Symmetric Chaos: How and Why

Mike Field and Martin Golubitsky

Symmetry is a powerful tool in our understanding of natural phenomena. Indeed, symmetries often appear at the heart of mathematical models of the physical world as they constitute a mathematical realization of spatial invariance. As examples of mathematical models that include symmetry as a basic modeling assumption, we cite the equations of fluid mechanics, elasticity, general relativity, electromagnetism, and superconductivity. In view of the assumptions of symmetry underpinning these equations, it is not so surprising that symmetry has often played a fundamental role in their analysis.

Mathematicians and physicists have long realized that when symmetries are present in equations, there are important relations between the possible symmetries of solutions and the group of symmetries of the equation. Typically, the symmetries of solutions can be interpreted as a pattern in the physical space realization of that solution. This observation is seen in the equilibria of differential equations and leads to the subject of spontaneous symmetry breaking. (We refer to [SG] for a nontechnical overview.)

The Faraday Experiment

A cup of coffee sitting on a table in a moving train shows that standing waves can form on the surface of a vibrating fluid. These waves were first studied and described by Michael Faraday in 1831 [Fa]. In experiments, waves are formed by acoustically vibrating a fluid layer in a container, usually with a square or circular cross-section, at a given amplitude and frequency. At low amplitudes, the surface of the fluid remains flat. As the amplitude is increased, the fluid surface deforms and waves appear. Because of the homogeneity of the fluid, the differential equations modeling this experiment have the symmetry of the cross-section.

In Figure 1, we show data from recent elegant work on the Faraday experiment performed by Bruce Gluckman in Jerry Gollub's laboratory at Haverford College [GAG, GMBG]. In these experiments, the deformation of the fluid surface is seen through shadowgraph pictures; these images provide a vivid illustration of patterns that can result from symmetry. The images are formed by recording, on a video camera, the intensity of light transmitted through the bottom of the container and refracted through the top surface. This arrangement has the effect that the image is bright only where the fluid surface is almost flat. See [GAG] for details.

On the mathematical side, the Faraday experiment is modeled by a periodically forced system of partial differential equations. From the point of view of dynamics, the experiment may be modeled by the *stroboscopic map*, that is, the map that records the position of the surface after each period of the forcing. Viewed in this way, the stroboscopic map undergoes a period doubling bifurcation when the flat surface (a fixed point of the stroboscopic map) loses stability as the frequency of vibration is increased. The shadowgraphs in Figure 1 are images in

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physical space taken at an amplitude just beyond the point of this period doubling bifurcation.

Symmetric Chaos

It is only fairly recently that the issue of symmetries of more complex dynamical states in dissipative systems has been discussed. Of course, a "complex dynamical state" is a pseudonym for an aperiodic chaotic attractor, and so we are really asking whether there is a subject of *symmetric chaos*. In particular, do there exist symmetric chaotic attractors? Can one even make sense of this terminology? It turns out that some simple numerical explorations strongly suggest that there are affirmative answers to these questions.

The numerical experiments that are performed are quite simple in character. We form a dynamical system by iterating a planar mapping f having $\mathbf{D}_m$ symmetry, where $\mathbf{D}_m$ denotes the dihedral group of order $2m$ (the symmetry group of the regular m-gon). As it is simplest to work in complex coordinates, we suppose that $f : \mathbf{C} \to \mathbf{C}$. The dynamics are produced by choosing an initial point $z_0 \in \mathbf{C}$ and iterating to form the trajectory $z_{n+1} = f(z_n)$. Thus, the images in Figure 2 are each pictures of a single trajectory.

