Math 145
Chaos Theory

Ralph Abraham
www.ralph-abraham.org

Math Dept, UCSC
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Meeting #9T, May 30

- Defining Inset, Basin, Attractor, …
  (From Foundations of Mechanics)
- Review from Meeting 8Th
  - Calculus on manifolds (Ch. 2)
  - Topological dynamics (Ch. 6, Sec. 1)
- New for Meeting 9T
  - Topological dynamics (Ch. 6, Sec. 3)

Chapter 2
Review

Let us now turn to the elaboration of these ideas when a vector field X is given on a manifold M.

Recall that a curve c at a point m of a manifold M is a C^1 map from an open interval I of $\boldsymbol{R}$ into M such that $0 \in I$ and $c(0) = m$. For such a curve we may assign a tangent vector at each point $c(\lambda), \lambda \in I$, by $c'(\lambda) = Tc(\lambda, 1)$.

2.1.1 Definition. *Let M be a manifold and $X \in \mathfrak{X}(M)$. An **integral curve of X at** $m \in M$ is a curve c at m such that $c'(\lambda) = X(c(\lambda))$ for each $\lambda \in I$.*

2.1.3 Definition. *Let M be a manifold and X a vector field on M. A* ***flow box of X at $m \in M$*** *is a triple (U_0, a, F), where*

(i) *$U_0 \subset M$ is open, $m \in U_0$, and $a \in \mathbf{R}$, $a > 0$ or $a = +\infty$;*
(ii) *$F: U_0 \times I_a \to M$ is of class C^∞, where $I_a = (-a, a)$;*
(iii) *for each $u \in U_0$, $c_u: I_a \to M$ defined by $c_u(\lambda) = F(u, \lambda)$ is an integral curve of X at u;*
(iv) *if $F_\lambda: U_0 \to M$ is defined by $F_\lambda(u) = F(u, \lambda)$, then for $\lambda \in I_a$, $F_\lambda(U_0)$ is open, and F_λ is a diffeomorphism onto its image.*

