APPENDIX A

TOPICS FROM TOPOLOGY

Probably the greatest difficulty this book presents to the nonmathematician is the reliance on point set topology. Although excellent references are available, the topics required cannot all be found in a single text, and the variation of notations and order in the different books presents a difficult challenge to an inexperienced reader. We assemble here for reference the topics needed, in a consistent notation used throughout this book. Some elementary direct proofs usually found as corollaries of more difficult theorems are also included. References to other texts are given instead of proofs, when this presents no problem to the reader. The exercises at the end also develop methods that are used in the text.

§A1. TOPOLOGICAL SPACES

A1.1. Definition. *A* **topological space** *is a set S together with a collection of subsets $\mathcal{O}$ called* **open sets** *such that*

(T1) $\phi \in \mathcal{O}$ *and* $S \in \mathcal{O}$;

(T1) *If* $U_1, U_2 \in \mathcal{O}$ *then* $U_1 \cap U_2 \in \mathcal{O}$;

(T3) *The union of any collection* U_α *of open sets is open.*

For such a topological space the **closed sets** *are the elements of*

$$\Gamma = \{A | \mathcal{C}A \in \mathcal{O}\}$$

where $\mathcal{C}$ *denotes the complement,* $\mathcal{C}A = S \setminus A = \{s \in S | s \notin A\}$. *(The closed sets then obey rules dual to those for open sets.)*

An **open neighborhood of a point** u *in a topological space S is an open set U such that $u \in U$. Similarly, for a subset A of S, U is an* **open neighborhood of** A *iff U is open and $A \subset U$.*

If A is a subset of a topological space S, the **relative topology** *on A is defined by*

$$\mathcal{O}_A = \{U \cap A | U \in \mathcal{O}\}$$

(which is clearly a topology on A).

Let S be a topological space. Then a **basis** *for the topology is a collection $\mathcal{B}$ of open sets such that every open set of S is a union of elements of $\mathcal{B}$. The topology is called* **first countable** *iff for each*

$u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u such that for any neighborhood U of u, there is an N so $U_N \subset U$. The topology is called **second countable** *iff it has a countable basis.*

Let S and T be topological spaces and $S \times T = \{(u, v) | u \in S$ and $v \in T\}$. The **product topology** *on $S \times T$ consists of all subsets that are unions of sets of the form $U \times V$ where U is open in S and V is open in T. Thus, these* **open rectangles** *form a basis for the topology.*

Let S be a topological space and $\{u_n\}$ a sequence of points in S. The sequence is said to **converge** *iff there is a point $u \in S$ such that for every neighborhood U of u, there is an N such that $n \geq N$ implies $u_n \in U$. We say that $\{u_n\}$* **converges to** *u, or u is a* **limit point** *of $\{u_n\}$.*

A1.2. Example. On the real line $\mathbf{R}$, the *standard topology* consists of the sets that are unions of open intervals (a, b). Then $\mathbf{R}$ is second countable, and hence first countable with a basis

$$\left\{\left(r_n - \frac{1}{m}, r_n + \frac{1}{m}\right) \middle| r_n \text{ is rational, } m \in \mathbf{N}, \text{ the positive integers}\right\}$$

The topology on the plane $\mathbf{R}^2$ is the product topology $\mathbf{R} \times \mathbf{R}$. In $\mathbf{R}$, the sequence $\{1/n\}$ converges to 0, but in the subspace $(0, 1]$, the sequence does not converge.

A1.3. Definition. *Let S be a topological space and $A \subset S$. Then the* **closure** *of A, denoted A^c, is the intersection of all closed sets containing A. The* **interior** *of A, denoted A^o is the union of all open sets contained in A. The* **boundary** *of A, denoted $bd(A)$ is defined by*

$$bd(A) = A^c \cap (\mathscr{C}A)^c$$

Thus, $bd(A)$ is closed, and $bd(A) = bd(\mathscr{C}A)$. Note A open iff $A = A^o$, and A closed iff $A = A^c$.

A1.4. Proposition. *Let S be a topological space and $A \subset S$. Then*

(i) $u \in A^c$ iff for every neighborhood U of u, $U \cap A \neq \phi$;

(ii) $u \in A^o$ iff there is a neighborhood U of u such that $U \subset A$;

(iii) $u \in bd(A)$ iff for every neighborhood U of u, $U \cap A \neq \phi$ and $U \cap (\mathscr{C}A) \neq \phi$.

This proposition follows at once from the definitions.

A1.5. Definition. *Let S be a topological space. A point $u \in S$ is called* **isolated** *iff $\{u\}$ is open. The unique topology in which every point is isolated is called the* **discrete topology** *($\mathcal{O} = 2^S$, the collection of all subsets). The topology in which $\mathcal{O} = \{\phi, S\}$ is called the* **trivial topology**.

