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THE MATHEMATICS OF EVOLUTION

Thanks, Bob. It's a pleasure to be here, back on the East Coast where I started. You probably think it's a warm day, but to a Californian it's bracingly and stimulatingly cool. Down here in the bowels of the earth, of course, it's cozy and warm. The bowels of the earth is a metaphor. Thinking of the earth as a living body goes back to "The Clan of the Cave Bear." But since James Lovelock's book, entitled "The Faces of Gaia: A Biography of Our Living Earth", we are seeing a revival, a pagan revival of the idea that everything, the whole thing is actually alive with creativity, love and life. If general evolution theory could be repackaged as a theory of creation, life, and love, I'd like to do it.

The lithosphere, the hydrosphere, the atmosphere, the biosphere, the socio-political-econosphere, the noosphere, these are all related and comprise a whole system, each in meaningful harmony and interaction with the others: a kind of revival of pagan Pythagorean theory of the harmony of the spheres. This is just another historical perspective, I guess.

The subject of these lectures is *From Cosmos to Consciousness*: *New Evolutionary Synthesis.* **We want to connect this with the old Pythagorean theory of the harmony of the spheres. Those who devote their lives to it don't do it simply because it's interesting. I can't speak for everyone, but I think they do it because it seems to be medicine for a disease that has afflicted our society since the scientific revolution. With the good comes the bad, and the bad is the specialization of the sciences -- the fragmentation of the University into the Multiversity of many different departments, each with its specialized jargon. None of them can speak with any of the others.**

A cure would be a general theory that would allow all these departments to join hands and create a level of understanding of the whole, which today exists only for the parts. We seek understanding of the whole because we feel it's necessary to participate consciously and with understanding in evolution. It's not that we're pessimistic for the future of humankind on this planet. Our future would be dark indeed if we failed to achieve an increase of understanding to help us to interact creatively with our own evolution so that the global problems that appear to be potentially fatal to Gaia, our living planet, can be eobviated. That's the whole idea. In spite of the apparent applicability of the evolution concept to any field of science and life, and to the whole of history, it remains as difficult as ever for a person in one speciality to speak to a person in another. A universal theory of evolution, while it seems to go in the direction of unifying the sciences and generating global understanding of the whole, doesn't really achieve this goal.

Therefore, the hope is mathematics. Mathematics is always the handmaiden of the sciences. Mathematics itself is not a science. Science is science. There is a great difference in the personality of a mathematician and that of

a scientist. They are different in every way. The role of mathematics in the service of science is just a small part of its entire existence. In the history of science since Copernicus, mathematics has played the role of compression. Of unification. Of providing a universal language for scientists to communicate their results to each other within a speciality, and without a speciality.

So now we come to the subject of this talk. March 13th. The Mathematics of Evolution: Equations that Generate Structured Catastrophe and Chaos. Mathematics has a mathematical model to offer, a strategy for making mathematical models for evolving systems. It's not easy to do, and it's only a beginning. But the possiblity exists now to unify all the sciences around the concept of evolution by using a certain kind of mathematics that evolved from the examples of evolution in all these fields. The daughter of the sciences that for hundreds of years has desparately tried to control, be controlled by, or live in harmony with the planet, is called complex dynamical systems theory. The part that provides the actual models for evolution is called bifurcation theory. I'm going to give some examples, maybe roughly six, if it goes according to plan. We'll start with two examples which are well worked out -- not finished, but they are easy to understand and are covered in detail in the literature, particularly in my own writings.
The other four that I'm going to propose are in the category of fantasy and science fiction. They show ways to extend these models, to modify them and to give a mathematical basis for ideas that have been presented in this series by the six preceding speakers.