We now discuss in more detail how we choose the polynomial mapping f. The map f is symmetric or *equivariant* if it maps pairs of symmetric points to pairs of symmetric points. That is, if for all $\sigma \in \mathbf{D}_m$ and $z \in \mathbf{C}$, we have the equivariance condition

$$f(\sigma z) = \sigma f(z).$$

It is a straightforward exercise to show that the general $\mathbf{D}_m$-equivariant polynomial may be written in the form

$$f(z) = p(z\bar{z}, Re(z)^m)z + q(z\bar{z}, Re(z)^m)\bar{z}^{m-1},$$

where p and q are real-valued polynomials, uniquely determined by f. For the purposes of numerical calculation, we truncate to

$$f_0(z) = (\lambda + \alpha z\bar{z} + \beta Re(z)^m)z + \gamma\bar{z}^{m-1},$$

where $\lambda, \alpha, \beta, \gamma \in \mathbf{R}$. A polynomial mapping with cyclic $\mathbf{Z}_m$ symmetry can be produced by adding the symmetry breaking term $i\omega z$.

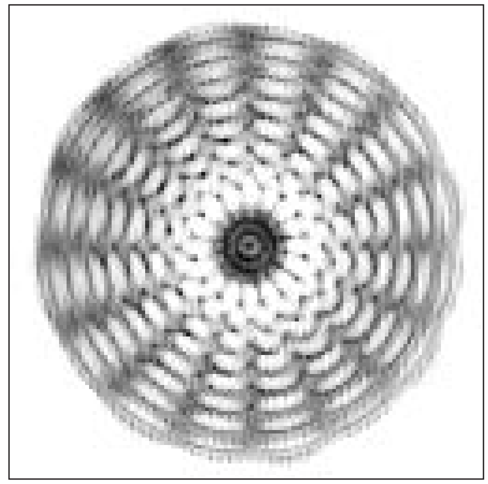
$$g(z) = (\lambda + i\omega + \alpha z\bar{z} + \beta Re(z)^m)z + \gamma\bar{z}^{m-1}.$$

The cover image, *Swirling Streamers*, has $\mathbf{Z}_4$ symmetry and was obtained by setting

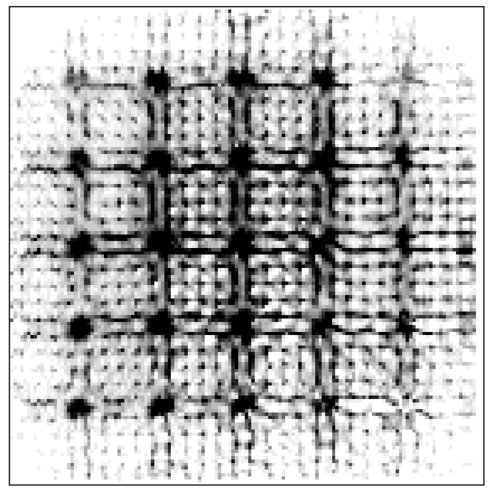
$$m = 4,\ \lambda = -1.86,\ \alpha = 2.0,$$
$$\beta = 0.0,\ \gamma = 1.0,\ \omega = 0.1.$$

This image represents a single trajectory of the dynamical system g. Starting from the initial condition $z_0 = 0.1 + 0.3214i$, we iterated the mapping 300,000,000 times, ignoring the first 1,000 iterates. The specific colors used in *Swirling Streamers* were chosen on aesthetic grounds, but the method of coloring is based on mathematical principles that we discuss below.

Numerical simulation [CG, KiS, FGa] generates a wide array of pictures that display an intriguing mix of symmetry and dynamics. A large number of color images may be found in the book *Symmetry in Chaos* by the authors [FG]. Figure 2 shows some examples of the apparently symmetric images that can be found by direct computation. (For each of these images, we computed 500,000 iterates. So as to ignore any transient effects, we did not plot the first thousand iterates.)



(a) Circular cell (r = 3.5 cm). Frequency = 81 Hz. e = 0.01.



(b) Square cell (1 = 10.5 cm). e = 0.01.

Figure 1 Shadowgraphs of patterns at onset for (a) circular geometry and (b) square geometry. Images courtesy of J. P. Gollub.

What Are We Simulating?