A subset A of S is called **dense** *in S iff $A^c = S$ and is called* **nowhere dense** *iff $\mathscr{C}(A^c)$ is dense in S.*

Thus, A is nowhere dense iff $(A^c)^0 = \phi$.

A1.6. Definition. *A topological space S is called* **Hausdorff** *iff each two distinct points have disjoint neighborhoods (that is, with empty intersection). Similarly, S is called* **normal** *iff each two disjoint closed sets have disjoint neighborhoods.*

Equivalent forms of Hausdorff are the following.

A1.7. Proposition. *(i) A space S is Hausdorff iff $\Delta_S = \{(u, u) | u \in S\}$ is closed in $S \times S$ in the product topology.*

(ii) A first countable space S is Hausdorff iff all sequences have at most one limit point.

Proof. If Δ_S is closed and u_1, u_2 are distinct, there is an open rectangle $U \times V$ containing (u_1, u_2) and $U \times V \subset \mathscr{C}\Delta_S$. Then in S, U and V are disjoint. The converse is similar, and we leave *(ii)* as an exercise. ∎

§A2. METRIC SPACES

A2.1. Definition. *Let $\hat{\mathbf{R}}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoined, and topology generated by the open intervals of the form (a, b) or $(a, +\infty]$. Let M be a set. A* **metric** *on M is a function $\rho: M \times M \to \hat{\mathbf{R}}^+$ such that*

(M1) $\rho(m_1, m_2) = 0$ *iff* $m_1 = m_2$;

(M2) $\rho(m_1, m_2) = \rho(m_2, m_1)$;

(M3) $\rho(m_1, m_3) \leq \rho(m_1, m_2) + \rho(M_2, m_3)$, *triangle inequality.*

For $\varepsilon \in \hat{\mathbf{R}}^+$, $\varepsilon > 0$, and $m \in M$, the ε **disk** *about m is defined by*

$$D_\varepsilon(m) = \{m' \in M | d(m', m) < \varepsilon\}$$

The collection of subsets of M that are unions of such disks is the **metric topology** *at (M, ρ). (It is easily verified that it is a topology on M.)*

Two metrics on a set are **equivalent** *iff they induce the same metric topology.*

Let M be a metric space with metric ρ and $\{u_n\}$ a sequence in M. Then $\{u_n\}$ is a **Cauchy sequence** *iff for all $\varepsilon > 0$, $\varepsilon \in \hat{\mathbf{R}}^+$, there is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $\rho(u_n, u_m) < \varepsilon$. (It is easily seen that a convergent sequence is a Cauchy sequence.) The space M is called* **complete** *iff every Cauchy sequence converges.*

A **pseudometric** *on a set M is a function $\rho: M \times M \to \hat{\mathbf{R}}^+$ that satisfies* (M2), (M3), *and*

$$\text{(PM1)} \qquad \rho(m, m) = 0 \qquad \textit{for all } m$$

The **pseudometric topology** *is defined exactly as the metric space topology.*

If M is a metric space (or pseudometric space) and $u \in M$, $A \subset M$ we define $\rho(u, A) = \inf\{\rho(u, v) | v \in A\}$, if $A \neq \phi$ and $\rho(u, \phi) = \infty$.

Clearly, metric spaces are first countable and Hausdorff.

A2.2. Proposition. *Every metric space is normal.*

Proof. Let A and B be closed, disjoint subsets of M, and let

$$U = \{u \in M | \rho(u, A) < \rho(u, B)\}$$

$$V = \{v \in M | \rho(v, A) > \rho(v, B)\}$$

It is easily verified that U and V are open, disjoint, and $A \subset U$, $B \subset V$. ∎

A2.3. Definition. *The* **standard metric** *on $\mathbf{R}^n$ is defined by*

$$\rho(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} (x^i - y^i)^2 \right)^{1/2}$$

where $\mathbf{x} = (x^1, \ldots, x^n)$.

§A3. CONTINUOUS MAPPINGS

A3.1. Definition. *Let S and T be topological spaces and $\varphi: S \to T$ be a mapping. Then φ is* **continuous at $\mathbf{u} \in \mathbf{S}$** *iff for every neighborhood V of $\varphi(u)$ there is a neighborhood U of u such that $\varphi(U) \subset V$. If, for every open set V of T, $\varphi^{-1}(V) = \{u \in S | \varphi(u) \in V\}$ is open in S, φ is* **continuous**. *(Thus, φ is continuous iff φ is continuous at each $u \in S$.)*

If $\varphi: S \to T$ is a **bijection** *(that is, one-to-one and onto), φ and φ^{-1} are continuous, then φ is a* **homeomorphism** *and S and T are* **homeomorphic**.