I came across a quote today in one of the museums. It's from a letter, I forget from whom, to Galileo. It says that with mathematics, there's no understanding without a picture. What I want to explain is bifurcation theory. It's an encyclopedia of different models of a certain type called response diagrams. I want to give you an example of some response diagrams. To really understand them takes more than an hour, but there are some simple pictures in the books. No complicated formulas, computer programs or anything else are needed to understand this. Dynamical systems theory is a rare example of a new branch of mathematics. Old branches of mathematics - in mathematics, as in anything else, there is evolution; there are branches, theories and departments that are young, medium, and old. Right now, most mathematicians are working on crossing the t's, dotting the i's of mature or post-fluorescent (?) branches of mathematics. The fact is that it gets more and more complicated once it's all worked out. Particularly when there's the habit - which might not be a natural habit - of trying to express mathematics in symbols. It starts out just as ideas, as this letter to Galileo said, back at the beginning of this branch of dynamics.

To begin with, I want to give an example of the response diagram that goes by the name of *Hopf Bifurcation*. This is an exemplary system worked out ten, fifteen, or maybe 20 years ago by William Smith, an applied mathematician in Canada. Fig.2 is Smith's model for puberty. It's a serious model living in the world of endocryne chemistry, blood chemistry, and hormone systems of mammalian reproductive mechanisms. The main idea, which is shared by complex dynamical systems theory and general systems theory or systems dynamics, is to decompose the whole system into parts or subsystems. But don't take this decomposition too seriously; the emphasis should always be on the whole system.

In this model there are three parts - three subsystems called the hypothalamus, the pituitary, and the gonads. These are connected by a short pipe carrying a large flow of blood that contains a chemical messenger called LHRH, which is emitted by the hypothalamus when it gets a certain idea. It gets this idea from consciousness, or from a clock deep in the system that lets out a mammoth pulse of this stuff. At the end of a very short pipe, called the portal ventil system, is the pituitary. The hypothalamus is inside the brain, the pituitary is outside. There is a barrier between the brain and the rest of the world. The pituitary emits into the general circulatory system, so it goes on a very long trip to the opposite end of the body until it gets to the gonads, and that's called LH. Of course it does a lot of other things too; this is a simplified model. We don't want our models to be too good because the hi-fi model would not be understandable. The gonads respond by sending out their messenger, which, depending on gender, is either estradiol or testosterone. It makes things happen. Among other things, it delivers a signal back to the hypothalamus. This is physiology, so far. No mathematics.

Then they call in Smith. Smith is in the mathematics department. There are these people who work out there in the neighboring veterinary school where they have bulls, cats and rats and things. They frequently take blood samples and assay them for these difficult molecues; then they report back and make charts and so on. Says Bill, we need a model for this; it just won't settle down, you know; it keeps bouncing. Oh, you mean oscillating, says Smith. Well, we have a theory about that. The theory of oscillation. He makes a model for the hypothalamus, which is in itself a response diagram. And then a model for the pituitary, which is a response diagram, and a model for the gonads, which is another one.

Now here's the first example of a response diagram for these subsystems. We will connect them up by making a response diagram for the whole system and then we will draw that down here. (Fig) We need more dimensions for it, because actually these three items are one-dimensional models and there are three of them, so we need a three-dimensional model. They are all identical. The response diagram has two different directions. One is called control variable, and the other the state variable. The idea for this subsystem is that the hypothalamus is going to be controlled externally by the level of estradiol or testosterone in the blood, which it senses by some sort of biochemical hormone receptor sensory system.

Within a subsystem, control is something that doesn't change by itself. Let's say it has a constant value such as this one and that's the lexout (?) of this two-dimensional picture. One dimension, within which each point is to be a model for a virtual state. These are the states. Within the state direction, unlike the control direction, motion takes place of itself. Like chemical reaction, or the motion of falling bodies, and in this one, what is intended is that the motion goes toward this red thing. For each setting of the control parameter there is a red point called the attractor. It attracts the virtual states nearby. Anything in the green zooms toward the red. What is the state in this case is the rate at which this LHRH is going to be released. The subsystem releases chemicals into the blood, and the state they produce could be too high to control the testosterone level in the blood, in which case it would go down. And if it was too low, it would go up, arriving at the

so-called homeostatic or equilibrium value - the red point, the attractor. All the control does is to adjust the level, the rate at which it is at equilibrium so the subsystem can do its job. The others are also a feedback system where LHRH encourages, and testosterone inhibits the function of the system. Besides the subsystems, there are also connections. In neuronet theory, which is on the forefront of artificial intelligence and neurobiology today, one thinks of the connections in the network as being more important than the internal dynamics of the various subsystems. When these simplest of all possible response diagrams are connected up with the plus-plus and minus in the simplest possible system, what results is a response diagram which is different.

