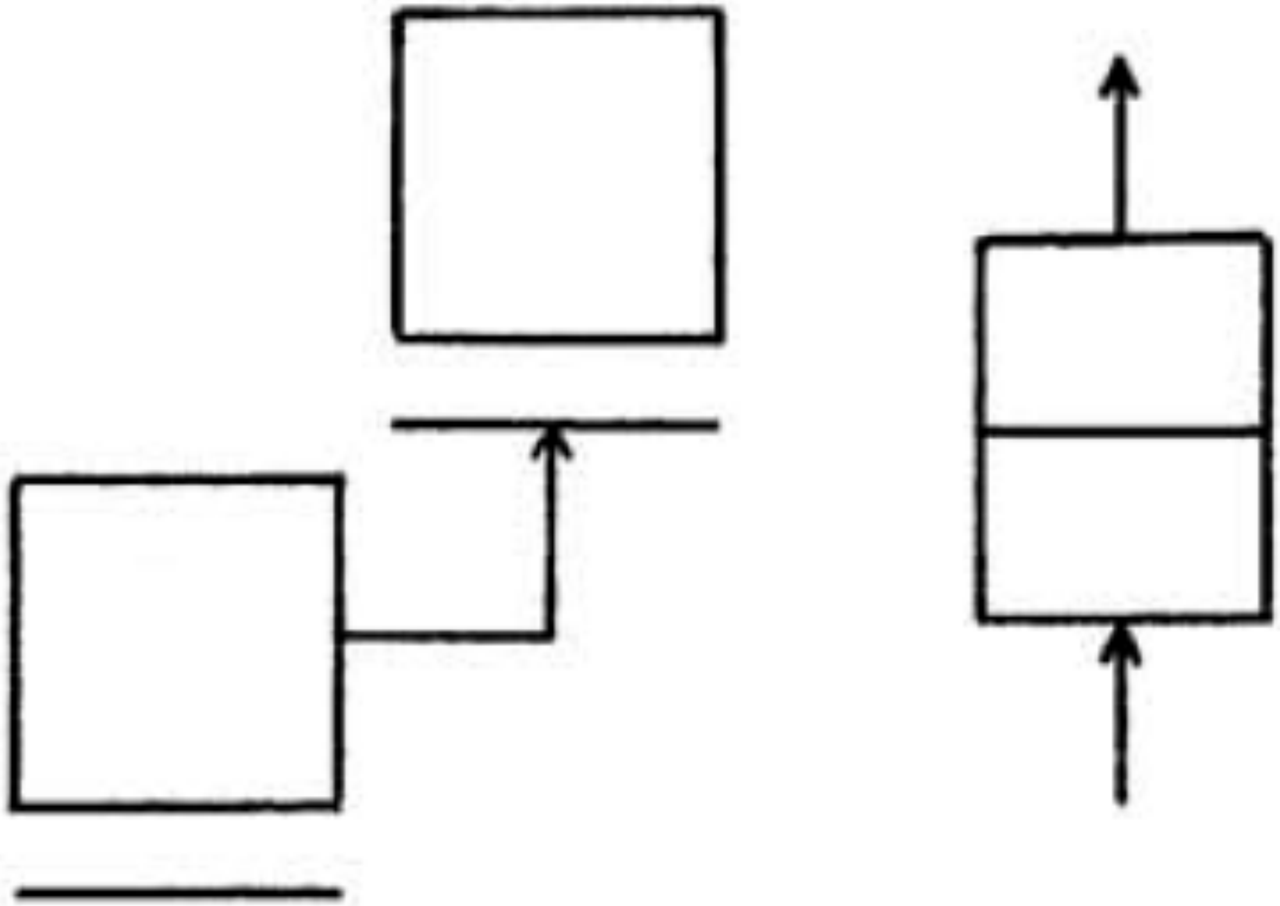


FIGURE 76. SERIAL COUPLING DIAGRAMS now may be constructed from these box symbols by connecting two of them by a directed line, representing a linking map from S^1 to C^2, as shown on the left. Alternatively, the simpler symbol on the right could be used.