It follows at once that $\varphi: S \to T$ is continuous iff the inverse image of every closed set is closed. The following is also useful.

A3.2. Proposition. *Let S and T be topological spaces and $\varphi: S \to T$. Then φ is continuous iff for every $A \subset S$, $\varphi(A^c) \subset \varphi(A)^c$.*

Proof. If φ is continuous, then $\varphi^{-1}(\varphi(A)^c)$ is closed. But $A \subset \varphi^{-1}(\varphi(A)^c)$ and hence $A^c \subset \varphi^{-1}(\varphi(A)^c)$, or $\varphi(A^c) \subset \varphi(A)^c$. Conversely, let $B \subset T$ be closed and $A = \varphi^{-1}(B)$. Then $A^c \subset \varphi^{-1}(B) = A$, so A is closed. ∎

From A1.4 we obtain the following.

A3.3. Proposition. *Let S be a first countable space and $A \subset S$. Then $u \in A^c$ iff there is a sequence of points of A that converge to u (in the topology on S).*

Continuity may be expressed in terms of sequences as follows:

A3.4. Proposition. *Let S and T be topological spaces with S first countable and $\varphi: S \to T$. Then φ is continuous iff for every sequence $\{u_n\}$ converging to u, $\{\varphi(u_n)\}$ converges to $\varphi(u)$, for all $u \in S$.*

We leave this to the reader. In the case that φ is continuous, note that the result follows at once from A3.2 and A3.3.

For metric spaces, note that $\varphi: M_1 \to M_2$ is continuous at $u_1 \in M_1$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(u_1, u_1') < \delta$ implies $\rho(\varphi(u_1), \varphi(u_1')) < \varepsilon$.

A3.5. Proposition. *Let M and N be metric spaces with N complete. Then the collection $C(M, N)$ of all continuous maps $\varphi: M \to N$ forms a complete metric space with the metric*

$$\rho^0(\varphi, \psi) = \sup\{\rho(\varphi(u), \psi(u)) | u \in M\}$$

We leave this as an exercise. Note that ∞ is an allowed value.

§A4. COMPACT SPACES

A4.1. Definition. *Let S be a topological space. Then S is called* **compact** *iff for every covering of S by open sets U_α (that is, $\bigcup_\alpha U_\alpha = S$) there is a finite subcovering. A subset $A \subset S$ is called* **compact** *iff A is compact in the relative topology. A space is called* **locally compact** *iff each point has a neighborhood whose closure is compact.*

It follows easily that a closed subset of a compact space is compact and that the continuous image of a compact space is compact.

The following is often convenient.

A4.2. Theorem (Bolzano–Weierstrass). *If S is a first countable space and is compact, then every sequence has a convergent subsequence.*

(The converse is also true in a metric space.)

Proof. Suppose $\{u_n\}$ contains no convergent subsequences. Then we may assume all points are distinct. Each u_n has a neighborhood $\mathcal{O}_n$ that contains no other u_m. From A3.3 $\{u_n\}$ is closed, so that $\{\mathcal{O}_n\}$ together with $\mathcal{C}\{u_n\}$ forms an open covering of S, with no finite subcovering. ∎

In a metric space, every compact subset is closed and bounded. In $\mathbf{R}^n$, the converse is also true (Dieudonné [1, §3.16]).

A4.3. Proposition. *Let S be a Hausdorff space. Then every compact subset of S is closed. Also, every compact Hausdorff space is normal.*

Proof. Let $u \in \mathcal{C}A$ and $v \in A$, where A is compact in S. There are disjoint neighborhoods of u and v and, since A is compact, there are disjoint neighborhoods of u and A. Thus $\mathcal{C}A$ is open. We leave the second part as an exercise. ∎

A4.4. Proposition. *Let S be a Hausdorff space that is locally homeomorphic to a locally compact Hausdorff space (that is, for each $u \in S$, there is a neighborhood of S homeomorphic, in the subspace topology, to an open subset of a locally compact Hausdorff space). Then S is locally compact. In particular, manifolds are locally compact. (The Hausdorff assumption is essential.)*

Proof. Let $U \subset S$ be homeomorphic to $\varphi(U) \subset T$. There is a neighborhood V of $\varphi(u)$ so $V^c \subset \varphi(U)$ and V^c is compact. (We leave this as an exercise; locally compact Hausdorff spaces are regular.) Then $\varphi^{-1}(V^c)$ is compact,

and hence closed in S. By A3.2 $\varphi^{-1}(V^c) \subset (\varphi^{-1}V)^c$. Thus $\varphi^{-1}(V)$ has compact closure $\varphi^{-1}(V)^c = \varphi^{-1}(V)^c$. ∎