For a while, when the control is full to the left, all three levels of blood chemistry remain constant. But for higher values of this control parameter, they oscillate. This is an example of an attractor which is not a point. When the control is set, you could start up a mammal.(WHAT?) If you give it massive injections to begin an experiment with fixed levels of the three chemicals and hormones in the blood stream, it would run rapidly toward this red loop and then go round and round it. So that means that the level in the blood of each of the three chemicals would be oscillating first higher then lower, higher, lower, and always return to the same value, always repeating the loop again, always in the same time, which is called the period of the oscillation. In this case it's six hours. Or maybe 12 hours, depending on whether it's a rat, human, or a bull. The control parameter has to do with the sensitivity of the hypothalamus to the negative feedback from the gonads. Which, as it turns out, can be different in different individuals. It is particularly different at different ages.

Somewhere, as this control is changed, the static equilibrium of the sensitivity of the hypothalamus to testosterone is gradually changed into this oscillation. This is a standard item in the literature of bifurcation theory, called a *Hopf bifurcation.* **It's also Smith's model for puberty. Normally, if you set out to do a job of applied mathematics and you come up with a response diagram, it's not as simple as this. It's not just one of the stock items out of the book on bifurcation theory. It might have a lot of these different bifurcation events. In this picture the bifurcation event is the value of this control parameter at which the point changes into a circle. Now, in this picture, I want to map evolution, I mean to say history, the history of an individual. It just happens that there is a relationship between the sensitivity c(?)and the age of the mammal, so that it gets more and more sensitive. That's just the unexplained function of aging in the hypothalamus.**

So the younger individual is here, H2, and the older individual is here. At some age, the history trajectory of this individual passes through the so-called bifurcation value, usually around 15 or so. That's puberty. If you take blood samples, you find that after puberty, along with all the other visible or perceptible signs of puberty, there is also this invisible one - the daily oscillation of blood chemistry. That's just an example from the literature. It's a really nice one. People came along after Smith and said, well, the hypothalamus can't be totally ignorant of what the pituitary is doing because they're extremely close together, and in the pipe that goes from the hypothalamus to the pituitary there's a feedback, because the flow is turbulent. In Smith's model, if you add this extra connection, the response

diagram changes radically. You get not one cycle but two -- one normal one and the other much lower, where the values of all the hormones are depressed and correspond more or less to ammenorea(?). You could make that connection if the data from the clinic agreed with the model, but if it didn't, you could fix the model a little bit so that it did. And this ammended model, which has an extra cylce down here, suggested to clinicians a treatment strategy for ammenorea which was tried, and it worked. Now I won't say it worked because the model is correct, but this is just an example of a real life case, where making a model and having what amounts to a map of the territory, provides you with intelligence to navigate from point A to point B -- from where you are to where you want to get -- from a pathological state to a desired one.

Blood chemistry may not be your favorite topic, but this model is a really good example because it is obviously tied to evolution. Where we would like to go with this theory is to make a model like this for the whole of history, showing changes in social transformations corresponding to catastrophes and chaotic states in this response diagram. A good thing about this model is that we see the history of an individual within the diagram, and it just so happens that this history, the normal history, just moves to the right, but that would not always be the case. If we took an individual and gave him an injection of something that blocked receptors at the thalamus so that it became less sensitive, the history of the individual might go back again to here, or it might go back again to there, and so on. We have to look at a reponse diagram as a map for navigation, and within that map we could move around in any direction. The usual relationship between the response diagram for a model and the target system, which may be in a process of evolution, is ???