It is reasonable to ask, however, whether the symmetry we see in these images is actually reflected in the underlying dynamics. Indeed, is the symmetry real or not? First, owing to round-off errors, the image that we see only approximates the real attractor, and so the computed image will *never* be exactly symmetric. Second, we believe that the attractors that we have pictured are actually chaotic aperiodic attractors. Yet, if a single trajectory has a nontrivial symmetry, then that trajectory must be periodic. (Proof: Suppose that σ is a symmetry and that $f^n(x) = \sigma x$ for some positive integer n. It follows easily by equivariance that $f^{kn}(x) = \sigma^k x$ for every positive integer k. Since the symmetry group is finite, $\sigma^k = I$ for some k and x is a point of period kn.)

In fact, the idealized object that we are attempting to approximate with our computations is the set of all limit points of the trajectory, that is the ω-limit set of the trajectory. Theoretical questions about symmetry must therefore deal with the ω-limit set rather than the actual trajectory.

We now discuss whether the symmetry that we see is really there. Assume that the ω-limit set A is stable (points near A stay near A under forward iteration). Let σ be a symmetry of the mapping and assume that $\sigma(A) \cap A \neq \emptyset$. Then $\sigma(A) = A$ [G]. Under these hypotheses on A, the symmetries that we see are exact symmetries of the underlying ω-limit set.

We use color to bring out additional details of the structure of the attractor. There is a natural mathematical idea suggesting how the attractor should be colored [FGa, FG]. Numerically, iterate the mapping a large number of times—say 30,000,000—and count the number of times each pixel is visited during the iteration process. Then color by number; pixels with the same color are therefore pixels that are equally likely to be visited. If we assume that the attractor has a Sinai-Bowen-Ruelle measure, then our recipe is an attempt at color coding the invariant measure on the attractor [R] and so our images illustrate the sym metric structure of invariant measures of attractors [DGN].

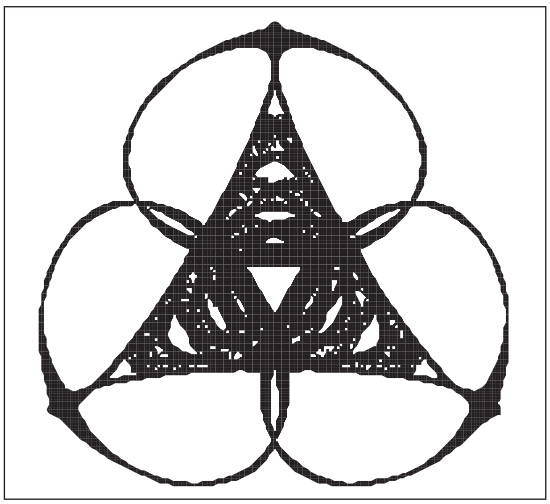


Figure 2(a) Images of symmetric chaos—triangular (m = 3) symmetry (= 1.52, Æ = -1.21, Ø = -0.091, = -0.8005, ! = 0).

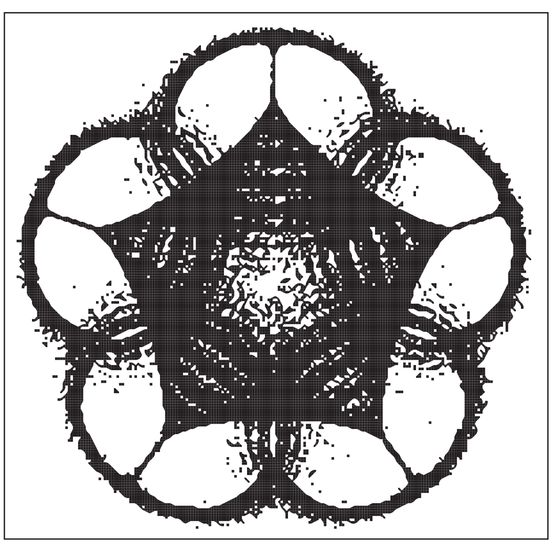


Figure 2(b) Images of symmetric chaos—pentagonal (m = 5) symmetry = 1.415, Æ = -1.21, Ø = 0.001, = -0.8005, ! = 0).