§A5. PARACOMPACT SPACES

A5.1. Definition. *Let S be a topological space. A covering $\{U_\alpha\}$ of S is called a* **refinement** *of a covering $\{V_i\}$ iff for every U_α there is a V_i such that $U_\alpha \subset V_i$. A covering $\{U_\alpha\}$ of S is called* **locally finite** *iff each point $u \in S$ has a neighborhood U such that U intersects only a finite number of U_α. A space is called* **paracompact** *iff every open covering of S has a locally finite refinement of open sets, and S is Hausdorff.*

A5.2. Theorem. *Second countable, locally compact Hausdorff spaces are paracompact. In particular, manifolds are paracompact.*

Proof. S is the countable union of open sets U_n such that U_n^c is compact and $U_n^c \subset N_{n+1}$. If W_α is a covering of S by open sets, and $K_n = U_n^c - U_{n-1}$ then we can cover K_n by a finite number of open sets each of which is contained in some $W_\alpha \cap U_{n+1}$, and is disjoint from U_{n-2}^c. The union of such collections yields the desired refinement of $\{W_\alpha\}$. ∎

A minor modification yields the following.

A5.3. Corollary. *Let M be an n manifold and $\{W_\alpha\}$ be an open covering. Then there is a locally finite refinement consisting of charts $\{V_i, \varphi_i\}$ such that $\varphi_i(V_i)$ is the disk of radius 3, and such that $\varphi_i^{-1}(D_1(\mathbf{0}))$ cover M.*

This is the critical lemma for C^∞ partitions of unity (§A6).

A5.4. Theorem. *Every paracompact space is normal. In particular, manifolds are normal.*

Proof. We first show that if A is closed and $u \in \mathscr{C}A$, there are disjoint neighborhoods of u and A (regularity). For each $v \in A$ let U_u, V_v be disjoint neighborhoods of u and v. Let W_α be a locally finite refinement of the covering $V_v, \mathscr{C}A$, and $V = \cup W_\alpha$, the union over those α so $W_\alpha \cap A \neq \phi$. A neighborhood U_0 of u meets a finite number of W_α. Let U denote the intersection of U_0 and the corresponding U_v. Then V and U are the required neighborhoods. The case for two closed sets proceeds somewhat similarly, so we leave the details for the reader. ∎

§A6. PARTITIONS OF UNITY

We wish to give here some proofs of theorems in the text.

A6.1. Proof of 11.3. Let $\mathscr{A}$ be an atlas on M and let $\{V_i, \varphi_i\}$ be a locally finite refinement with the properties of A5.3. From 8.6 we have a function $h_i \in \mathscr{F}(M)$ whose support lies in V_i. Let

$$g_i(u) = \frac{h_i(u)}{\Sigma_i h_i(u)}$$

(the sum is finite). These are the required functions. ∎

A6.2. Proof of 8.7. More generally, let A and B be two closed sets. Then by normality, there is an atlas $\{U_\alpha, \varphi_\alpha\}$ such that $U_\alpha \cap A \neq \phi$ implies $U_\alpha \cap B = \phi$. Let $\{V_i, \varphi_i, g_i\}$ be a subordinate partition of unity and $h = \Sigma g_i$, where the sum is over those i for which $V_i \cap A \neq \phi$. Then h is one on A and zero on B. ∎

This theorem is a smooth version of a topological theorem for normal spaces (Urysohn's lemma; see J. Kelley [1, p. 115]). For a more general discussion of partitions of unity, see Lang [1, pp. 25 ff.], and J. Kelley [1, p. 171].

§A7. CONNECTEDNESS

A7.1. Definition. *A topological space S is* **connected** *iff ϕ and S are the only subsets of S that are both open and closed. A subset of S is connected iff it is connected in the relative topology. A* **component** *A of S is a nonempty connected subset of S such that the only connected subset of S containing A is A; S is called* **locally connected** *iff each point has a connected neighborhood.*

It follows easily that the continuous image of a connected set is connected. Equivalent forms of the definition follow at once.

A7.2. Proposition. *A space S is not connected iff either of the following holds.*

(i) There is a nonempty proper subset of S that is both open and closed.

(ii) S is the disjoint union of two nonempty open sets.

(iii) S is the disjoint union of two nonempty closed sets.

Also, we have the following.

A7.3. Proposition. *Let S be a connected space and $f: S \to \mathbf{R}$ be continuous. Then f assumes every value between any two values $f(u)$, $f(v)$.*

Proof. Suppose $f(u) < a < f(v)$ and f does not assume the value a. Then $U = \{u_0 | f(u_0) < v\}$ is both open and closed. ∎

It is also true that disks in $\mathbf{R}^n$ are connected. See Dieudonné [1, §19 and §20].