I have another example. This one is less medical. It's from the cover of my book. I like this one a lot -- not only the event itself, but also the drawing. That's why I put it on the cover of the book. There are six stages I want to explain in these blow-ups. You can do this experiment yourself with a glass of water and a spoon. People frequently use a cup of coffee, I don't know why. Here is the idealized cocktail glass with a stirrer inside. This is the experiment: usually the glass is rotated one way and the rod inside rotates the other way. You might have two knobs, but we don't want to make it too complicated, so here is the outer cylinder. The glass is fixed and the stirring rod is rotating. That's one way to stir cocktails. This is the control knob; in this case the experimentalist gets to actually control the control knob where in complex systsm we frequently have no control. They use sophisticated equipment for this experiment in labs. It is the most studied of all hydordynamical experiments exhibiting turbulence.

Here are three different views of the same thing: this one is a photograph of the actual cylider; here is the bar in the middle, and the fluid is pink. We are looking at the outside of the cylinder, which is transparent. The fluid is inside; it's not being stirred now, so there's nothing going on. That's A. Then B is a picture that shows what is happening if you are a molecule of the water. It's not moving, it's just sitting there. Then you turn the knob. Now in A the cylinder still looks the same. We know that if you sprinkle dust in the water, you will see B, which shows that molecules are rotating around because they're being stirred. And on the outside of the cylinder there are some dots. They are not moving because the water is sticking to the glass. On the inside cylinder they're moving with the speed of the cylinder, which is whatever you selected -- not too fast, a slow rotation, the so-called *Lanaler* *flow,* **because they're concentric cylinders of different radii. From the outside to the inside each cylider is rotating as a rigid cylinder, but it consists of water. Fluid dynamics belonged to physics for a long time, but the physicists couldn't understand turbulence, so they rejected it and gave it to the engineers. Now something surprising happens. You stir a little faster and you get an example of morphogenisis, pattern formation, dissipated structure, evolution of complexity. Very amazing. It was discovered a long time ago, 1834 or so, by Taylor; they're called** *Taylor cells*. **Looking at the side of the fluid through the outer cylinder, you see these stripes called** *the barber pole pattern.* **Actually they are pretty much horizontal stripes. If you look again at the dust in the water, you see that there is a stack of these annually(?) where the water is spiraling. In one it spirals around one way and in the next it spirals around the other way. At the interface between two of these annually(?) the spirals actually match up, and there's no friction between the layers.**

Amazingly, the height of these cylinders is fixed. I mean you can jiggle it, kick it, whatever, it divides up to some number, say seven of these so-called Taylor cells and that's it. You end up with a very stable structure. What hapens between the previous one and this one is what is called a bifurcation, where one structure disappears and another one appears. Why, is not explained by this model. It is just a phenomenological model. That's a bifurcation event. If I stir it a little faster, these cells are still there the same as before, and the same thing is going on in them, as you can tell by looking at the dust, but they're wavy. They are called wavy vortices. They're vortices or eddies, it's still spiralling around. The only thing that's changed now is that the boundary between any two of them is wavy instead of flat, that's another bifurcation.

This representation down here is another space. This is the fluid flow, that's what is going on in a given instant: the state of the system, the fluid in the cylinder is it's velocity vector field. This means that each point in it has a certain speed and direction; different point, different speed and direction. In a given instant all those speeds and directions consist of one state of the system. What mathematics is good for is that it will shrink all of that complexity to a point, and we'll see that point as being in some other space where it's only a point. PLEASE ELABORATE!

There are three different kinds of bifurcations, I mean there are a zillion bifurcations, but they all fall into these three classes. There's the so-called subtle bifucation which is the first one I showed you, the heart bifurcation. Look at what happens here. This is the navigational map. When you turn the control knob this way past the exact critical value, this steadiness, if you happened to be stirring coffee would be like this: you're stirring slowly, then you stir a little faster and the top of the coffee begins going. When it first starts going, it starts oscillating very slightly. You can't even see it. Then you stir a liitle faster and the oscillation increases a little more. Finally, you can see it. That's why it's called a subtle bifurcation. But it does start at the same point each time you increase the control, and it stops at exactly the same point when you decrease the control. That is characteristic of subtle bifurcation. Now there is another whole class of bifurcations, called explosions: a little attracter jumps up and instantly becomes a big one. A little oscillation goes (?) and becomes a big one.