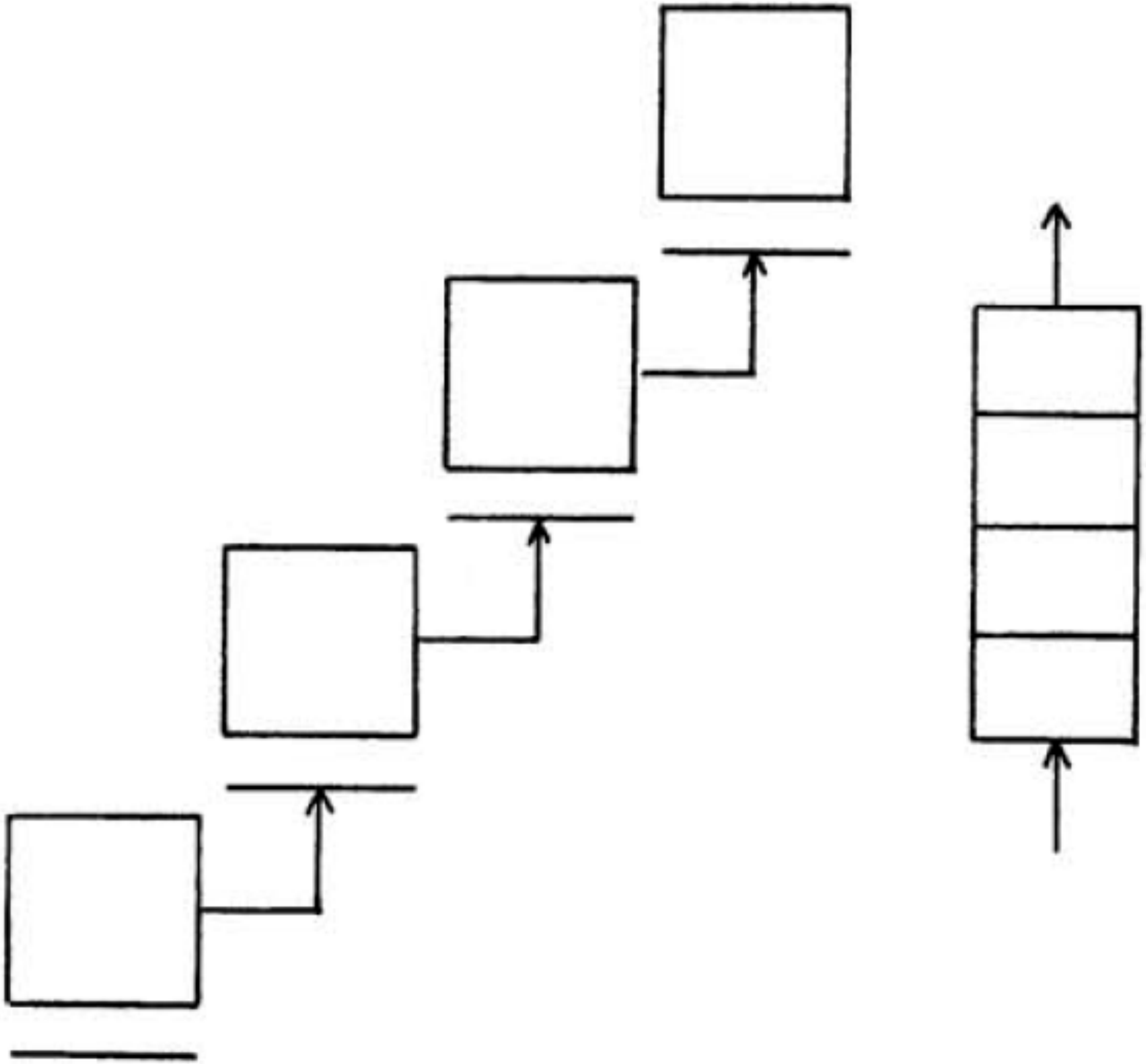


FIGURE 77. HIERARCHICAL SYSTEMS may be modeled by a sequence of serially coupled dynamical systems with controls, as shown here in two equivalent symbols.

Examples

- 2D Periodic
  - Rayleigh
  - Van der Pol
  - Duffing
  - Volterra-Lotka
- 3D Chaotic
  - Ueda
  - Lorenz
  - Rossler
  - Shaw

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End