A7.4. Proposition. *Let S be a topological space and $B \subset S$ be connected. Then*

(i) if $B \subset A \subset B^c$, then A is connected;

(ii) if B_α are connected and $B_\alpha \cap B \neq \phi$, then $B \cup (\bigcup_\alpha B_\alpha)$ is connected.

Proof. If A is not connected, A is the disjoint union of $U_1 \cap A$ and $U_2 \cap A$ where U_1 and U_2 are open on S. Then from A1.4 (*i*), $U_1 \cap B \neq \phi$, $U_2 \cap B \neq \phi$, so B is not connected. We leave (*ii*) as an exercise. ∎

A7.5. Corollary. *The components of a topological space are closed. Also, S is the disjoint union of its components. If S is locally connected, the components are open as well as closed.*

Note that manifolds are locally connected.

A7.6. Proposition. *Let S be a first countable compact Hausdorff space and $\{A_n\}$ a sequence of closed, connected subsets of S with $A_n \subset A_{n-1}$. Then $A = \bigcap_1^\infty A_n$ is connected.*

Proof. As S is normal, if A is not connected, A lies in two disjoint open subsets U_1 and U_2 of S. If $A_n \cap \mathscr{C}U_1 \cap \mathscr{C}U_2 \neq \phi$ for all n, then there is a sequence $u_n \in A_n \cap \mathscr{C}U_1 \cap \mathscr{C}U_2$ with a subsequence converging to u. As $A_n, \mathscr{C}U_1, \mathscr{C}U_2$ are closed, $u \in A \cap \mathscr{C}U_1 \cap \mathscr{C}U_2$, a contradiction. Hence some A_n is not connected. ∎

In A7.6, the first countable assumption is not essential.

An intuitively more appealing, but less convenient, definition of connectedness is the following.

A7.7. Definition. *Let S be a topological space and $I = [0, 1] \subset \mathbf{R}$. An* **arc** *$\varphi$ in S is a continuous mapping $\varphi: I \to S$. If $\varphi(0) = u$, $\varphi(1) = v$ we say φ* **joins** *u and v; S is called* **arcwise connected** *iff every two points in S can be joined by an arc in S. A space is called* **locally arcwise connected** *iff each point has an arcwise connected neighborhood (in the relative topology).*

The relationship with connectedness is the following.

A7.8. Proposition. *Every arcwise connected space is connected. If a space is connected and locally arcwise connected, it is arcwise connected. In particular, a manifold is connected iff it is arcwise connected.*

Proof. If S is arcwise connected and not connected, write $S = U_1 \cup U_2$, where U_1 and U_2 are nonempty, disjoint, and open. Let $u_1 \in U_1$ and $u_2 \in U_2$ and let φ be an arc joining u_1 and u_2. Now $\varphi(I)$ is connected, and since $\varphi(I) \cap U_i \neq \phi$, $\varphi(I) \cap U_1 \cap U_2 \neq \phi$. Hence $U_1 \cap U_2 \neq \phi$, a contradiction. For the second part, let $u \in S$ and U denote all points that can be joined to u by an arc. An easy argument shows U and $\mathscr{C}U$ are open and so $U = S$, by A7.2. ∎

It is often convenient to use the following.

A7.9. Theorem. *Every manifold is metrizable. That is, there is a metric that induces the manifold topology.*

This can be proven in several ways. First, manifolds can be embedded in Euclidean space (3.5), and the metric on $\mathbf{R}^{2n+1}$ induces one on the manifold. Second, as we saw in 11.13, every manifold possesses a Riemannian metric, and it can be shown that the metric $\rho(m_1, m_2) = \inf\{\text{length } \varphi | \varphi \text{ is a piecewise smooth curve joining } m_1 \text{ and } m_2\}$ induces the manifold topology (Milnor [1, p. 62]). Finally, the result also follows from general metrization theorems, such as: A second countable space is metrizable iff it is normal. See J. Kelley [1, p. 125] and A5.4.

§A8. HAUSDORFF METRICS

A8.1. Definition. *Let S be a metric space with metric ρ, and 2^S denote the set of all subsets of S. For $a \in S$ and $B \subset S$, $B \neq \phi$, define*

$$\rho(a, B) = \inf\{\rho(a, b) | b \in B\}$$

and for $A, B \subset S$, $A, B \neq \phi$

$$\vec{\rho}(A, B) = \sup\{\rho(a, B) | a \in A\}$$

As this is not symmetric, we further define

$$\bar{\rho}(A, B) = \sup\{\vec{\rho}(A, B), \vec{\rho}(B, A)\}$$

If $A \neq \phi$ and $B = \phi$ we define $\rho(a, B) = \infty$ and $\bar{\rho}(A, B) = \infty$. Finally, define $\bar{\rho}(\phi, \phi) = 0$. We call $\bar{\rho}$ the **Hausdorff metric**.