There is a third kind called catastrophes. What happens in catastrophes is that one of these attracters will actually vanish. It disappears into the blue, and the system finds itself nowhere near its equilibrium; it has to zoom off to another state. So here is our current level we have waiting (?). One way to understand this map is to look at a single point in the fluid in the cylinder and look at its velocity, but only in a single direction, let's say north. Then you get a number and then the velocity there is increasing and decreasing because the waves between the cells are actually slowly rotating. So when you look at a spot, one of these waves might go past you and then the velocity will change. When all of this is mapped into this space called superspace, the entire velocity vector field is going around one of these circles. What has happened here between the previous situation which had flat boundaries between the Taylor cells, corresponded to a point in superspace because the velocity vector field for the fluid is moving. The velocity vetor field is not; so it is regarded as a single point in superspace. Whereas, when you have wavey (?) then it's going around like that. The size of this oscillation gives an idea of the full range of the velociti's variation at a given point, and it is bigger when the waves between Taylor cells are big. So, here again we have an example of this (?); this same phenomenon appears again.

The good thing about mathematics is that it gives you a model that is applicable to many different experimental environments and domains. Even with history, with models for consciousnes and so on, we always see the same events.

What happens is that after stirring between the Taylor cells you get bumps. Then you start stirring a little faster and these bumps start going back and forth like this. It's like a second oscillation on top of the first one, and it causes a sqeezing off of some of these cells. So, instead of having an entire (?), you just have a piece of one. These pieces of cells are then very unhappy, and they sort of move around, which causes a very erratic behavior in the fluid called mild turbulence. This sequence of different (?) is a model for the onset of turbulence as seen in this particular laboratory experiment. Finally, in the book it's called "white water everywhere! Hang on to your kayak!" It's just all white in the cylinder, and this is the artist's representation of what's going on with the dust particles in the water. Mathematically it's represented as this object which is kind of a thickened innertube called *a bagel attractor.*

You've all seen the familiar six guys and the pictures that were below it now put above the superspace and then connecting up between the images with the fantasy of what goes on.?????? Then there is this red, which is the so-called locus of attraction, and this is a bifurcation diagram for that experiment. Some of the details have not been worked out experimentally, so I just asked my artist partner Chris Shaw to fill in as he saw fit. So here, nothing is rotating then the Taylor cells developed with the flat top which is called the double fold bifurcation. Double fold bifurcation is a catastrophic bifurcation. When you turn the control knob the rate of rotating the inner clyinder this way, this attracter disappears which corresponds to the (?) flow with no cells, and then the system zooms quickly up to this upper one which corresponds to the (?) flow with Taylor cells. They say 'Oh, that was interesting; let us turn that control backwards and see if we can make these Taylor cells go alway.' But you know how spirals are: they're very robust.

They don’t like to go away. So you turn it back, and you see that it does disapear eventually and it zooms back down to the cylindrical (?) flow. But not at the same point; so there’s a (?) loop. (?) in experimental situations is always the sign sent to shore up a catastrophic bifurcation. Whereas here when the (?) vortexes become wavey vortexes there’s a (?) bifurcation. Oscillation starts very gradually; there’s a subtle bifurcation and there’s no (?) there. And here’s the onset of chaos, a mild chaos, and white water. Those are the two examples well worked out in the literature.

How can you make a model for that with the white froth everywhere? Even in Niagara Falls? This has a very simple mathematical model, a so-called low dimensional attractor; in chaos there is order. It’s not exactly my subject this evening; I don’t want to go on about chaos because, despite being quoted in the books as saying that this is the most important thing since the wheel, I actually don’t think it’s a very big thing. Chaos is a double negative. It’s only important now because people rejected it for the past six thousand years and now, thank God, we’ve got it back again. But once we’ve got it back, it seems so natural. The miraculous thing is that the model is so simple.
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