Some Mathematical Questions

Of course, the pictures by themselves are only approximations to conjectured entities. Indeed, it is notoriously difficult to prove the existence of Sinai-Bowen-Ruelle measures on invariant sets arising from *explicit* mappings or differential equations. Nevertheless, the images do suggest some interesting mathematical questions, questions that are spawned by the computer exploration.

1. Is every subgroup of the symmetry group of the dynamical system a symmetry subgroup of an attractor? Does this answer depend on the type of dynamical system that is being considered: noninvertible, diffeomorphism, or differential equation?
2. How do symmetry types of attractors change as parameters in the dynamical system are varied?
3. Is it possible to develop effective numerical algorithms that determine the symmetry groups of attractors for mappings and flows in higher dimensions?

It turns out, perhaps surprisingly, that there are restrictions on the symmetry groups of attractors for mappings and flows [MDG, AM, FMN]. For example, it is not possible for a continuous planar mapping having hexagonal $\mathbf{D}_6$ symmetry to have an attractor with triangular $\mathbf{D}_3$ symmetry. Generally, for a finite group Γ acting on $\mathbf{R}^n$, a subgroup $\Sigma \subset \Gamma$ is *admissible* if there is a continuous Γ-equivariant map $f : \mathbf{R}^n \to \mathbf{R}^n$ having an attractor with symmetry subgroup precisely Σ. (For example, all subgroups of $\mathbf{D}_6$ except $\mathbf{D}_3$ are admissible [KiS].) Ashwin and Melbourne [AM] have given necessary and sufficient *algebraic* conditions for admissibility.

For diffeomorphisms and flows with finite symmetry, there are additional restrictions on admissibility [FMN]. The corresponding questions for continuous group symmetry are still unresolved, though some progress has been made [ACS, M, FGN]. Answers to all of these questions come in two parts: the discovery of computable algebraic invariants that determine inadmissibility and the explicit construction of dynamical systems and attractors that establish admissibility.

Changes in symmetry type as parameters are varied were noted in the first numerical explorations of symmetric dynamical systems [CG]—leading to the term *symmetry increasing bifurcation*. These symmetric *crises* [GOY] have been studied recently by Dellnitz [D, DH], both from the numerical point of view of determining algorithms for computing these bifurcations, and from the theoretical point of view of determining how changes in symmetry actually occur.

Turning to our final question, it has been shown, using ergodic theory and some elementary representation theory, that it is possible to compute a number that determines whether a given symmetry fixes an attractor arising (numerically) in a mapping or a system of differential equations [BDG, DGN, GN, T, KrS].

Patterns on Average

We return now to discuss the physical relevance of attractor symmetries. In order to do this, we have to make a connection between the symmetry of an attractor in phase space and the manifestation of that symmetry in physical space. For an equilibrium this correspondence is straightforward (a symmetry of the equilibrium in phase space is a symmetry of the equilibrium state in physical space). For time-periodic solutions, the connection between phase space symmetries and physical space symmetries is also well understood. But, for complicated dynamical states, the issue has not been raised until recently.

The correspondence proceeds as follows: Let $u(x,t)$ be a (time-dependent) solution to a (partial) differential equation and let A be the "attractor" associated with u in the state space of the differential equation. Form the time-average

$$U(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x,t)dt$$

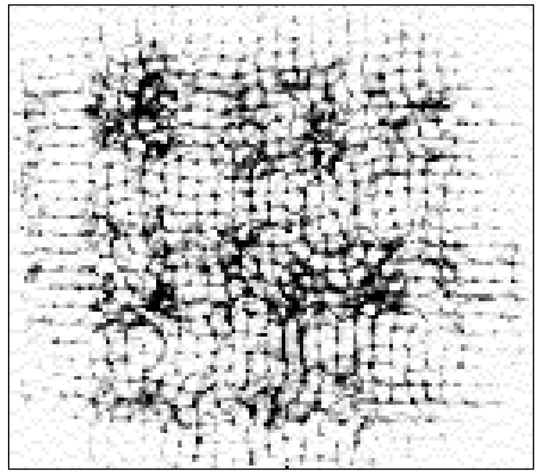
at each point in space. Suppose that the symmetry σ (which acts in space and hence on state space) leaves the attractor A invariant; that is, $\sigma(A) = A$. Under reasonable hypotheses [DGM, DGN], we find that