A8.2. Proposition. *If ρ is a metric on S, then $\bar{\rho}$ is a pseudo-metric on 2^S.*

Proof. Clearly, $\bar{\rho}: 2^S \times 2^S \to \hat{\mathbf{R}}^+ = [0, \infty]$, $\bar{\rho}(A, A) = 0$ and $\bar{\rho}$ is symmetric. For the triangle inequality, it is sufficient to show $\vec{\rho}(A, C) \leq \vec{\rho}(A, B) + \vec{\rho}(B, C)$ for $A, B, C \in 2^S$. But $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$ implies $\rho(a, C) \leq \rho(a, b) + \rho(b, C) \leq \rho(a, b) + \vec{\rho}(B, C)$. Hence $\rho(a, C) \leq \rho(a, B) + \vec{\rho}(B, C)$, which yields the result. The cases A, B, or $C = \phi$ are easily checked. ∎

As metric spaces are normal, it follows easily that on the closed subsets of S, $\bar{\rho}$ is a metric. The Hausdorff metric provides a measure of the uniform closeness of sets, as noted in §26. For further details, see Hausdorff [1], J. Kelley [1, p. 131], or Michael [1, p. 152].

Continuity of a map $f: S \to 2^S$ can be rephrased as follows.

A8.3. Proposition. *Let S be a metric space and $\bar{\rho}$ the Hausdorff metric on 2^S. Then $f: S \to 2^S$ is continuous at $u_0 \in S$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(u, u_0) < \delta$ implies:*

(i) for all $a \in f(u)$, there is a $b \in f(u_0)$ so $\rho(a, b) < \varepsilon$; that is

$$f(u) \subset \bigcup_{b \in f(u_0)} D_\varepsilon(b)$$

and

(ii) for all $b \in f(u_0)$, there is an $a \in f(u)$ such that $\rho(b, a) < \varepsilon$; that is, $f(u_0) \subset \bigcup_{a \in f(u)} D_\varepsilon(a)$.

This proposition follows at once from the definitions of continuity and the Hausdorff metric.

§A9. QUOTIENT TOPOLOGIES; THE TORUS AND KLEIN BOTTLE

A9.1. Definition. *Let S be a set. An* **equivalence relation** $\sim$ *on S is a binary relation such that for all $u, v, w \in S$,*

(i) $u \sim u$,

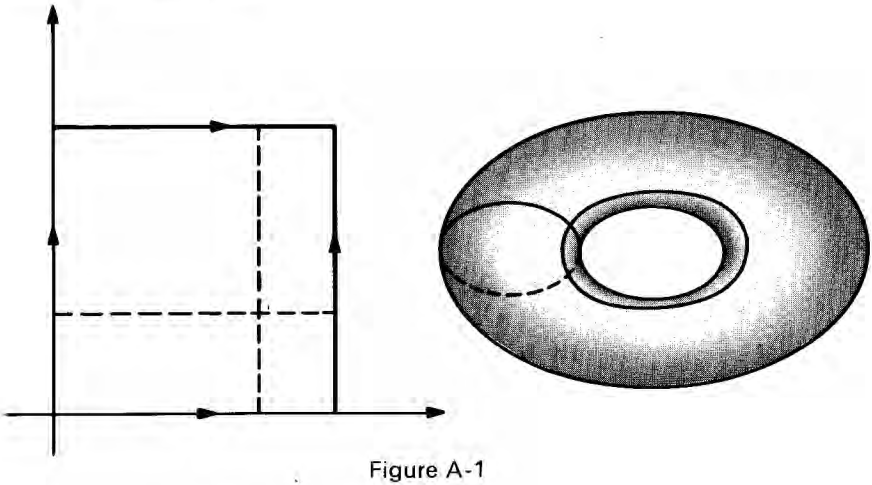

Figure A-1

(*ii*) $u \sim v$ *iff* $v \sim u$, *and*

(*iii*) $u \sim v$ *and* $v \sim w$ *implies* $u \sim w$.

The **equivalence class** *containing* u, *denoted* $|u|$, *is defined by*

$$[u] = \{v \in S | u \sim v\}$$

The set of equivalence classes is denoted $S/\sim$, *and the mapping* $\pi: S \to S/\sim ; u \to [u]$ *is called the* **canonical projection**

It follows easily that S is the disjoint union of its equivalence classes.