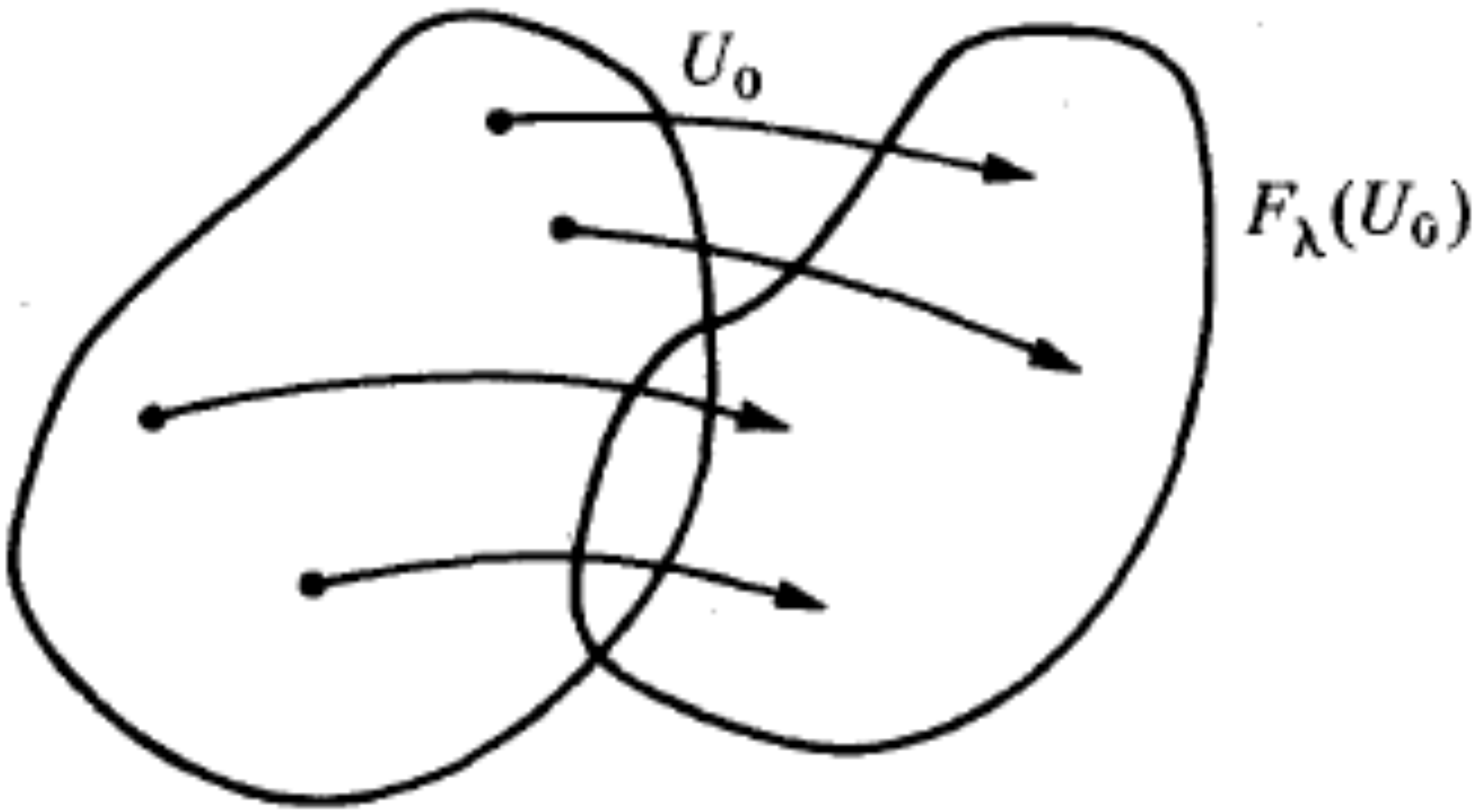

Figure 2.1-1

2.1.13 Definition *Given a manifold M and a vector field X on M, let $\mathfrak{D}_X \subset M \times \mathbf{R}$ be the set of $(m, \lambda) \in M \times \mathbf{R}$ such that there is an integral curve $c: I \to M$ of X at m with $\lambda \in I$. The vector field X is* **complete** *if $\mathfrak{D}_X = M \times \mathbf{R}$. Also, a point $m \in M$ is called* **σ complete**, *where $\sigma = +, -,$ or $\pm$, if $\mathfrak{D}_X \cap (\{m\} \times \mathbf{R})$ contains all (m, t) for $t > 0$, $t < 0$, or $t \in \mathbf{R}$, respectively.*

Thus, X is complete iff each integral curve can be extended so that its domain becomes $(-\infty, \infty)$.

Chapter 6, Section 1
Review

6.1.1 Definition. *Suppose X is a vector field on the manifold M with integral $F\colon \mathcal{D}_X \subset M \times \mathbf{R} \to M$. Then the λ^σ* ***limit set*** *of m, $\sigma = +, -$,* or *$\pm$, is defined by*

$$\lambda^+(m) = \bigcap_{n \in \mathbf{Z}} cl\left\{F\left[(\{m\} \times (n, +\infty)) \cap \mathcal{D}_X\right]\right\}$$

$$\lambda^-(m) = \bigcap_{n \in \mathbf{Z}} cl\left\{F\left[(\{m\} \times (-\infty, n)) \cap \mathcal{D}_X\right]\right\}$$

and

$$\lambda^\pm(m) = \lambda^+(m) \cup \lambda^-(m)$$

where $\mathbf{Z}$ denotes the integers. Also, let $\Lambda_X^\sigma = \cup\{\lambda^\sigma(m) \mid m \in M\}$, $\sigma = +, -, \pm$, and $\Lambda_X = \Lambda_X^\pm$.

The λ^+ (resp. λ^-) limit set is sometimes called the ω (resp. α) limit set.

6.1.6 Definition. *An* ***equilibrium point*** *(rest point, critical point) of X is a point $m \in M$ such that $F_t(m) = m$ for all $t \in \mathbf{R}$. A* ***periodic point*** *of X is a point $m \in M$ such that for some $\tau > 0$, $F_{t+\tau}(m) = F_t(m)$ for all $t \in \mathbf{R}$, and the* ***period*** *of m is the smallest $\tau > 0$ satisfying this condition. A* ***closed orbit*** *is the orbit of a periodic (nonequilibrium) point. A* ***critical element*** *of X is either a set $\{m\}$, where m is an equilibrium, or a closed orbit. The set of all critical elements of X is denoted by Γ_X.*

Chapter 6, Section 3

6.3.1 Notation. *Let M be a manifold and X a vector field on M, with integral $F: \mathfrak{D}_X \subset M \times \mathbf{R} \to M$. For $(m,t) \in \mathfrak{D}_X$ let $m_t = F(m,t)$. Then for each $m \in M$, let*

$$m_+ = \cup \{ m_t | (m,t) \in \mathfrak{D}_X, t \geqslant 0 \}, \quad \text{the + orbit of } m$$

$$m_- = \cup \{ m_t | (m,t) \in \mathfrak{D}_X, t \leqslant 0 \}, \quad \text{the − orbit of } m$$

$$m_\pm = m_+ \cup m_-, \quad \text{the full orbit of } m$$

These will be denoted m_σ, where σ can be $+$, $-$, or $\pm$.

In either case above, we define

$$S^{\sigma}(m)=\{m'\in M|\lambda^{\sigma}(m')\subset cl(m_{\sigma})\}$$

and

$$A^{\sigma}(m)=\left\{m'\in M|m' \text{ is } \sigma \text{ complete and } \lim_{t\to\sigma\infty}\rho(m_t,m'_t)=0\right\}$$

if m is σ complete, and $A^{\sigma}(m)=\{m\}$ otherwise. $S^{+}(m)$ and $S^{-}(m)$ are known as the **inset** and **outset** of m, respectively.

6.3.5 Definition. *A subset $A \subset M$ is an* **attractor** *of a complete vector field $X \in \mathfrak{X}(M)$ if it is closed, invariant, and has an open neighborhood $U_0 \subset M$ that is (i) positively invariant, and (ii) for each open neighborhood V of A ($A \subset V \subset U_0 \subset M$) there is $\tau > 0$ such that $U_t = F_t(U_0) \subset V$ for all $t \geqslant \tau$. An attractor $A \subset M$ is* **stable** *if for every neighborhood U_0 of $A \subset M$ there is a neighborhood V of $A \subset M$ such that $V_t \subset U_0$ for all $t \geqslant 0$. If $A \subset M$ is an attractor, the* **basin** *of A is the union of all open neighborhoods of A satisfying (i) and (ii) above.*

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End