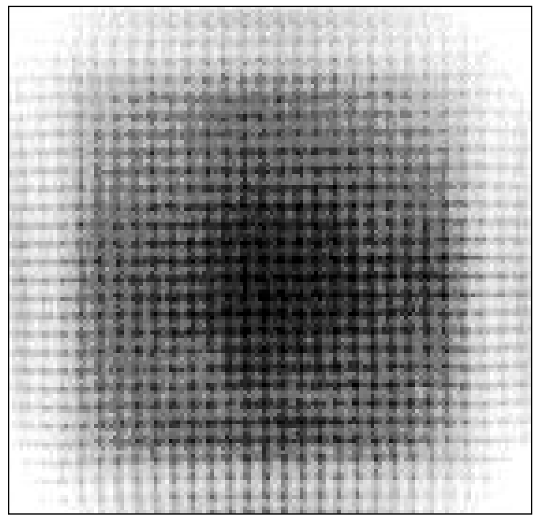
$$U(\sigma x) = U(x).$$

Thus, the time-average has potentially more symmetry (and a more well-defined regular pattern) than the solution $u(\cdot,t)$ at a given instant of time.

This implication, suggested by computer simulations and the images of symmetric attractors, is illustrated in the Faraday experiment of Gluckman *et al.* [GMBG, GAG]. A shadowgraph image, when the forcing amplitude is large, is shown in Figure 3(a). Observe the lack of a regular pattern and the absence of instantaneous time symmetry. In Figure 3(b), a time-average of the intensity of the transmitted light is shown. Note the regular symmetric pattern that has appeared in the time-average. As often happens, the experiments have raised interesting issues not anticipated by theory. In this case, the apparent periodicity in the time-average was unexpected [GAG]. Recently, Bosch *et al.* [BLW] have observed transitions between symmetry types in the Faraday experiment.

We see, therefore, that for chaotic attractors, there are two types of attractor symmetry [MDG]: those that fix the attractor pointwise, the *instantaneous symmetries*, and those that fix the attractor setwise, the *average symmetries.*





Figures 3(a) and 3(b) Square Cell (l=10.5 cm). e=1.00.

Figure 3 (a) Instantaneous and (b) time-average patterns for chaotic dynamics in square geometry. Images courtesy of J. P. Gollub.

Concluding Remarks

What we have surveyed here is a small body of theoretical mathematics that was inspired by asking what happens when symmetry and chaotic dynamics are mixed. Insight and intuition were initially developed using some (relatively simple) computer simulations. With the ever-growing availability of personal computers and graphics workstations, this type of question will surely become more central to mathematical research. A side benefit of this particular exploration is a set of rather striking images that combine the regularity of symmetry with the complexity of chaos.

The workstation program *prism* (Program for the Interactive Study of Maps) that we used to form the (monochrome) images shown in Figure 2, is available (for workstations using Motif) by anonymous ftp. E-mail Mike Field (`mf@uh.edu`) or Marty Golubitsky (`mg@uh.edu`) for additional information.

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Editor's Note

"Mathematics and Symmetry" is the theme for Mathematics Awareness Week, April 23-29, 1995. Mike Field and Martin Golubitsky will design the MAW poster and postcard visuals using the algorithms for iterative procedures as described in this article. For more information, contact the Joint Policy Board for Mathematics at 202-234-9570 or browse the MAA gopher for information on MAW.