A9.2. Definition. *Let* S *be a topological space and* $\sim$ *an equivalence relation on* S. *Then* $\{U \subset S/\sim | \pi^{-1}(U)$ *is open in* $S\}$ *is called the* **quotient topology** *on* $S/\sim$. *Similarly, if* $S/\sim$ *has a topology, we can induce one on* S *by* $\{\pi^{-1}(U) | U$ *is open in* $S/\sim\}$. *These are clearly topologies.*

A9.3. Example. Consider $\mathbf{R}^2$ and the relation $\sim$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \qquad \text{iff} \qquad a_1 - b_1 \in \mathbf{Z}$$

and $a_2 - b_2 \in \mathbf{Z}$ ($\mathbf{Z}$ denotes the integers).

Then $\mathbf{T}^2 = \mathbf{R}^2/\sim$ is called the 2-**torus.** In addition to the quotient topology, it inherits a group structure in the usual way:

$$[(a_1, a_2)] + [(b_1, b_2)] = [(a_1, a_2) + (b_1, b_2)]$$

The n-dimensional torus $\mathbf{T}^n$ is defined in a similar manner.

The torus $\mathbf{T}^2$ may be obtained in two other ways. First, let $\square$ be the unit square in $\mathbf{R}^2$ with the subspace topology. Define $\sim$ by $\mathbf{x} \sim \mathbf{y}$ iff any of the following hold:

(i) $\mathbf{x} = \mathbf{y}$;

(ii) $x_1 = y_1, x_2 = 0, y_2 = 1$;

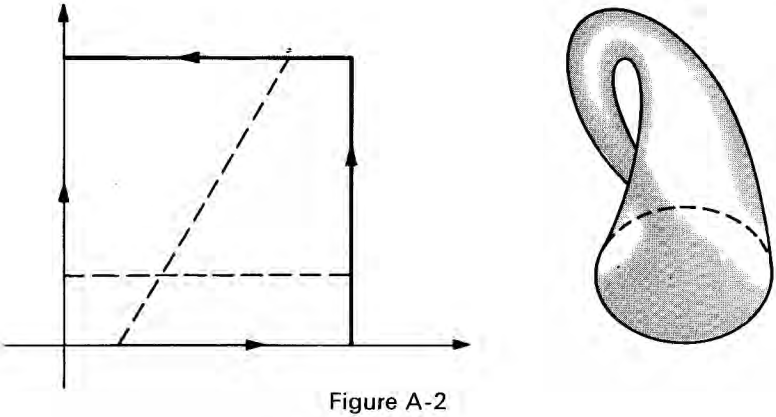

Figure A-2

(iii) $x_1 = y_1, x_2 = 1, y_2 = 0$;
(iv) $x_2 = y_2, x_1 = 0, y_1 = 1$; or
(v) $x_2 = y_2, x_1 = 1, y_1 = 0$;
as indicated in Figure A-1.

Then $\mathbf{T}^2 = \square/\sim$.

Second, define $T^2 = S^1 \times S^1$, as shown in Figure A-1.

A9.4. Example. The Klein bottle is obtained by reversing one of the orientations on $\square$, as indicated in Figure A-2.

Then $\mathbf{K}^2 = \square/\sim$ (the equivalence relation indicated) is the Klein bottle. Although it is realizable as a submanifold of $\mathbf{R}^4$, it is convenient to picture it in $\mathbf{R}^3$ as shown. Notice that $\mathbf{K}^2$ is not orientable and does not inherit a group structure from $\mathbf{R}^2$, as did $\mathbf{T}^2$.

§A10. SIMPLY CONNECTED SPACES

A10.1. Definition. *Let Z be a topological space and $c: [0, 1] \to Z$ a continuous map such that $c(0) = c(1) = p \in Z$. We call c a* **loop** *in Z based at p. The loop c is called* **contractible** *if there is a continuous map $H: [0, 1] \times [0, 1] \to Z$ such that $H(t, 0) = c(t)$ and $H(t, 1) = p$ for all $t \in [0, 1]$. (See Figure A-3.)*

Roughly speaking, a loop is contractible when it can be shrunk continuously to p. The study of loops leads naturally to homotopy theory. In fact the loops at p can easily be made into a group called the **fundamental group**.

A10.2. Definition. *A space Z is called* **simply connected** *iff every loop in Z is contractible.*

In the plane $\mathbf{R}^2$ there is an alternate approach to simple connectedness, by way of the Jordan curve theorem: namely, that every simple (nonintersecting) loop in $\mathbf{R}^2$ divides $\mathbf{R}^2$ (that is, its complement has two components). The unbounded component of the complement is called the interior, and a

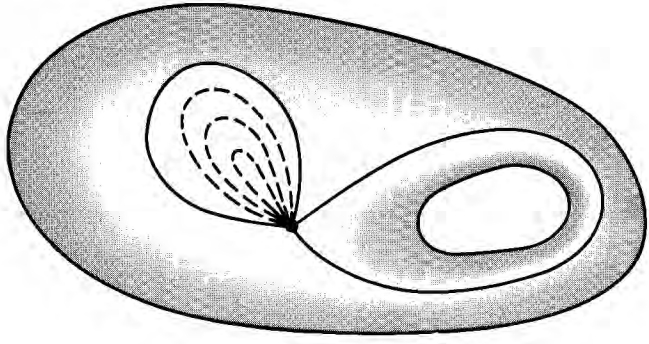

Figure A-3

subset A of $\mathbf{R}^2$ is simply connected iff the interior of every loop in A lies in A.

For manifolds it can be shown that a connected manifold is simply connected iff every closed one form is exact. (The proof is a long story.)

§A11. RESIDUAL SETS

A11.1. Definition. *Let X be a topological space and $A \subset X$. Then A is called* **residual** *iff A is the intersection of a countable family of open dense subsets of X. A space X is called a* **Baire space** *iff every residual set is dense.*

Recall that $B \subset X$ is nowhere dense iff $(B^c)^o = \phi$, so that $X \backslash A$ is residual iff A is the union of a countable collection of nowhere dense closed sets. Clearly, a countable intersection of residual sets is residual.

A11.2. Lemma. *Let X be a locally Baire space; that is, each point $x \in X$ has a neighborhood U such that U^c is a Baire space. Then X is a Baire space.*

Proof. Let $A \subset X$ be residual;

$$A = \bigcap_1^\infty O_n$$

where $O_n^c = X$. Then

$$A \cap U^c = \bigcap_1^\infty (O_n \cap U^c)$$

Now $O_n \cap U^c$ is dense in U^c for if $u \in U^c$ and $u \in O$ then $O \cap U \neq \phi$ and $O \cap U \cap O_n \neq \phi$. Hence $U^c \subset A^c$ and so $A^c = X$. ∎

Then the *Baire category theorem* is as follows.

A11.3. Theorem. *Every complete pseudometric space is a Baire space.*

Proof. Let $U \subset X$ be open and $A = \bigcap_1^\infty O_n$ be residual. By A1.4 (*i*) we must

show $U \cap A \neq \phi$. Now as $O_n^c = X$, $U \cap O_n \neq \phi$ and so we can choose a disk of diameter less than one, say V_1, such that $V_1^c \subset U \cap O_1$. Proceed inductively to obtain $V_n^c \subset U \cap O_n \cap V_{n-1}$ where V_n has diameter $< 1/n$. Let $x_n \in V_n^c$. Clearly $\{x_n\}$ is a Cauchy sequence, and by completeness has a convergent subsequence with limit point x. Then

$$x \in \bigcap_1^\infty V_n^c \qquad \text{and} \qquad U \cap O_n \neq \phi \qquad \blacksquare$$

For the case of a locally compact regular space, see J. Kelley [1, p. 200].

A11.4. Corollary. *Every manifold is a Baire space.*

This follows at once from A11.2 and A11.3 (see A4.4).

Exercises

AA. Let S and T be sets and $f : S \to T$. Show that f is a bijection iff there is a mapping $g : T \to S$ such that $f \circ g$ and $g \circ f$ are the identity mappings on T and S, respectively.

AB. Let X and Y be topological spaces with Y Hausdorff. Then show that, for any continuous maps $f, g : X \to Y$, $\{x \in X | f(x) = g(x)\}$ is closed. Hint: Consider the mapping $x \rightsquigarrow (f(x), g(x))$.

AC. (i) Prove A3.4 and A3.5.

(ii) Prove A1.7 (*ii*), A4.3, and A4.4. Also, in a Hausdorff space, single points are closed.

(iii) Complete the proofs for A5.2, A5.3, and A5.4.

AD. Define a *topological manifold* as a space locally homeomorphic to $\mathbf{R}^n$. Find a topological manifold that is not Hausdorff and not locally compact. Hint: Consider $\mathbf{R}$ with "extra origins."

AE. Prove A7.2, A7.4 (*ii*) and a corresponding statement for arcwise connected spaces. Show that the continuous image of a connected (resp. arcwise connected) space is connected (resp. arcwise connected).

AF. Let M be a topological space and $H : M \to \mathbf{R}$ continuous. Suppose $e \in H(M)$. Then show $H^{-1}(e)$ divides M; that is, $M \setminus H^{-1}(e)$ has at least two components.

AG. (i) Show that $\bar{\rho}$ in A8.1 is not symmetric, by an example.

(ii) Prove that $\bar{\rho}$ is a metric on the closed subsets (including ϕ).

(iii) Express the definition of uniform convergence of a sequence of (real) functions in terms of the Hausdorff metric.

(iv) If X is a compact metric space and ρ and ρ' are equivalent metrics, then show $\bar{\rho}$ and $\bar{\rho}'$ are equivalent.

(v) In case X is not compact show that (iv) can fail.